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TREATMENT OF QUANTUM MOTIONS BY PATH INTEGRAL APPROACH

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Toute personne interessée est cordialement invitée

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Contents

Chapter 1

Introduction

The path integral formulation of quantum mechanics is a technique that generalizes the action principle of classical mechanics. It replaces the classical notion of a unique trajectory by taking all the contributions over all possible paths connecting two space-time points (x', t') and (x'', t''), it is a functional integral, over an infinity of possible trajectories to compute the quantum amplitude.

The basic idea of the path integral formulation can be traced back to Norbert Wiener in his attempt to solve the problems in diffusion and Brownian motion, then in 1933 Dirac extended this idea to the use of the Lagrangian in quantum mechanics. In 1948 Richard Feynman had completed and developed the method to have a functional integration formula. Some preliminaries were worked out earlier, in the course of his doctoral thesis work by John Archibald Wheeler.

The physical intuition came from the two-slit experiment. Each time an electron hits the screen, and it is not possible to tell which slit the electron has gone through. After repeating the same experiment several times, a fringe pattern gradually appears on the screen, proving that there is an interference between two waves, one from a slit, the other from the second slit. As a conclusion, it should be a summation of amplitudes of such waves, wave for each path. The generalization of this idea for all possible paths, which means more slits, each of which contributing an amplitude, was the main idea of path integration.

This formulation has proven crucial to the subsequent development of theoretical physics because it is manifestly symmetric between time and space. Unlike previous methods, the path-integral allows a physicist to easily change coordinates between very different canonical descriptions of the same quantum system.

The path integral also relates quantum and stochastic processes, and this provided the basis for the grand synthesis of the 1970s which unified quantum field theory with the statistical field theory. The Schrödinger equation is a diffusion equation with an imaginary diffusion constant, and the path integral is an analytic continuation of a method for summing up all possible random walks. For this reason, path integrals were used in the study of Brownian motion and diffusion then it was great to introduce in quantum mechanics.

The aim of this thesis is to illustrate the path integral technique on concrete problems of quantum mechanics; essentially quantum systems with position-time dependent coefficient. In recent years, the treatment of these systems becomes very intensive because of their important applications in various areas of the material sciences and condensed matter physics. Special applications of these models are achieved in the study of the physical potentials of semiconductors, quantum well, quantum dots, metal clusters and quantum liquids ..etc. Many approaches have been used for studying these systems, the main ones are the supersymmetric quantum mechanics, potential algebras and path integral, and the goal is obtaining the energy spectra and/or the wave functions.

In the second chapter, we present a description of non-relativistic quantum systems according to Feynman's path integral and we show how can this technique be presented in phase space as a functional integral(The propagator) related to the Hamiltonian (The Hamiltonian form), and in configuration space as a functional integral related to the Lagrangian (The Lagrangian form). Then, the propagator characterized by the quadratic action is fully expressed by the classical trajectory.

In the third chapter, we present a way toward obtaining the propagator in the framework of path integrals of general time-dependent systems. The treatment is mainly based on the use of explicitly time-dependent transformations which permit to transform the propagator into a new propagator.

In the fourth chapter, we present a way toward obtaining the propagator in the framework of path integrals of a time-dependent harmonic oscillator with both mass and frequency being arbitrary functions of time. The treatment is mainly based on the use of explicitly time-dependent transformations which permit to transform the propagator for the time-dependent system to a new propagator with constant mass and frequency. We illustrate the general procedure by considering some models of varying mass and frequency.

In the fifth chapter, we present the problem of a particle with time-dependent mass in coulomb potential in the framework of path integrals. The treatment is mainly based on the use of explicitly time-dependent transformations which permit to transform the propagator a new propagator with constant mass which is easy to be treated.

In this sixth chapter, we present the problem of a particle with a position-time dependent mass via path integral in phase space, where we use a point canonical and time transformations to absorb the time dependence of the Hamiltonian. Then by translating the momentum and performing another time transformation, this transforms the problem to that of constant mass. Then, we present some examples.

In the seventh chapter, the problem of a particle in an infinite square well potential will be discussed in the presence of some chosen potentials, a canonical space-time transformation will be performed to solve such problem, where they will be reduced to solvable ones.

The purpose of the last chapter is to find the path integral solution for a non-relativistic particle of electric charge (-e) and mass μ subjected to the influence of a field created by a dyon whose electric and magnetic charges are Q and g, respectively. The main objective is to solve this problem for a quite general vector potential of magnetic monopole elegantly and simply.

Chapter 2

Path integral

The path integral is a technique that is equivalent to the Schroedinger equation and the other standards formulations, which offers a new manner on treating quantum mechanical problems.

In this chapter, we will introduce some basic notions of path integration, that are given by Feynman. We will try to find an expression of this path integration in quantum mechanics in configuration and phase spaces. For simplicity, we will try to find its one-dimensional version and a generalization can be easily done.

2.1 Quantum action principle

In quantum mechanics, as in classical mechanics, the Hamiltonian is the generator of time-translations. This means that the state at the current time and the state at a slightly later time can be related by the Hamiltonian operator $\hat{H(t)}$, for states with definite energy.

The Hamiltonian is a function of the position and momentum at one time, where the Lagrangian is a function, for infinitesimal time separations, is a function of the position and velocity. The relation between the two (The Hamiltonian and The Lagrangian) is given by a Legendre transform. To find the Legendre Transformation, we need to determine the classical equations of motion, which can be found by looking for the conditions that make the action an extremum

In quantum mechanics, we can not know which trajectory the particle will choose since all the trajectories have the same probability, so there is no preferred trajectory than the other one. For each trajectory, we have a corresponding action $S(path)$.

2.2 Feynman formulation

Feynman showed that quantum action was, for most cases of interest, simply equals to the classical action. He proposed the following postulates to find an equivalent version of quantum mechanics:

1− The probability for an event is given by the modulus length squared of a complex number called the "probability amplitude".

2− The probability amplitude is given by adding together the contributions of all paths in configuration space.

3− The contribution of a path is proportional to $e^{iS/\hbar}$, where S is the action given by the time integral of the Lagrangian along the path.

To find the probability amplitude for a given process one adds up the amplitudes $e^{iS/\hbar}$s for each possible path in between the initial and final states. In calculating the probability amplitude for a single particle to go from one space-time point to another, it is correct to include the set of all possible paths in which the particle can take. The path integral assigns to all these paths amplitudes with equal weights but varying phases. Contributions from paths wildly different from the classical trajectory may be suppressed by interference since they vary quickly they cancel each other.

Feynman showed that this formulation of quantum mechanics is equivalent to the canonical approach to quantum mechanics when the Hamiltonian is at most quadratic in the momentum. An amplitude according to Feynman's principles will also obey the Schrödinger equation for the Hamiltonian corresponding to the given action.

2.3 Foundation and concepts of path integral

2.4 Probability amplitude

The probability of reaching a space-time point form an initial space-time by two possible paths is not the sum of probabilities over those paths but the sum of the amplitudes related to those paths. It is convenience to present the amplitudes of wave-functions by complexes numbers, and take $P(q)$ is the absolute square of the transition amplitude $\phi(q)$ form a space-time point to another. By definition the total amplitude of $\phi(q)$ is the sum of amplitudes over the two paths, then we can write

$$\begin{aligned} &P = |\phi(q)| \\ &\phi(q) = \phi(q)_1 + \phi(q)_2 \\ &P_i = |\phi(q)_i| \end{aligned} \tag{2.1}$$

2.5 Transition amplitude

We will see how total amplitude can be found for a particle translates from a space time-point $q(x, t)$ to another $q'(x', t')$ by considering all possible paths. In classical mechanics, the only possible path is that of a minimal action S, but in quantum mechanics, since we have the Heisenberg rule the meaning of path is undefined and it is meaningless, by another word we can not say that the particle will choose this path or the other one under any condition. We can say that all paths have the same probability but with different actions(phase). We call the sum of those amplitudes over all possible paths the propagator $K(q', q)$ for a particle going from q to q' and we write

$$K(q', q) = \sum_{over\ all\ possible\ paths} \phi(x(t)) \tag{2.2}$$

2.6 The propagator

Let us recall the from Schrödinger's point of view that the operators do not depend on time but the wave functions are so doing. In the configuration space (The vector space spanned by the position eigenfunctions$\{|x\rangle\}$) and let $|\Psi(t)\rangle$ be the wave function that can be represented as

$$\Psi(x,t) = \langle x|\Psi(t)\rangle \tag{2.3}$$

As it defined, that the evolution of the state $|\Psi(t)\rangle$ through time can be given using the evolution operator $U(t_f, t_i)$

$$|\Psi(t_f)\rangle = U(t_f, t_i)|\Psi(t_i)\rangle, \tag{2.4}$$

where

$$U(t_f, t_i) = U(t_f, t')U(t', t_i). \tag{2.5}$$

Since the vector space $\{|x\rangle\}$ is orthonormal, then with (2.4) one will find that

$$\Psi(x_f, t_f) = \int dx_i \langle x_f|U(t_f, t_i)|x_i\rangle\Psi(x_i, t_i). \tag{2.6}$$

By defining the kernel $\langle x_f|U(t_f, t_i)|x_i\rangle$ as the transition amplitude we can write that

$$k(x_f, t_f; x_i, t_i) = \langle x_f|U(t_f, t_i)|x_i\rangle, \tag{2.7}$$

where $k(x_f, t_f; x_i, t_i)$ is the propagator.
The propagator (2.7) can be decomposed for small segments of time ε's, $\varepsilon = (t_f - t_i)/(N+1)$, where N is a natural number. Using (2.7) and (2.5) we can write

$$\begin{aligned} k(x_f, t_f; x_i, t_i) = \langle x_f|U(t_f, t_i)|x_i\rangle = \langle x_f|U(t_f, t_N)U(t_N, t_{N-1})...\\ ...U(t_2, t_1)U(t_1, t_i)|x_i\rangle \end{aligned} \tag{2.8}$$

with $t_f = t_{N+1}$ and $t_i = t_0$.

Using the orthogonal properties of the vector space $\{|x\rangle\}$ then

$$k(x_f, t_f; x_i, t_i) = \prod_{n=1}^{N} \int dx_n \prod_{n=1}^{N+1} k(x_n, t_n; x_{n-1}, t_{n-1}), \tag{2.9}$$

where the elementary propagator $k(x_n, t_n; x_{n-1}, t_{n-1})$ is expressed as a function of the Hamiltonian H as

$$k(x_n, t_n; x_{n-1}, t_{n-1}) = \langle x_n | e^{-\frac{i}{\hbar}\varepsilon H} | x_{n-1} \rangle. \tag{2.10}$$

2.7 Path integral

Let us assume the system with the mass m subjected in the potential $V(x)$. The Hamiltonian that describes this system is

$$H = \frac{1}{2m} p^2 + V(x) \tag{2.11}$$

This is not the general case because m and $V(x)$ can be time-dependent, but for simplicity and for the calculation to go smoothly we will consider the time-independent case.

The evolution operator, then, can be given by

$$U(t) = exp(-\frac{i}{\hbar} Ht). \tag{2.12}$$

We are interested on the propagator (or the matrix element of the evolution operator $U(t)$) then

$$\begin{aligned} k(x_f, t_f; x_i, t_i) =& \langle x_f | exp(-\frac{i}{\hbar} Ht) | x_i \rangle \\ & \prod_{n=1}^{N} \int dx_n \prod_{n=1}^{N+1} k(x_n, t_n; x_{n-1}, t_{n-1}) \end{aligned} \tag{2.13}$$

Let us find the expression of the propagator $k(x_n, t_n; x_{n-1}, t_{n-1})$ which is the propagator of particle moving from the point x_{n-1} to the point x_n through the infinitesimal time interval ε, we have

$$\begin{aligned} k(x_n, t_n; x_{n-1}, t_{n-1}) =& \langle x_n | exp(-\frac{i}{\hbar} H\varepsilon) | x_{n-1}\rangle \\ & \langle x_n | exp(-\frac{i}{\hbar}(\frac{1}{2m}p^2 + V(x))\varepsilon) | x_{n-1}\rangle. \end{aligned} \tag{2.14}$$

Using Campbell-Baker-Hausdorff relation one would find that

$$\begin{aligned} k(x_n, t_n; x_{n-1}, t_{n-1}) =& \langle x_n | exp(-\frac{i}{2m\hbar} p^2\varepsilon) exp(-\frac{i}{\hbar} V(x)\varepsilon) \\ & \text{x} exp(-\frac{i}{2m\hbar}[p^2, V(x)]\varepsilon^2) | x_{n-1}\rangle. \end{aligned} \tag{2.15}$$

We keep just those terms of order ε which means that

$$\begin{aligned} k(x_n, t_n; x_{n-1}, t_{n-1}) \backsimeq & \langle x_n | exp(-\frac{i}{2m\hbar} p^2\varepsilon) exp(-\frac{i}{\hbar} V(x)\varepsilon) | x_{n-1}\rangle \\ =& \int dp_n \langle x_n | exp(-\frac{i}{\hbar} V(x)\varepsilon) | p_n\rangle \langle p_n | exp(-\frac{i}{2m\hbar} p^2\varepsilon) | x_{n-1}\rangle \\ =& \int dp_n exp(-\frac{i}{2m\hbar} p^2\varepsilon) exp(-\frac{i}{\hbar} V(x)\varepsilon) \langle x_n || p_n\rangle \langle p_n || x_{n-1}\rangle \\ =& \int \frac{dp_n}{2\pi\hbar} exp(-\frac{i}{2m\hbar} p_n^2\varepsilon) exp(-\frac{i}{\hbar} V(x_n)\varepsilon)\rangle exp(\frac{i}{\hbar} p_n(x_n - x_{n-1})\varepsilon), \end{aligned} \tag{2.16}$$

then after some arrangements

$$k(x_n, t_n; x_{n-1}, t_{n-1}) = \int \frac{dp_n}{2\pi\hbar} exp(\frac{i}{\hbar}(p_n(x_n - x_{n-1}) - H_n)\varepsilon), \tag{2.17}$$

where

$$H_n = \frac{1}{2m\hbar} p_n^2 + V(x_n). \tag{2.18}$$

Inserting this in (2.13) we will find that

$$k(x_f, t_f; x_i, t_i) = \prod_{n=1}^{N} \int dx_n \prod_{n=1}^{N+1} \int \frac{dp_n}{2\pi\hbar} exp(\frac{i}{\hbar}(p_n(x_n - x_{n-1}) - H_n)\varepsilon). \tag{2.19}$$

This is the discrete form of the path integral. In a compact form, it can be expressed as as

$$k(x_f,t_f;x_i,t_i) = \int D[x(t)]D[p(t)]exp(\frac{i}{\hbar}\int dt(p\dot{q}-H)). \tag{2.20}$$

This expression is the expression of path integral in phase space and it is given as a function of the Hamiltonian.
There is another expression of this in the configuration space which can be reached by taking the expression (2.17) and completing the square and making the integral over p

$$\begin{aligned} k(x_n,t_n;x_{n-1},t_{n-1}) &= \int \frac{dp_n}{2\pi\hbar}exp(\frac{i}{\hbar}(p_n(x_n-x_{n-1})-H_n)\varepsilon) \\ &= \sqrt{\frac{m}{2\pi i\hbar\varepsilon}}exp(\frac{i}{\hbar}\frac{m}{2\epsilon}(x_n-x_{n-1})^2-\varepsilon V(x_n)) \\ &= \sqrt{\frac{m}{2\pi i\hbar\varepsilon}}exp(\frac{i}{\hbar}S_n), \end{aligned} \tag{2.21}$$

then

$$k(x_f,t_f;x_i,t_i) = \lim_{N\to\infty}\prod_{n=1}^{N}\sqrt{\frac{m}{2\pi i\hbar\varepsilon}}\int dx_n exp(\frac{i}{\hbar}\frac{m}{2\epsilon}(x_n-x_{n-1})^2-\varepsilon V(x_n)). \tag{2.22}$$

We defined the functional integral measure to be D$[x(t)]$ to be

$$\mathrm{D}[x(t)] = \lim_{N\to\infty}\prod_{n=1}^{N}\sqrt{\frac{m}{2\pi i\hbar\varepsilon}}dx_n. \tag{2.23}$$

Using this we can write our final expression of path integral in configuration space as

$$k(x_f,t_f;x_i,t_i) = \int \mathrm{D}[x(t)]exp(\frac{i}{\hbar}S) \tag{2.24}$$

this expression explains the contribution of each possible path between the binging point x_i and the arrival point x_f.

2.8 The quadratic action

The quadratic system is the simples systems that can be exactly evaluated by path integral(non-time-dependent system), it is a system given by a quadratic form in coordinates and velocities. This system is used as a first approximation for studying some no exactly evaluated systems, moreover, it provides a good example to show mathematical tools used in the path integral.

Between tow points (x_i, t_i) and (x_f, t_f) on a manifold there excites a classical path say $x_{cl}(t)$. Then, a trajectory $x(t)$ connecting these tow points can be given as a fluctuation from the classical trajectory $x(t) = x_{cl}(t) + \eta(t)$, with $\eta(t_i) = \eta(t_f) = 0$. We can consider this as a functional variable transformation. The Taylor development can be written as

$$S[x] = S[x_{cl}] + \int dt \frac{\delta S}{\delta x(t)} \eta(t) + \frac{1}{2} \int dt dt' \frac{\delta^2 S}{\delta x(t) \delta x(t')} \eta(t) \eta(t') + ... \tag{2.25}$$

The second term it vanished since our calculation at the classical path, which means that

$$k(x_f, t_f; x_i, t_i) = e^{\frac{i}{\hbar} S[x_{cl}]} \int D[x(t)] exp(\frac{i}{2\hbar} \int dt dt' \frac{\delta^2 S}{\delta x(t) \delta x(t')} \eta(t) \eta(t')) \tag{2.26}$$

The action is chosen to have the following form

$$S[x] = \int dt (\frac{a}{2} \dot{x}^2 + b x \dot{x} + c x^2 + d \dot{x} + e x f), \tag{2.27}$$

where a, b, c, d, e and f are constants. This will make the propagator to have the following form

$$k(x_f, t_f; x_i, t_i) = e^{\frac{i}{\hbar} S[x_{cl}]} F(t_f - t_i) \tag{2.28}$$

where $F(t_f - t_i)$ is a function of the interval $T = t_f - t_i$ because $\eta(t_f) = \eta(t_i) = 0$. This result is very important and its explaining implies the dependence of the quadratic path integration on the classical trajectory.

2.9 Conclusion

We have presented a new description of non-relativistic quantum systems according to the Feynman path integral technique. And we have shown that the path integral can be put under two prescriptions. The first called the Hamiltonian form (the integral path in the phase space), and the second is the Lagrangian form. Moreover, we have been able to prove that the propagator characterized by quadratic action can be completely expressed by the classical trajectory.

We conclude that the path integral technique has the following advantages:
- It is more intuitive, and its point of view is global; instead of considering amplitudes of probabilities for a state at a given space-time point, we associate a probability amplitude with each possible path between tow space-time points(start-end).
- It allows certain formal manipulations, in particular the space-time transformations and the canonical quantifications.

Chapter 3

The time-dependent systems

3.1 Introduction

During the past decades so much interest has been paid to the subject of time-dependent systems. This comes from the important of theses systems and their applications in various areas of physics[1, 2, 3, 4] is the main reason for intensive studies. There are various methods to solve such systems, like the time-dependent canonical transformations method, the path integral approach, the evolution operator method, the direct integration of equations of motion, or dynamical invariant method. In this chapter we will follow path integral technique and perform space time transformations to the propagator, where will simplify the system under consideration.

3.2 Explicitly Time-Dependent Transformation

In this section, we develop a method of calculating path integral for non-relativistic quantum systems with time-dependent mass in general time-dependent potentials by using explicitly time-dependent space-time transformations technique. For this purpose, we start from the one-dimensional path integral formulation according to

$$K(x'',t'';x',t') = \int D[x(t)] e^{\frac{i}{\hbar}\int \left(\frac{m(t)}{2}\dot{x}^2 - V(x,t)\right)dt}, \tag{3.1}$$

$$= \lim_{N\to\infty} \int \prod_{j=1}^{N-1} dx_j \prod_{j=1}^{N} \left(\frac{m(t_j)}{2\pi i\hbar\epsilon}\right)^{1/2} exp\Big(\frac{i}{\hbar}\sum_{j=1}^{N} S(j,j-1)\Big), \tag{3.2}$$

where $S(j,j-1) = \frac{m(t_j)}{2\epsilon}(x_j - x_{j-1})^2 - \epsilon V(x_j,t_j)$ is the short-time classical action.

Now we consider an explicitly time-dependent coordinate transformation defined by the function $x = h(q,t)$.
With the mid-point consideration, the action depends not only on coordinate mid-point $\bar{q_j} = q_j + \Delta q_j/2$ but also on time mid-point $\bar{t_j} = t_j + \Delta t_j/2$ due to the explicit time-dependence of the potential.

Introducing this transformation and keeping all terms O(ϵ), the measure transforms as follows

$$\prod_{j=1}^{N}\left(\frac{m(t_j)}{2\pi i\hbar\epsilon}\right)^{1/2}\prod_{j=1}^{N-1} dx_j = \Big(h'(q_f,t_f)h'(q_i,t_i)\Big)^{-1/2}\prod_{j=1}^{N}\left(\frac{m(t_j)}{2\pi i\hbar\epsilon}\right)^{1/2} \times \prod_{j=1}^{N}\Big(h'(q_j,t_j)h'(q_{j-1},t_{j-1})\Big)^{1/2}\prod_{j=1}^{N-1} dq_j. \tag{3.3}$$

In the case of time-dependent transformation, the Taylor expansion around the mid-point of functions $m(t_j)$ and $h(q_j,t_j)$ is desirable because it gives a manageable expression of the propagator:

$$m(t_j) \simeq m(\bar{t_j}) - \dot{m}(t_j)\frac{\Delta t_j}{2}, \tag{3.4}$$

and

$$h'(\bar{q}_j - \frac{\Delta q_j}{2}, \bar{t}_j - \frac{\Delta t_j}{2}) \simeq h'(\bar{q}_j,\bar{t}_j) - \frac{\partial h'(\bar{q}_j,\bar{t}_j)}{\partial q}\frac{\Delta q_j}{2} - \frac{\partial h'(\bar{q}_j,\bar{t}_j)}{\partial t}\frac{\Delta t_j}{2} + + \frac{1}{2!}\frac{\partial^2 h'(\bar{q}_j,\bar{t}_j)}{\partial q^2}\frac{(\Delta q_j)^2}{4}. \tag{3.5}$$

After inserting them into Eq.(3.3) and taking into account all contributions up to first order in ϵ, the measure changes as

$$\prod_{j=1}^{N}\left(\frac{m(t_j)}{2\pi i\hbar\epsilon}\right)^{1/2}\prod_{j=1}^{N-1}dx_j = \Big(h'(q_f,t_f)h'(q_i,t_i)\Big)^{-1/2}\prod_{1}^{N}h'(\bar{q}_j,\bar{t}_j)\Big(1-\frac{\Delta q_j^2}{8}\big(\frac{h''^2(\bar{q}_j,\bar{t}_j)}{h'^2(\bar{q}_j,\bar{t}_j)}-\frac{h'''(\bar{q}_j,\bar{t}_j)}{h'(\bar{q}_j,\bar{t}_j)}\big)\Big)$$
$$\times\prod_{j=1}^{N}\left(\frac{m(\bar{t}_j)}{2i\pi\hbar\epsilon}\right)^{1/2}exp\Big(-\frac{\dot{m}(\bar{t}_j)}{m(\bar{t}_j)}\frac{\Delta t_j}{4}\Big)\prod_{1}^{N-1}dq_j, \qquad (3.6)$$

also in the new variable the kinetic energy term has the form

$$\begin{aligned} exp\Big(\frac{im(t_j)}{2\hbar\epsilon}(x_j-x_{j-1})^2\Big) \simeq & exp\Big[\frac{im(t_j)}{2\hbar\epsilon}\Big(h'^2(\bar{q}_j,\bar{t}_j)\Delta q_j^2+\dot{h}^2(\bar{q}_j,\bar{t}_j)\Delta t_j^2+ \\ & 2h'(\bar{q}_j,\bar{t}_j)\dot{h}(\bar{q}_j,\bar{t}_j)\Delta q_j\Delta t_j+h'(\bar{q}_j,\bar{t}_j)h'''(\bar{q}_j,\bar{t}_j)\frac{\Delta q_j^4}{12}\Big)\Big] \\ \simeq & \; exp\Big[\frac{im(\bar{t}_j)}{2\hbar\epsilon}\Big(h'^2(\bar{q}_j,\bar{t}_j)\Delta q_j^2+\dot{h}^2(\bar{q}_j,\bar{t}_j)\Delta t_j^2+ \\ & 2h'(\bar{q}_j,\bar{t}_j)\dot{h}(\bar{q}_j,\bar{t}_j)\Delta q_j\Delta t_j+h'(\bar{q}_j,\bar{t}_j)h'''(\bar{q}_j,\bar{t}_j)\frac{\Delta q_j^4}{12}\Big)- \\ & \frac{i}{4\hbar}\dot{m}(\bar{t}_j)h'^2(\bar{q}_j,\bar{t}_j)\Delta q_j^2\Big]. \qquad (3.7) \end{aligned}$$

Here a dot and primes denote derivatives with respect to the time and to the coordinate q, respectively.

Bringing together these two relations the transformed path integral can be written as

$$\begin{aligned} K(x'',t'';x',t') = & \Big(h'(q_f,t_f)h'(q_i,t_i)\Big)^{-1/2}\int\prod_{1}^{N-1}dq_j\prod_{j=1}^{N}\left(\frac{m(\bar{t}_j)}{2i\pi\hbar\epsilon}\right)^{1/2}exp\Big(-\frac{\dot{m}(\bar{t}_j)}{m(\bar{t}_j)}\frac{\Delta t_j}{4}\Big) \\ & \times\prod_{1}^{N}h'(\bar{q}_j,\bar{t}_j)\Big(1-\frac{\Delta q_j^2}{4}\Big(\frac{h''^2(\bar{q}_j,\bar{t}_j)}{h'^2(\bar{q}_j,\bar{t}_j)}-\frac{h'''(\bar{q}_j,\bar{t}_j)}{h'(\bar{q}_j,\bar{t}_j)}\Big)\Big)exp\Big[\frac{im(t_j)}{2\hbar\epsilon}h'^2(\bar{q}_j,\bar{t}_j)\Delta q_j^2 \\ & +\frac{im(\bar{t}_j)}{2\hbar\epsilon}\dot{h}^2(\bar{q}_j,\bar{t}_j)\epsilon+\frac{im(\bar{t}_j)}{\hbar\epsilon}h'(\bar{q}_j,\bar{t}_j)\dot{h}(\bar{q}_j,\bar{t}_j)\Delta q_j+\frac{im(\bar{t}_j)}{\hbar\epsilon}h'(\bar{q}_j,\bar{t}_j)h'''(\bar{q}_j,\bar{t}_j)\frac{\Delta q_j^4}{24\epsilon} \\ & -\frac{i}{4\hbar}\dot{m}(\bar{t}_j)h'^2(\bar{q}_j,\bar{t}_j)\Delta q_j^2-\frac{i}{\hbar}V(\bar{q}_j,\bar{t}_j)\epsilon\Big], \qquad (3.8) \end{aligned}$$

or else

$$K(x'',t'';x',t') = \Big(h'(q_f,t_f)h'(q_i,t_i)\Big)^{-1/2}\int\prod_{j=2}^{N}\Delta q_j\prod_{j=1}^{N}\Big(\frac{m(\bar{t}_j)}{2i\pi\hbar\epsilon}\Big)^{1/2}$$
$$\times\prod_{j=1}^{N}h'(\bar{q}_j,\bar{t}_j)exp\Big(\frac{im(\bar{t}_j)}{2\hbar\epsilon}h'^2(\bar{q}_j,\bar{t}_j)\Delta q_j^2+\frac{im(\bar{t}_j)}{2\hbar\epsilon}\dot{h}^2(\bar{q}_j,\bar{t}_j)\epsilon$$
$$+\frac{im(\bar{t}_j)}{\hbar\epsilon}h'(\bar{q}_j,\bar{t}_j)\dot{h}(\bar{q}_j,\bar{t}_j)\Delta q_j+\frac{im(\bar{t}_j)}{\hbar\epsilon}h'(\bar{q}_j,\bar{t}_j)h'''(\bar{q}_j,\bar{t}_j)\frac{\Delta q_j^4}{24\epsilon}$$
$$-\frac{\Delta q_j^2}{4}\Big(\frac{h''(\bar{q}_j,\bar{t}_j)^2}{h'(\bar{q}_j,\bar{t}_j)^2}+\frac{h'''(\bar{q}_j,\bar{t}_j)}{h'(\bar{q}_j,\bar{t}_j)}\Big)-\frac{\dot{m}(\bar{t}_j)}{m(\bar{t}_j)}\frac{\epsilon}{4}$$
$$-\frac{i}{4\hbar}\dot{m}(\bar{t}_j)h'^2(\bar{q}_j,\bar{t}_j)\Delta q_j^2-\frac{i}{\hbar}V(\bar{q}_j,\bar{t}_j)\epsilon\Big), \tag{3.9}$$

where it seemed useful to replace the integration over upper positions q_j by the integration over intervals Δq_j thanks to the identity

$$\int\prod_{j=1}^{N-1}dq_j=\int\prod_{j=2}^{N}d(\Delta q_j). \tag{3.10}$$

According to the McLaughlin-Shulman procedure [13] we replace the terms Δq_j^2 and Δq_j^4 appearing in the action by making the substitutions:
$\Delta q_j^2 \to \frac{i\hbar\epsilon}{m(\bar{t}_j)h'^2(\bar{q}_j,\bar{t}_j)}$ and $\Delta q_j^4 \to -3\frac{\hbar^2\epsilon^2}{m^2(\bar{t}_j)h'^4(\bar{q}_j,\bar{t}_j)}$.
Then the propagator admits the following continuous form

$$K(q'',t'';q',t') = \Big(h'(q''.t'')h'(q'.t')\Big)^{-1/2}\int h'(q,t)D[q(t)]exp\Big[\frac{i}{\hbar}\int\Big(\frac{m(t)}{2}h'^2(q.t)\dot{q}^2-$$
$$\frac{m(t)}{2}\dot{h}^2(q,t)-m(t)\dot{h}(q,t)h'(q,t)\dot{q}-\frac{\hbar^2}{8m(t)}\frac{h''^2(q,t)}{h'^4(q,t)}-V(q,t)\Big)dt\Big]. \tag{3.11}$$

We emphasize that the kinetic energy term in (3.11) has unconventional form. This can be fixed up by appropriately chosen time-transformation, but instead if the transformation $x = h(q,t)$ is linear, other transformations will not be necessary as in the case of the time-dependent harmonic oscillator which will be treated in the following section. Also to reach a more convenient form of the path integral (3.11), it is preferable to eliminate the term proportional to $\dot{q}$. For this purpose, let us define a function $F(q,t)$ as

$$F(q,t)=\int^{q}m(t)\dot{h}(z,t)h'(z,t)dz. \tag{3.12}$$

By making use of the relation

$$\frac{dF(q,t)}{dt} = \frac{\partial F(q,t)}{\partial q}\dot{q} + \frac{\partial F(q,t)}{\partial t}, \tag{3.13}$$

we can perform the following replacement in the action

$$\int m(t)\dot{h}(q,t)h'(q,t)\dot{q}dt = F(q'',t'') - F(q',t') - \int \frac{\partial F(q,t)}{\partial t}dt. \tag{3.14}$$

Finally, the insertion of this result into (3.11) enables us to present the path integral for a system with time-dependent mass and furthermore subjected to the action of a time-dependent potential in a simpler form

$$K(q'',t'';q',t') = \Big(h'(q''.t'')h'(q'.t')\Big)^{-1/2}\frac{g(q_f,t_f)}{g(q_i,t_i)}\int h'(q,t)D[q(t)]exp\Big[\frac{i}{\hbar}\int\Big(\frac{m(t)}{2}h'^2(q.t)\dot{q}^2 - \frac{m(t)}{2}\dot{h}^2(q,t) - \frac{\hbar}{i}\frac{\dot{g}(q,t)}{g(q,t)} - \frac{\hbar^2}{8m(t)}\frac{h''^2(q,t)}{h'^4(q,t)} - V(q,t)\Big)dt\Big], \tag{3.15}$$

where

$$g(q,t) = exp\Big(\frac{i}{\hbar}F(q,t)\Big). \tag{3.16}$$

Here we note that unlike time-dependent models studied by some authors, the time-dependence of our system Eq. (3.1) is more general.

3.3 Point Canonical transformation

We consider the time dependent system defined by the following Hamiltonian

$$H(p,q,t) = \frac{1}{2m(t)}p^2 + V(q,t). \tag{3.17}$$

The propagator corresponding to this system can be written in the phase space as Ref. [6]

$$K(q'',p'',t'';,q',p',t') = \int \frac{D[q(t)]D[p(t)]}{2\pi\hbar}e^{\frac{i}{\hbar}\int_{t'}^{t''}dt(p\dot{q}-H(p,q,t))} \tag{3.18}$$

or in the discrete form as

$$K(q'',p'',t'';,q',p',t') = \lim_{n\to\infty}\prod_{i=1}^{n}\int_{-\infty}^{\infty}dq_i\prod_{i=1}^{n+1}\int_{-\infty}^{\infty}\frac{dp_i}{2\pi\hbar}$$
$$\prod_{i=1}^{n+1}e^{\frac{i}{\hbar}(p_i(q_i-q_{i-1})-\frac{\epsilon p_i^2}{2m(t_i)}-\epsilon V(q,t)} \tag{3.19}$$

where

$$\epsilon = t_i - t_{i-1} = \frac{t''-t'}{n+1}, \quad q' = q_0, t' = t_0, \quad q'' = q_{n+1}, t'' = t_{n+1} \tag{3.20}$$

Dealing with the problem (3.19) by a straight way and find its explicit expression is not evident, since the mass and frequency are time-dependent. To be able to solve this we need to make some coordinates transformations $(p,q)\to(P,Q)$ which will simplify the problem in order to be explicitly evaluated for many systems.
By taking the following time-dependent canonical transformations

$$\begin{aligned} q &= f(t)Q \\ p &= \frac{P}{f(t)} \end{aligned} \tag{3.21}$$

where the generating function is

$$F(q,P,t) = \frac{qP}{f(t)} \tag{3.22}$$

The Hamiltonian (3.17) becomes

$$\begin{aligned} H(Q,P,t) =& H(p,q,t) + \frac{\partial F(P,q,t)}{\partial t} \\ =& \frac{1}{2m(t)f(t)^2}P^2 + V(Q,t) - \frac{\dot{f}(t)}{f(t)}QP. \end{aligned} \tag{3.23}$$

Then the exponent in (3.18) is expressed in terms of the new phase-space coordinates

$$\begin{aligned} \int_{t'}^{t''}\Big(p\dot{q} - H(q,p,t)\Big)dt = \int_{t'}^{t''}\Big(P\dot{Q} - \frac{1}{2m(t)f(t)^2}P^2 - V(Q,t) + \\ + \frac{\dot{f}(t)}{f(t)}QP\Big)dt. \end{aligned} \tag{3.24}$$

Also the measure under this transformations as Ref. [17]

$$\frac{dp_{n+1}}{2\pi\hbar}\prod_{i=1}^{n}\int_{-\infty}^{\infty}\frac{dq_i dp_i}{2\pi\hbar} = \frac{dP_{n+1}}{2\pi\hbar f(t_{n+1})}\prod_{i=1}^{n}\int_{-\infty}^{\infty}\frac{dQ_i dP_i}{2\pi\hbar}$$
$$= \frac{dP_{n+1}}{2\pi\hbar\sqrt{f(t_{n+1})f(t_0)}}e^{-\frac{1}{2}\ln\frac{f(t_{n+1})}{f(t_0)}}\prod_{i=1}^{n}\int_{-\infty}^{\infty}\frac{dQ_i dP_i}{2\pi\hbar}. \quad (3.25)$$

Using relations (3.24) and (3.25) and after some arrangements, the propagator Eq.(3.18) takes the following expression

$$K(q'',p'',t'';q',p',t') = \frac{1}{\sqrt{f(t'')f(t')}}\int\frac{D[Q(t)]D[P(t)]}{2\pi\hbar}$$
$$e^{\frac{i}{\hbar}\int_{t'}^{t''}dt(P\dot{Q}-\frac{1}{2m(t)f(t)^2}P^2-V(Q,t)+\frac{\dot{f(t)}}{f(t)}QP)} \quad (3.26)$$

At this point the system still time-dependent and more than that we have an extra term $\frac{\dot{f(t)}}{f(t)}QP$ linear in $\bar{P}$. To remove it we make a shift in the momentum

$$P = \bar{P} + g(t)Q, \quad (3.27)$$

where $g(t)$ is a time dependent function. By inserting this in (3.26) we find the following result

$$K(q'',p'',t'';q',p',t') = \frac{1}{\sqrt{f(t'')f(t')}}exp\frac{i}{2\hbar}(g(t'')Q''^2 - g(t')Q'^2)$$
$$\bar{K}(\bar{P}'',Q'',t'';\bar{P}',Q',t'), \quad (3.28)$$

where the new propagator $\bar{K}(Q'',\bar{P}'',t'';Q',\bar{P}',t')$ has the form

$$\bar{K}(\bar{P}'',Q'',t'';\bar{P}',Q',t') = \int\frac{D[Q(t)]D[\bar{P}(t)]}{2\pi\hbar}exp\frac{i}{\hbar}\int_{t'}^{t''}dt\Big[\bar{P}\dot{Q} - \frac{1}{2m(t)f^2(t)}\bar{P}^2$$
$$-\Big(\frac{g^2(t)}{2m(t)f^2(t)} - \frac{\dot{f}(t)}{f(t)}g(t) + \frac{\dot{g}(t)}{2}\Big)Q^2 - V(Q,t)$$
$$+\Big(\frac{\dot{f(t)}}{f(t)} - \frac{g(t)}{m(t)f^2(t)}\Big)Q\bar{P}\Big]. \quad (3.29)$$

Since $g(t)$ is an arbitrary function we will choose it such that

$$\frac{\dot{f(t)}}{f(t)} - \frac{g(t)}{m(t)f^2(t)} = 0. \tag{3.30}$$

Then the propagator Eq.(3.29) becomes

$$\begin{aligned}\bar{K}(\bar{P}'', Q'', t''; \bar{P}', Q', t') = \int \frac{D[Q(t)]D[\bar{P}(t)]}{2\pi\hbar} exp\frac{i}{\hbar}\int_{t'}^{t''} dt(\bar{P}\dot{Q} - \frac{1}{2m(t)f(t)^2}\bar{P}^2 \\ -\frac{1}{2}m(t)f(t)^2(\frac{\dot{m}(t)}{m(t)}\frac{\dot{f}(t)}{f(t)} + \frac{\ddot{f}(t)}{f(t)})Q^2 - V(Q,t))\end{aligned} \tag{3.31}$$

At this level the function $f(t)$ can be chosen to absorb the time varying mass $m(t)$ or to bring the problem to solvable one.

3.4 Conclusion

In this chapter we have performed an explicitly time-dependent quantum mechanical transformation at the midpoint for the Lagrangian propagator, and point canonical time-dependent transformation for the Hamiltonian propagator. Therefore we have shown that the technique of explicitly time-dependent space-time transformations is a necessary tool in path integral to treat explicitly time-dependent problems or to simplify them, that they can not be in the event exactly evaluated. We have obtained a general formula for the propagator for any quantum system with time-dependent mass and potential simultaneously in both configuration and phase spaces. This will be so helpful in the next two chapters where the time-dependent harmonic oscillator and time dependent coulomb system will be investigated respectively.

Chapter 4

The time dependent harmonic oscillator

4.1 Introduction

A great deal of attention has been paid to the subject of time- dependent Hamiltonians. The main reason for intensive studies of these quantum systems is due to their important applications in various fields of physics, such as quantum optics [1], cosmology [2], nano-technologies [3] and plasma physics [4]. The harmonic oscillator with time-dependent frequency, or with explicitly time-dependent mass, or both simultaneously is the most commonly mechanical system used in this area. These problems have received considerable interest [5-11] and have been solved by various methods, such as the time-dependent canonical transformation method, the path integral approach, the evolution operator method, the direct integration of equations of motion, and dynamical invariant method.

Looking through the literature one finds that an explicit expression for the propagator could not be obtained for all time varying mass-functions because the procedure involves the solutions of non-linear differential equations. This is the reason why only few cases of varying mass has been solved. As mentioned above we can cite the following cases: the strongly pulsating mass [7], the exponentially time-dependent mass [6], the power-low mass [8] and some other examples are given in Ref.[12].

In this paper we will present a way toward obtaining the propagator in the framework of path integrals of time-dependent harmonic oscillator with both mass and frequency being arbitrary functions of time. The treatment is mainly based on the use of explicitly time-dependent transformations which permit to transform the propagator for the time-dependent system to a new propagator with constant mass and frequency. We illustrate the general procedure by considering some models of varying mass and frequency.

4.2 The harmonic oscillator with time-dependent mass and frequency in configuration space

The general time-dependent Lagrangian for a harmonic oscillator is given by

$$L(x,\dot{x},t) = \frac{m(t)}{2}\dot{x}^2 - \frac{1}{2}m(t)\omega^2(t)x^2, \tag{4.1}$$

where $m(t) = m_0 f(t)$ and $\omega(t)$ are well-behaved functions of time. By using explicitly space-time transformations such that $x = c(t)q$, we can write after following the same steps given above, the propagator corresponding to the Lagrangian Eq.(4.1) as:

$$\begin{aligned} K(q',t';q'',t'') =& (c(t'')c(t'))^{-1/2}\xi(q'',t'')\xi^*(q',t')\int D[q(t)] \\ &\times exp\Big\{\frac{i}{\hbar}\int(\frac{m_0}{2}\dot{q}^2 - \frac{m_0}{2}\Omega^2(t)q^2)dt\Big\}, \end{aligned} \tag{4.2}$$

where $\xi(q,t) = exp\frac{im_0}{2\hbar}\frac{\dot{c}(t)}{c(t)}q^2$ and $\Omega^2(t) = \frac{\ddot{c}(t)}{c(t)} - 2\frac{\dot{c}^2(t)}{c^2(t)} + \omega^2(t)$. We have chosen the function $c(t)$ such that $c^2(t)f(t) = 1$, which reduces the problem to that of the well-known of the harmonic oscillator with constant mass and time dependent frequency. The propagator of this system is given by(See Ref.[65])

$$\begin{aligned} K(q',t';q'',t'') = \frac{\Xi(q'',t'')\Xi^*(q',t')}{sin(\gamma(t'')-\gamma(t'))} exp\Big\{\frac{im_0}{2\hbar sin(\gamma(t'')-\gamma(t'))}\Big[(\dot{\gamma}^2(t'')q''^2 + \\ + \dot{\gamma}^2(t')q'^2)cos(\gamma(t'')-\gamma(t')) - 2\sqrt{\dot{\gamma}(t'')\dot{\gamma}(t')}q''q'\Big]\Big\}, \end{aligned} \tag{4.3}$$

where $\Xi(q,t) = \left(\frac{m_0 i^{-1}\dot{\gamma}(t)}{2\pi\hbar c(t)}\right)^{1/2} exp\frac{im_0}{2\hbar}(\frac{\dot{c}(t)}{c(t)} + \frac{\dot{\mu}(t)}{\mu(t)})q^2$, and the functions $\gamma(t)$ and $\mu(t)$ satisfy the following coupled differential equations

$$\ddot{\mu} - \mu\dot{\gamma}^2 + \Omega^2(t)\mu = 0$$
$$\mu\ddot{\gamma} + 2\dot{\mu}\dot{\gamma} = 0 \tag{4.4}$$

4.3 Applications

4.3.1 Example 1

We consider the problem of the harmonic oscillator that has a mass of the form $m(t) = m_0(\alpha e^{\lambda t} + \beta e^{-\lambda t})^2$, where m_0 is a real number, α and β are complex numbers and λ can be either pure real number or pure complex number, such that $m(t)$ has a physical meaning. The Lagrangian corresponding to this system is

$$L(x,\dot{x},t) = \frac{m_0}{2}(\alpha e^{\lambda t} + \beta e^{-\lambda t})^2\dot{x}^2 - \frac{m_0}{2}(\alpha e^{\lambda t} + \beta e^{-\lambda t})^2\omega^2(t)x^2. \tag{4.5}$$

We choose the frequency $\omega(t)$ to be a constant function of time. The propagator of this system is given by

$$K(x'',t'';,x',t') = \int D[x(t)]e^{\frac{i}{\hbar}\int L(x,\dot{x},t)dt}. \tag{4.6}$$

By using the transformation $x = c(t)q$, where $c(t) = (\alpha e^{\lambda t} + \beta e^{-\lambda t})^{-1}$ and following the procedure detailed above one can write the propagator as

$$K(x'',t'';,x',t') = (c(t'')c(t'))^{-1/2}e^{\frac{im_0}{2\hbar}(\frac{\dot{c}(t'')}{c(t'')}q''^2 - \frac{\dot{c}(t')}{c(t')}q'^2)}k(q'',t'';,q',t'), \tag{4.7}$$

where $k(q'',t'';q',t')$ is the propagator corresponding to the Lagrangian $L(q,\dot{q},t) = \frac{m_0}{2}\dot{q}^2 - \frac{m_0}{2}\Omega^2q^2$ which can be exactly expressed

$$k(q'',t'';q',t') = \sqrt{\frac{m_0\Omega}{2\pi\hbar sin(\Omega(t''-t'))}}exp\frac{im_0\Omega}{2\hbar sin(\Omega(t''-t'))}\Big((q''^2 + q'^2)cos(\Omega(t''-t')) - 2q''q'\Big), \tag{4.8}$$

where $\Omega = \sqrt{\omega^2 - \lambda^2}$.

4.3.2 Particular cases

1. The exponentially changing mass $m(t) = m_0 e^{2\lambda t}$

By putting $\beta = 0$ and $\alpha = 1$ we can get this case and the propagator in this case is

$$K(x'', t''; x', t') = \left(\frac{m_0 \Omega e^{\lambda(t''+t')}}{2\pi i\hbar sin(\Omega(t''-t'))} \right)^{1/2} e^{\frac{\lambda m_0}{2i\hbar}(e^{2\lambda t''}x''^2 - e^{2\lambda t'}x'^2)} \times exp\Big\{ \frac{im_0\Omega}{2\hbar sin(\Omega(t''-t'))} \Big[(e^{2\lambda t''}x''^2 + e^{2\lambda t'}x'^2) cos(\Omega(t''-t')) + - 2e^{\lambda(t''+t')}x''x' \Big] \Big\}. \tag{4.9}$$

This result coincides exactly with that given in [6].

2. The strongly pulsating mass $m(t) = m_0 cos^2(\sigma t + \delta)$

To get this case we put $\alpha = \frac{1}{2}e^{i\delta}$,$\beta = \frac{1}{2}e^{-i\delta}$ and $\lambda = i\sigma$, where σ and δ are real numbers. By replacing these quantities in Eqs.(4.7) and (4.8), we obtain the following expression of the propagator

$$K(x'', t''; x', t') = \left(\frac{m_0 \Omega cos(\sigma t'' + \delta) cos(\sigma t' + \delta)}{2\pi i\hbar sin(\Omega(t''-t'))} \right)^{1/2} \times e^{\frac{\sigma m_0}{2i\hbar}(sin(2\sigma t''+2\delta)x''^2 - sin(2\sigma t'+2\delta)x'^2)} \times exp\Big\{ \frac{im_0\Omega}{2\hbar sin(\Omega(t''-t'))} \Big[(cos^2(\sigma t''+\delta)x''^2 + cos^2(\sigma t'+\delta)x'^2) cos(\Omega(t''-t')) - 2cos(\sigma t''+\delta)cos^2(\sigma t'+\delta)x''x' \Big] \Big\}, \tag{4.10}$$

which is the same result given in [7] if we choose $\delta = 0$.

3. The mass $m = m_0 cosh^2(\lambda t + \vartheta)$

This case by putting $\alpha = \frac{1}{2}e^{\vartheta}$ and $\beta = \frac{1}{2}e^{-\vartheta}$. The propagator is

$$K(x'', t''; x', t') = \left(\frac{m_0 \Omega cosh(\lambda t'' + \vartheta) cosh(\lambda t' + \vartheta)}{2\pi i\hbar sinh(\Omega(t'' - t'))} \right)^{1/2}$$
$$\times e^{\frac{\lambda m_0}{2i\hbar}(sinh(2\lambda t''+2\vartheta)x''^2 - sinh(2\lambda t'+2\vartheta)x'^2)}$$
$$\times exp\Big\{ \frac{i m_0 \Omega}{2\hbar sinh(\Omega(t'' - t'))} \Big[(cosh^2(\lambda t'' + \vartheta)x''^2 +$$
$$+ cosh^2(\lambda t' + \vartheta)x'^2) cosh(\Omega(t'' - t')) -$$
$$2cosh(\lambda t'' + \vartheta) cosh^2(\lambda t' + \vartheta) x'' x' \Big] \Big\} \tag{4.11}$$

4.3.2.1 Example 2

The second example will be the harmonic oscillator with the mass $m(t) = m_0 t^2 (\alpha e^{\lambda/t} + \beta e^{-\lambda/t})^2$ and the time dependent frequency $\omega(t) = \dfrac{\omega_0}{t^2}$, where m_0 and ω_0 are real numbers, α and β are complex numbers and λ can be either pure real number or pure complex number, such that $m(t)$ has a physical meaning. To find the propagator corresponding to this system we will follow the same procedure as before and tack the transformation

$$x = \frac{1}{t(\alpha e^{\lambda/t} + \beta e^{-\lambda/t})} q, \tag{4.12}$$

which will lead to the problem of the harmonic oscillator with a constant mass m_0 and a time-dependent frequency $\Omega_0 = \frac{\sqrt{\omega_0^2 + \lambda^2}}{t^2}$. The propagator in this case can be expressed as

$$K(x'', t'';, x', t') = \Big((t' t'' (\alpha e^{\lambda/t'} + \beta e^{-\lambda/t'})(\alpha e^{\lambda/t''} + \beta e^{-\lambda/t''}) \Big)^{1/2}$$
$$\times e^{\left\{ \frac{-i m_0}{2\hbar} \left[(\frac{1}{t''} + \frac{\lambda}{t''^2} \frac{-\alpha e^{\lambda/t''} + \beta e^{-\lambda/t''}}{\alpha e^{\lambda/t''} + \beta e^{-\lambda/t''}}) q''^2 - (\frac{1}{t'} + \frac{\lambda}{t'^2} \frac{-\alpha e^{\lambda/t'} + \beta e^{-\lambda/t'}}{\alpha e^{\lambda/t'} + \beta e^{-\lambda/t'}}) q'^2 \right] \right\}}$$
$$\times k(q'', t'';, q', t'), \tag{4.13}$$

where the propagator $k(q'', t'';, q', t')$ has the following form

$$k(q', t'; q'', t'') = \int D[q(t)] exp\Big\{ \frac{i}{\hbar} \int (\frac{m_0}{2}\dot{q}^2 - \frac{m_0}{2}\frac{\Omega_0^2}{t^4} q^2) dt \Big\}. \tag{4.14}$$

To find the exact expression of the propagator (4.14) we will make another transformation $q = ty$, then follow that by a time-transformation $d\tau = dt/t^2$, then putting everything together the propagator (4.13) will take the final form

$$K(x'',t'';x',t') = \left(\frac{m\Omega_0(\alpha e^{\lambda/t'}+\beta e^{-\lambda/t'})(\alpha e^{\lambda/t''}+\beta e^{-\lambda/t''})}{2\pi i sin(\Omega_0(\frac{1}{t''}-\frac{1}{t'}))}\right)^{1/2} \times e^{\left\{\frac{-im_0}{2\hbar}\left[\frac{\lambda}{t''^2}\frac{-\alpha e^{\lambda/t''}+\beta e^{-\lambda/t''}}{\alpha e^{\lambda/t''}+\beta e^{-\lambda/t''}}q''^2-\frac{\lambda}{t'^2}\frac{-\alpha e^{\lambda/t'}+\beta e^{-\lambda/t'}}{\alpha e^{\lambda/t'}+\beta e^{-\lambda/t'}}q'^2\right]\right\}} \times e^{\left\{\frac{im_0\Omega_0}{2\hbar sin(\Omega_0(\frac{1}{t''}-\frac{1}{t'}))}\left[(\frac{q''^2}{t''^2}+\frac{q'^2}{t'^2})cos(\Omega_0(\frac{1}{t''}-\frac{1}{t'}))-2\frac{q''}{t''}\frac{q'}{t'}\right]\right\}}. \quad (4.15)$$

These tow examples presented here are more generalized than those given in the literature,. The same problems with an inverse quadratic potential can be exactly solved by the following the same steps and choosing the same transformations.

4.4 The harmonic oscillator with time-dependent mass and frequency in phase space

4.4.1 The propagator and the transformation

Consider a quantum problem of a harmonic oscillator in which the Hamiltonian that explicitly depending on time

$$H(p,q,t) = \frac{1}{2m(t)}p^2 + \frac{1}{2}m(t)\omega^2(t)q^2. \tag{4.16}$$

The propagator corresponds to this system can be written in the phase space as Ref. [6]

$$K(q'',p'',t'';q',p',,t') = \int \frac{D[q(t)]D[p(t)]}{2\pi\hbar} e^{\frac{i}{\hbar}\int_{t'}^{t''} dt(p\dot{q}-H(p,q,t))}. \tag{4.17}$$

Dealing with the expression (4.17) by a straight way and find its explicit expression is not evident and it is a problem, since the mass and frequency are time-dependent. To be able of solving this problem we need to make some coordinate transformations $(p,q) \to (P,Q)$ which will simplify (4.17) to another form which may be explicitly evaluated for some chosen systems.

4.4.2 The canonical transformations

To treat the problem given by the Hamiltonian Eq.(4.16) we need to do some transformations and represent it in a new phase space coordinates.
By taking the following time-dependent canonical transformations

$$\begin{aligned} q &= f(t)Q \\ p &= \frac{P}{f(t)}, \end{aligned} \tag{4.18}$$

and following the same steps given in chapter 3 we can find that

$$K(q'',p'',t'';q',p',,t') = \frac{1}{\sqrt{f(t'')f(t')}} exp\frac{i}{2\hbar}(g(t'')Q''^2 - g(t')Q'^2)$$
$$\bar{K}(Q'',\bar{P}'',t'';Q',\bar{P}',t'), \tag{4.19}$$

where the new propagator $\bar{K}(Q'',\bar{P}'',t'';Q',\bar{P}',t')$ is

$$\bar{K}(Q'',\bar{P}'',t'';Q',\bar{P}',t') = \int \frac{D[Q(t)]D[\bar{P}(t)]}{2\pi\hbar} exp\frac{i}{\hbar}\int_{t'}^{t''} dt(\bar{P}\dot{Q} - \frac{1}{2m(t)f(t)^2}\bar{P}^2$$
$$-\frac{1}{2}m(t)f(t)^2(\frac{\dot{m}(t)}{m(t)}\frac{\dot{f}(t)}{f(t)} + \frac{\ddot{f}(t)}{f(t)} + \omega^2(t))Q^2). \tag{4.20}$$

To simplify this problem $f(t)$ will be chosen such that

$$\frac{\dot{m}(t)}{m(t)}\frac{\dot{f}(t)}{f(t)} + \frac{\ddot{f}(t)}{f(t)} + \omega^2(t) = \frac{\varpi^2}{m(t)^2 f(t)^4}, \tag{4.21}$$

where ϖ is a constant.
Following by the time transformation $d\tau = dt/m(t)f(t)^2$ the propagator (4.20) will be

$$\bar{K}(Q'',\bar{P}'',t'';Q',\bar{P}',t') = \int \frac{D[Q(\tau)]D[\bar{P}(\tau)]}{2\pi\hbar} exp\frac{i}{\hbar}\int_{\tau'}^{\tau''} d\tau(\bar{P}\dot{Q} - \frac{1}{2}\bar{P}^2$$
$$-\frac{\varpi^2}{2}Q^2), \tag{4.22}$$

which is the propagator of the simple harmonic oscillator and it is exactly evaluated.

4.4.3 Applications

4.4.3.1 Example 1

As a first example we will treat the system described by the following time dependent Hamiltonian

$$H(p,q,t) = \frac{1}{2m_0(\alpha e^{\lambda t} + \beta e^{-\lambda t})^2}p^2 + \frac{1}{2}m_0(\alpha e^{\lambda t} + \beta e^{-\lambda t})^2\omega^2 q^2, \tag{4.23}$$

where α and β are complex numbers and λ can be a pure real or pure complex number with $m(t)$ has a physical meaning. As it seems that when α or β is zero the system will be that of exponentially time dependent mass [1], or when $\alpha = \beta = 1/2$ and λ is pure complex, the system will be that os the strong pulsating mass [7]. To deal with this system we will choose $f(t)$ such that; $f(t)^2 m(t) = m_0$ or

$$f(t) = \frac{1}{(\alpha e^{\lambda} + \beta e^{-\lambda})}, \tag{4.24}$$

then the canonical transformations in this case will be

$$q = \frac{Q}{(\alpha e^{\lambda} + \beta e^{-\lambda})} \tag{4.25}$$

$$p = (\alpha e^{\lambda} + \beta e^{-\lambda})P, \tag{4.26}$$

with the generating function

$$F(q, P, t) = (\alpha e^{\lambda} + \beta e^{-\lambda})qP, \tag{4.27}$$

and clearly $g(t)$ is

$$g(t) = -m_0\lambda\frac{(\alpha e^{\lambda} - \beta e^{-\lambda})}{(\alpha e^{\lambda} + \beta e^{-\lambda})}. \tag{4.28}$$

Following the same steps detailed in the last chapter we can find that

$$\bar{K}(Q'', \bar{P}'', t''; Q', \bar{P}', t') = \int \frac{D[Q(t)]D[\bar{P}(t)]}{2\pi\hbar} exp\frac{i}{\hbar}\int_{t'}^{t''} dt(\bar{P}\dot{Q} - \frac{1}{2m_0}\bar{P}^2 - \frac{1}{2}m_0(\omega^2 - \lambda^2)Q^2) \tag{4.29}$$

$$= \sqrt{\frac{m_0\Omega}{2\pi\hbar sin(\Omega(t''-t'))}} exp\frac{im_0\Omega}{2\hbar sin(\Omega(t''-t'))}((Q''^2 + Q'^2)cos(\Omega(t''-t')) - 2Q''Q'), \tag{4.30}$$

where $\Omega = \sqrt{\omega^2 - \lambda^2}$. Then the propagator of this system will be deduced easily as

$$K(q'', p'', t''; q', p', t') = \sqrt{\frac{m_0\Omega}{2\pi\hbar f(t'')f(t')sin(\Omega(t''-t'))}}exp\frac{i}{\hbar}(\dot{g}(t'')Q''^2 - \dot{g}(t')Q'^2)$$
$$\times\, exp\frac{im_0\Omega}{2\hbar sin(\Omega(t''-t'))}((Q''^2 + Q'^2)cos(\Omega(t''-t')) - 2Q''Q'). \tag{4.31}$$

This will be identical to that result given in [7], when we choose $\alpha = \beta = \frac{1}{2}$ and λ as pure imaginary constant, and to that given in [6] when α or β is zero and λ is pure real.

4.4.3.2 Example 2

We present here a new system which will be exactly evaluated following the steps presented above. This system is more general than those given in the literature,where the mass and frequency will be time-dependent as it shown in the following Hamiltonian

$$H(p, q, t) = \frac{1}{2m_0}\left(\frac{t_0}{t}\right)^{2\alpha} p^2 + \frac{1}{2}m_0\omega_0^2\left(\frac{t}{t_0}\right)^{2(\alpha+\beta)} q^2, \tag{4.32}$$

where α and β are constants. This example is more generalized than those given in[14] . The propagator related to this problem is

$$K(q'', p'', t''; q', p', t') = \int \frac{D[q(t)]D[p(t)]}{2\pi\hbar}exp\frac{i}{\hbar}\int_{t'}^{t''} dt(p\dot{q} - \frac{1}{2m_0}\left(\frac{t_0}{t}\right)^{2\alpha} p^2 +$$
$$- \frac{1}{2}m_0\omega_0^2\left(\frac{t}{t_0}\right)^{2(\alpha+\beta)} q^2). \tag{4.33}$$

To deal with this problem we need to take the following Canonical transformations

$$q = \left(\frac{t}{t_0}\right)^{-\alpha} Q$$
$$p = \left(\frac{t}{t_0}\right)^{\alpha} P. \tag{4.34}$$

The generating function related to this transformation is

$$F_1(q,P,t) = \left(\frac{t}{t_0}\right)^{\alpha} qP, \tag{4.35}$$

and $g_1(t)$ is

$$g_1(t) = -\frac{m_0\alpha}{t}. \tag{4.36}$$

Using all of this we can find that

$$K(q'',p'',t'';q',p',t') = \left(\frac{t''t'}{t_0^2}\right)^{\alpha/2} exp\frac{-im_0\alpha}{\hbar}(\frac{Q''^2}{t''} - \frac{Q'^2}{t'})\bar{K}(P'',Q'',t'';P',Q',t'), \tag{4.37}$$

where $\bar{K}(P'',Q'',t'';P',Q',t')$ is the propagator given by

$$\begin{aligned}\bar{K}(Q'',P'',t'';Q',P',t') = \int \frac{D[Q(t)]D[P(t)]}{2\pi\hbar} exp\frac{i}{\hbar}\int_{t'}^{t''} dt(P\dot{Q} - \frac{1}{2m_0}P^2 + \\ -\frac{1}{2}m_0(\frac{-\alpha^2+\alpha}{t^2} + \omega_0^2\left(\frac{t}{t_0}\right)^{2\beta})Q^2).\end{aligned} \tag{4.38}$$

At this level the system transformed to that of a constant mass and varied frequency, which is not easily evaluated. To be able of finding the exact expression of the propagator Eq.(4.38) we need to take another canonical transformations

$$\begin{aligned} Q &= \sqrt{t}J_\mu\left(\frac{t^{\beta+1}t_0^{-\beta}\omega_0}{(\beta+1)}\right)\bar{Q} \\ P &= \frac{\bar{P}}{\sqrt{t}J\mu\left(\frac{t^{\beta+1}t_0^{-\beta}\omega_0}{(\beta+1)}\right)}, \end{aligned} \tag{4.39}$$

where $J_\mu(x)$ is Bessel's function of the first kind and $\mu = \frac{(\alpha-\frac{1}{2})(-1+2\alpha)}{(\beta+1)(-1+2\alpha)})$.
The generating function for this transformation $F_2(Q,\bar{P},t)$ is

$$F_2(Q,\bar{P},t) = \frac{\bar{P}Q}{\sqrt{t}J_\mu\left(\frac{t^{\beta+1}t_0^{-\beta}\omega_0}{(\beta+1)}\right)}, \tag{4.40}$$

and

$$g_2(t) = m_0(\frac{1}{2\sqrt{t}}J_\mu\left(\frac{t^{\beta+1}t_0^{-\beta}\omega_0}{(\beta+1)}\right) + \sqrt{t}\frac{dJ_\mu\left(\frac{t^{\beta+1}t_0^{-\beta}\omega_0}{(\beta+1)}\right)}{dt})\sqrt{t}J_\mu\left(\frac{t^{\beta+1}t_0^{-\beta}\omega_0}{(\beta+1)}\right). \tag{4.41}$$

Then easily we can find

$$K(q'',p'',t'';q',p',t') = \frac{(t''t')^{-1/2}\left(\frac{t''t'}{t_0^2}\right)^{\alpha/2} exp\frac{-im_0\alpha}{\hbar}(\frac{Q''^2}{t''}-\frac{Q'^2}{t'})exp\frac{i}{\hbar}(\dot{g}_2(t'')\bar{Q}''^2-\dot{g}_2(t')\bar{Q}'^2)}{\sqrt{J_\mu\left(\frac{t''^{\beta+1}t_0^{-\beta}\omega_0}{(\beta+1)}\right)J\left(\frac{(\alpha-\frac{1}{2})(-1+2\alpha)}{(\beta+1)(-1+2\alpha)}),\frac{t'^{\beta+1}t_0^{-\beta}\omega_0}{(\beta+1)}\right)}} \times \int\frac{D[\bar{Q}(t)]D[\bar{P}(t)]}{2\pi\hbar}e^{\frac{i}{\hbar}\int_{t'}^{t''}dt(\bar{P}\dot{\bar{Q}}-\frac{1}{2M(t)}\bar{P}^2)}, \tag{4.42}$$

where $M(t) = m_0 t J_\mu\left(\frac{t^{\beta+1}t_0^{-\beta}\omega_0}{(\beta+1)}\right)^2$. Following by the time transformation $d\tau = \frac{m_0 dt}{M(t)}$, this will lead to an exact expression of the propagator (4.42), then

$$K(q'',p'',t'';q',p',t') = \frac{\left(\frac{2\pi i\hbar t''t'}{m_0}\int_{t'}^{t''}\frac{m_0dt}{M(t)}\right)^{-1/2}\left(\frac{t''t'}{t_0^2}\right)^{\alpha/2} exp\frac{-im_0\alpha}{\hbar}\left(\left(\frac{t''}{t_0}\right)^{2\alpha}\frac{q''^2}{t''}-\left(\frac{t'}{t_0}\right)^{2\alpha}\frac{q'^2}{t'}\right)}{\sqrt{J_\mu\left(\frac{t''^{\beta+1}t_0^{-\beta}\omega_0}{(\beta+1)}\right)J_\mu\left(\frac{t'^{\beta+1}t_0^{-\beta}\omega_0}{(\beta+1)}\right)}} \times exp\frac{im_0}{\hbar}(\frac{\dot{g}_2(t'')}{M(t'')}\left(\frac{t''}{t_0}\right)^{2\alpha}q''^2-\frac{\dot{g}_2(t')}{M(t')}\left(\frac{t'}{t_0}\right)^{2\alpha}q'^2) \times exp\frac{im_0^2}{\hbar\int_{t'}^{t''}\frac{m_0dt}{M(t)}}\left(\left(\frac{t''}{t_0}\right)^{\alpha}\frac{q''}{\sqrt{M(t'')}}-\left(\frac{t'}{t_0}\right)^{\alpha}\frac{q'}{\sqrt{M(t')}}\right)^2. \tag{4.43}$$

4.5 The propagator for the harmonic oscillator with time-dependent mass and frequency in phase space using delta functional

4.5.1 The harmonic oscillator and the propagator

let suppose the following time dependent Hamiltonian

$$H(p,q,t) = \frac{1}{2m(t)}p^2 + \frac{1}{2}m(t)\omega^2(t)q^2. \tag{4.44}$$

The propagator corresponds to this system can be written in the phase space as

$$K(q'',p'',t'';q',p',t') = \int \frac{D[q(t)]D[p(t)]}{2\pi\hbar} e^{\frac{i}{\hbar}\int_{t'}^{t''} dt(p\dot{q}-H(p,q,t))}. \tag{4.45}$$

This propagator is not exactly evaluated for any arbitrary time dependent mass or frequency, because this will lead to non-linear differential equations. To deal with this system firstly we will absorb the quadratic term of q, by taking the following transformation

$$p = P + f(t)q, \tag{4.46}$$

where $f(t)$ is an arbitrary function. The propagator (4.45) under this transformation will have the following form

$$K(q'',p'',t'';q',p',t') = e^{\frac{i}{2\hbar}(f(t'')q''^2 - f(t')q'^2)}\tilde{K}(P'',q'',t'';P',q',t'), \tag{4.47}$$

where $\tilde{K}(P'',q'',t'';P',q',t')$ is the propagator that has the following expression

$$\tilde{K}(q'',P'',t'';q',P',t')) = \int \frac{D[q(t)]D[P(t)]}{2\pi\hbar} e^{\frac{i}{\hbar}\int_{t'}^{t''} dt(P\dot{q}-\tilde{H}(P,q,t))dt}, \tag{4.48}$$

and the new Hamiltonian $\tilde{H}(P,q,t)$ is

$$\tilde{H}(q,P,t) = \frac{P^2}{2m(t)} + \frac{f(t)}{m(t)}Pq + \frac{1}{2}(f(t)^2/m(t) + \dot{f}(t) + m(t)\omega^2(t))q^2. \tag{4.49}$$

Since $f(t)$ is an arbitrary function we will chose it such that the quadratic term in the new Hamiltonian disappears

$$f(t)^2/m(t) + \dot{f}(t) + m(t)\omega^2(t) = 0. \tag{4.50}$$

Then $\tilde{K}(P'', q'', t''; P', q', t')$ will be

$$\tilde{K}(q'', P'', t''; q', P', t') = \int \frac{D[q(t)]D[P(t)]}{2\pi\hbar} e^{\frac{i}{\hbar}\int_{t'}^{t''} dt(P\dot{q} - \frac{P^2}{2m(t)} - \frac{f(t)}{m(t)}Pq)dt}. \tag{4.51}$$

To deal with this propagator we will take the following canonical transformations

$$q = g(t)\bar{Q}$$
$$P = \frac{\bar{P}}{g(t)}, \tag{4.52}$$

with the generating function $F(\bar{P}, q, t)$

$$F(\bar{P}, q, t) = \frac{q\bar{P}}{g(t)}. \tag{4.53}$$

Then (4.51) will be

$$\tilde{K}(q'', P'', t''; q', P', t') = \frac{1}{\sqrt{g(t'')g(t')}} \int \frac{D[\bar{Q}(t)]D[\bar{P}(t)]}{2\pi\hbar} e^{\frac{i}{\hbar}\int_{t'}^{t''} dt(\bar{P}\dot{\bar{Q}} - \frac{\bar{P}^2}{2m(t)g(t)^2} - (-\frac{\dot{g}(t)}{g(t)} + \frac{f(t)}{m(t)})\bar{P}\bar{Q}dt}. \tag{4.54}$$

Since $g(t)$ is an arbitrary function it will be chosen such that the second term in the Hamiltonian will be zero or

$$\frac{\dot{g}(t)}{g(t)} - \frac{f(t)}{m(t)} = 0. \tag{4.55}$$

In the exponent by integrating the first term by part and do the functional integral over q we get the following condition

$$\delta(\dot{\bar{P}}), \tag{4.56}$$

which implies that $\bar{P}$ should be a constant. Then the propagator Eq.(4.54) will take the form

$$\tilde{K}(q'',P'',t'';q',P',t') = \frac{1}{\sqrt{g(t'')g(t')}} \int \frac{D[\bar{P}(t)]}{2\pi\hbar}\delta(\dot{\bar{P}})e^{\frac{i}{\hbar}(\bar{P}''\bar{Q}''-\bar{P}'\bar{Q}'+\int_{t'}^{t''}\frac{-\bar{P}^2}{2m(t)g(t)^2}dt)}$$
$$= \frac{1}{\sqrt{g(t'')g(t')}} \int \frac{d\bar{P}}{2\pi\hbar}\; e^{\frac{i}{\hbar}(\bar{P}(\bar{Q}''-\bar{Q}')-\bar{P}^2\int_{t'}^{t''}\frac{dt}{2m(t)g(t)^2})}. \quad (4.57)$$

Using the identity

$$\int_{-\infty}^{+\infty} e^{-ax^2+bx}dx = \sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}}, \quad (4.58)$$

one can find that

$$\tilde{K}(q'',P'',t'';q',P',t') = \sqrt{\frac{1}{2\pi\hbar i g(t'')g(t')\int_{t'}^{t''}\frac{dt}{m(t)g(t)^2}}}exp\frac{i}{2\hbar}\frac{(\bar{Q}''-\bar{Q}')^2}{\int_{t'}^{t''}\frac{dt}{m(t)g(t)^2}}. \quad (4.59)$$

By plugging this into Eq.(4.47) we will find the expression of the system Eq.(8.4)

$$\tilde{K}(q'',P'',t'';q',P',t') = \sqrt{\frac{1}{2\pi\hbar i g(t'')g(t')\int_{t'}^{t''}\frac{dt}{m(t)g(t)^2}}}e^{\frac{i}{2\hbar}(f(t'')q''^2-f(t')q'^2)}$$
$$\times\, exp\frac{i}{2\hbar}\frac{(\bar{Q}''-\bar{Q}')^2}{\int_{t'}^{t''}\frac{dt}{m(t)g(t)^2}}, \quad (4.60)$$

which is the desired result

4.5.2 The Models

We would like to present a class to time dependent Harmonic oscillator with constant mass and varied frequency, and we will follow the way that given above to find the exact propagator of the related system. Let us present the following Hamiltonian.

$$H(p,q,t) = \frac{1}{2}p^2 + \frac{1}{2}\left(\frac{a\ddot{k}(t)}{b-ak(t)}\right)q^2 \quad (4.61)$$

where $k(t)$ is an arbitrary function, a and b are constants. We will deal with those systems such that function $(b-ak(t))^{-2}$ has a definite integration. Toward finding the exact propagator related to this system we will chose the function $f(t)$

Eq.(4.46) to be

$$f(t) = \frac{-a\dot{k}(t)}{-ak(t)+b}. \tag{4.62}$$

Then the propagator related to this system can has the following expression

$$K(q'',p'',t'';q',p',t') = e^{\frac{i}{2\hbar}\left(\frac{-a\dot{k}(t'')}{-ak(t'')+b}q''^2 - \frac{-a\dot{k}(t')}{-ak(t')+b}q'^2\right)} \int \frac{D[q(t)]D[P(t)]}{2\pi\hbar} e^{\frac{i}{\hbar}\int_{t'}^{t''} dt\left(P\dot{q} - \frac{P^2}{2} - \frac{-a\dot{k}(t)}{-ak(t)+b}Pq\right)dt}. \tag{4.63}$$

Then we will present the following canonical transformations

$$q = (-ak(t)+b)\bar{Q}$$
$$P = \frac{\bar{P}}{-ak(t)+b}. \tag{4.64}$$

This will lead to a new expression to the propagator Eq.(4.63)

$$\begin{aligned} K(q'',p'',t'';q',p',t') &= \frac{e^{\frac{i}{2\hbar}\left(\frac{-a\dot{k}(t'')}{-ak(t'')+b}q''^2 - \frac{-a\dot{k}(t')}{-ak(t')+b}q'^2\right)}}{\sqrt{(-ak(t'')+b)(-ak(t')+b)}} \int \frac{d\bar{P}'}{2\pi\hbar} e^{\frac{i}{\hbar}\left(\bar{P}'\left(\frac{q''}{-ak(t'')+b} - \frac{q'}{-ak(t')+b}\right) - \bar{P}'^2\int_{t'}^{t''}\frac{dt}{(-ak(t)+b)^2}\right)} \\ &= \frac{e^{\frac{i}{2\hbar}\left(\frac{-a\dot{k}(t'')}{-ak(t'')+b}q''^2 - \frac{-a\dot{k}(t')}{-ak(t')+b}q'^2\right)}}{\sqrt{2\pi\hbar i(-ak(t'')+b)(-ak(t')+b)\int_{t'}^{t''}\frac{dt}{(-ak(t)+b)^2}}} \\ & exp\frac{i}{2\hbar}\frac{\left(\frac{q''}{-ak(t'')+b} - \frac{q'}{-ak(t')+b}\right)^2}{\int_{t'}^{t''}\frac{dt}{(-ak(t)+b)^2}}. \end{aligned} \tag{4.65}$$

From here it is clear why we have toke the condition $(b - ak(t))^{-2}$ has a definite integration.

4.5.3 Examples

- $\omega = \frac{\omega_0}{\sqrt{2}cosh(\omega_0 t)}$
  The related function for this frequency is $k(t) = tanh(\omega_0 t) + \frac{b}{a}$.
- $\omega = \omega_0\frac{\sqrt{ve^{\omega_0 t}-1}}{ve^{\omega_0 t}+1}$
  The related function for this frequency is $k(t) = \frac{1}{e^{-\omega_0 t}+v} + \frac{b}{a}$.

- $\omega = \omega_0 e^{r\frac{t}{t_0}}$

  The related function for this frequency is $k(t) = BesselJ(0, \frac{e^{r\frac{t}{t_0}} t_0 \omega_0}{r})$.

- $\omega = \omega_0 \left(\frac{t}{t_0}\right)^r$

  . The related function for this frequency is $k(t) = \sqrt{t} BesselJ(\frac{1}{2r+2}, \frac{\omega_0 t_0^{-r} t^{r+1}}{r+1}) + \frac{b}{a}$. This example $(b - ak(t))^{-2}$does not has a definite integration for all values of r.

Where r, v and t_0 are constants with t_0 has the dimension of time those are some examples that can be exactly evaluated following the steps given above.

4.6 Conclusion

The study of harmonic oscillators with time-dependent mass has assumed because it is very important in different areas of physics like plasma physics, cosmology, quantum optics etc. Looking through the literature one can notice, in this context, that the path integral method has been used to solve exactly some problems with specific time-dependent mass like exponentially varying mass, strongly pulsating mass, growing mass ...etc. In this work we have used a space-time transformations in phase and configuration spaces to treat the problem and find the propagators of new generalized examples. In this chapter, we have studied a general model of explicitly time-dependent quantum problems by path integrals. The treatment is based on the use of some time-dependent transformations. The problem treated in both configuration and phase space, we used space-time transformations in configuration space and point canonical transformations in phase space, that leads to a considerable simplification in computation and gives unambiguous results in comparison with already existing methods. We have derived the wave functions, expressed in terms of the Hermite polynomials, by simply use of the Mehlers formula. We also have considered interesting explicitly solvable cases where we have presented some new examples of harmonic oscillators with time-dependent mass and frequency for which exact propagators have could be evaluated providing us normalized wave functions.

Chapter 5

Particle with time-dependent mass in coulomb potential

5.1 Introduction

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The exact expression of the propagator for the time-dependent systems of Harmonic oscillator has been studied by many [5, 6], but looking through the literature, one would find that such problem, in fact, is evaluated just for few examples [5–11]. The difficulty to find the exact propagator for the time-dependent systems is that the calculation, in fact, involves solutions of non-linear differential equations.

Generally, time-dependent quantum problems, in fact, have been studied by many; in Ref. [15] Sobhan and Hassanabadi investigate Bohr Hamiltonian in the presence of time-dependent Manning–Rosen, harmonic oscillator and double-ring shaped potentials using Lewis–Riesenfeld dynamical invariant method. Using the same method, the authors could treat Davydov–Chaban Hamiltonian in the presence of time-dependent potential [16], which is one of the most important topics in physics; Grosch in Ref. [17], by using path integral technique, could find the exact solution of some systems; in Ref. [18], Lew is–Riesenfeld dynamical invariant and time evolution operator methods (to evaluate the quantum many-body systems in presence of time-dependent potential and electric fields) have been used. The time-dependent Coulomb problem has also been studied [17, 19, 20] Here, via path integral [17], we find a generalization to class of time-dependent potentials where a space-time transformations transforming the system to the stationary one is used.

In this work, we will focus on the problem of a particle with a time-dependent mass subjected to the Coulomb and the inverse quadratic potentials in two dimensions via path integral. We will use a linear space-time transformations to reduce it to a stationary problem then treat it in polar coordinates, which finally leads (as it will be clear later) to the corresponding wave and Green's functions and the related energies

5.2 The space-time transformations

The problem with a particle with an arbitrary time-dependent varying mass does not has an exact solution via path integral yet, this come from the difficulties of finding an exact propagator using the direct treatment or the way of the transformations. We will propose the problem with an exponential time-dependent mass which is very important and can represents many physical systems. The related Lagrangian is

$$\begin{aligned} L(x,\dot{x},t) =& \frac{1}{2}m(t)(\dot{x}^2+\dot{y}^2) - \frac{k}{\sqrt{x^2+y^2}} - \frac{g}{m(t)(x^2+y^2)} + \\ & - \frac{\hbar^2}{2m(t)}\left(\frac{\kappa(\kappa-1)}{x^2} + \frac{\lambda(\lambda-1)}{y^2}\right) \end{aligned} \tag{5.1}$$

Where the mass $m(t) = m_0 e^{\lambda t}$, with α and m_0 are constants. With the conditions $\kappa, \lambda > 1$

The propagator related to this system in configuration space is

$$k(x'', x'; T) = \int D[x(t)] e^{\frac{i}{\hbar}\int L(x,\dot{x},t)dt} \tag{5.2}$$

As it mentioned that the exact expression of this propagator may not be evaluated directly since it is time-dependent. To deal with such system and remove this difficulties it would be better if we take a transformation to absorb the time from the problem and make it stationary. The relevant transformation that may be chosen is

$$\begin{aligned} x &= e^{-\lambda t}\xi \\ y &= e^{-\lambda t}\zeta \end{aligned} \tag{5.3}$$

Then

$$\begin{aligned} exp(\frac{ie^{\lambda t_j}}{\hbar\epsilon}(x_j - x_{j-1})^2) \simeq exp\frac{i}{\hbar}(\frac{m_0}{2}e^{-\lambda\bar{t}_j}\Delta\xi_j^2 + \frac{m_0}{2}\lambda^2 e^{-\lambda\bar{t}_j}\bar{\xi}_j^2 + \\ - m_0\frac{\lambda}{\epsilon}e^{-\lambda\bar{t}_j}\bar{\xi}_j\Delta\xi_j + m_0\frac{\lambda}{4\epsilon}e^{-\lambda\bar{t}_j}\Delta\xi_j^2) \end{aligned} \tag{5.4}$$

and the same for the variable ζ. for the measure we have

$$\begin{aligned} \prod_{j=1}^{N}\left(\frac{m(t_j)}{2\pi i\hbar\epsilon}\right)\prod_{j=1}^{N-1} dx_j dy_j =& (\xi_i\zeta_i\xi_f\zeta_f)^{-1/2}e^{\lambda(t_i+t_f)}\prod_{1}^{N} e^{-2\lambda\bar{t}_j} \\ &\times\prod_{j=1}^{N}\left(\frac{m_0 e^{\lambda\bar{t}_j}}{2i\pi\hbar\epsilon}\right) exp(\lambda\frac{\Delta t_j}{2})\prod_{1}^{N-1} d\xi_j d\zeta_j \end{aligned} \tag{5.5}$$

then using the correction $\Delta q_j^2 \to \frac{i\hbar\epsilon}{m_0 e^{-\lambda\bar{t}_j}}$. this will lead to following expression of the propagator

$$\begin{aligned} k(x'', x'; T) =& \xi_i\zeta_i\xi_f\zeta_f)^{-1/2}e^{\lambda(t_i+t_f)}\prod_{j=1}^{N}\left(\frac{m_0 e^{-\lambda\bar{t}_j}}{2i\pi\hbar\epsilon}\right) exp(\lambda\frac{\Delta t_j}{2})\prod_{1}^{N-1} d\xi_j d\zeta_j \\ &\times e^{\frac{i}{\hbar}\sum_{j=1}^{N}(\frac{m_0}{2}e^{-\lambda\bar{t}_j}\frac{\Delta\xi_j^2}{\epsilon} + \frac{m_0}{2}\lambda^2 e^{-\lambda\bar{t}_j}\bar{\xi}_j^2\epsilon - m_0\lambda e^{-\lambda\bar{t}_j}\bar{\xi}_j\Delta\xi_j - \frac{\hbar^2(\kappa(\kappa-1))}{2m_0 e^{-\lambda\bar{t}_j}\bar{\xi}_j^2}\epsilon)} \\ &\times e^{\frac{i}{\hbar}\sum_{j=1}^{N}(\frac{m_0}{2}e^{-\lambda\bar{t}_j}\frac{\Delta\zeta_j^2}{\epsilon} + \frac{m_0}{2}\lambda^2 e^{-\lambda\bar{t}_j}\bar{\zeta}_j^2\epsilon - m_0\lambda e^{-\lambda\bar{t}_j}\bar{\zeta}_j\Delta\zeta_j - \frac{\hbar^2(\lambda(\lambda-1))}{2m_0 e^{-\lambda\bar{t}_j}\bar{\zeta}_j^2}\epsilon)} \\ &\times e^{-\frac{i}{\hbar}\sum_{j=1}^{N}\frac{k}{e^{-\lambda\bar{t}_j}\sqrt{\bar{\xi}_j^2+\bar{\zeta}_j^2}}\epsilon + \frac{g}{m_0 e^{-\lambda\bar{t}_j}(\bar{\xi}_j^2+\bar{\zeta}_j^2)}\epsilon} \end{aligned} \tag{5.6}$$

using the same trick given above in equation (3.15) we have, $F(q,t) = \lambda/2e^{-\lambda t}q^2$, and $g(q,t) = exp(\frac{i}{\hbar}\frac{\lambda}{2}e^{-\lambda t}q^2)$. Then the propagator (5.6) will have the following form

$$
\begin{aligned}
k(x'', x'; T) &= (\xi_i\zeta_i\xi_f\zeta_f)^{-1/2} e^{\lambda(t_i+t_f)} exp(\frac{i}{\hbar}\frac{\lambda}{2}(e^{-\lambda t_f}(\xi_f^2+\zeta_f^2) - e^{-\lambda t_i}(\xi_i^2+\zeta_i^2)) \\
&\times \prod_{j=1}^{N}\left(\frac{m_0 e^{-\lambda \bar{t}_j}}{2i\pi\hbar\epsilon}\right)\prod_{1}^{N-1} d\xi_j d\zeta_j e^{\frac{i}{\hbar}\sum_{j=1}^{N}(\frac{m_0}{2}e^{-\lambda\bar{t}_j}\frac{\Delta\xi_j^2}{\epsilon} - \frac{\hbar^2(\kappa(\kappa-1))}{2m_0 e^{-\lambda\bar{t}_j}\bar{\xi}_j^2}\epsilon)} \\
& e^{\frac{i}{\hbar}\sum_{j=1}^{N}(\frac{m_0}{2}e^{-\lambda\bar{t}_j}\frac{\Delta\zeta_j^2}{\epsilon} - \frac{\hbar^2(\lambda(\lambda-1))}{2m_0 e^{-\lambda\bar{t}_j}\bar{\zeta}_j^2}\epsilon)} e^{-\frac{i}{\hbar}\sum_{j=1}^{N}\frac{k}{e^{-\lambda\bar{t}_j}\sqrt{\bar{\xi}_j^2+\bar{\zeta}_j^2}}\epsilon + \frac{g}{m_0 e^{-\lambda\bar{t}_j}(\bar{\xi}_j^2+\bar{\zeta}_j^2)}\epsilon} \\
&= (\xi_i\zeta_i\xi_f\zeta_f)^{-1/2} e^{\lambda(t_i+t_f)} exp(\frac{i}{\hbar}\frac{\lambda}{2}(e^{-\lambda t_f}(\xi_f^2+\zeta_f^2) - e^{-\lambda t_i}(\xi_i^2+\zeta_i^2)) \\
& \int D[\xi(t)]D[\zeta(t)] exp\frac{i}{\hbar}\int(\frac{m_0}{2}e^{-\lambda t}(\dot{\xi}^2+\dot{\zeta}^2) - \frac{k}{e^{-\lambda t}\sqrt{\xi^2+\zeta^2}} + \\
& - \frac{g}{m_0 e^{-\lambda t}(\xi^2+\zeta^2)} - \frac{\hbar^2(\kappa(\kappa-1))}{2m_0 e^{-\lambda t}\xi^2} - \frac{\hbar^2(\lambda(\lambda-1))}{2m_0 e^{-\lambda t}\zeta^2})dt
\end{aligned} \tag{5.7}
$$

At this step the problem is still time dependent, and to deal with that we need to make time transformation $t \rightarrow s$

$$e^{\lambda t}dt = ds \tag{5.8}$$

where we have that $e^{\lambda \bar{t}_j}\epsilon = \tau_j$, then under this transformations the propagator (5.7) will be

$$
\begin{aligned}
k(x'', x'; T) &= (\xi_i\zeta_i\xi_f\zeta_f)^{-1/2} e^{\lambda(t_i+t_f)} exp(\frac{i}{\hbar}\frac{\lambda}{2}(e^{-\lambda t_f}(\xi_f^2+\zeta_f^2) - e^{-\lambda t_i}(\xi_i^2+\zeta_i^2))) \\
& \prod_{j=1}^{N}\left(\frac{m_0}{2i\pi\hbar\tau_j}\right)\prod_{1}^{N-1} d\xi_j d\zeta_j exp[\frac{i}{\hbar}\sum_{j=1}^{N}(\frac{m_0}{2}\frac{\Delta\xi_j^2+\Delta\zeta_j^2}{\tau_j} - \frac{k}{\sqrt{\bar{\xi}_j^2+\bar{\zeta}_j^2}}\tau_j + \\
& - \frac{g}{m_0(\bar{\xi}_j^2+\bar{\zeta}_j^2)}\tau_j - \frac{\hbar^2(\kappa(\kappa-1))}{2m_0\bar{\xi}_j^2}\tau_j - \frac{\hbar^2(\lambda(\lambda-1))}{2m_0\bar{\zeta}_j^2}\tau_j]
\end{aligned} \tag{5.9}
$$

Then the problem transformed to that of a particle with a constant mass m_0 subjected in a Coulomb and inverse quadratic potentials.

5.3 Propagator in polar coordinates

The evaluation of the propagator (5.9) is not easy, the difficulties come from the non-separation of the variables ξ and ζ in Coulomb and the inverse quadratic potential terms, so thinking of treating the problem in polar coordinate (r, θ) it may make it somehow simpler because the variables r and θ will be related by a multiplication in both of the mentioned potential terms.

The propagator in this coordinates where $\xi = r cos\theta$ and $\zeta = r sin\theta$ reads as

$$\mathbf{k}(\vec{r}_f, \vec{r}_i; T) = \lim_{N\to\infty} \prod_{j=0}^{N} \left(\frac{m}{2\pi i\hbar\tau_j} \right) \int \prod_{j=1}^{N-1} r_j dr_j d\theta_j exp(\frac{i}{\hbar} \sum_{j=1}^{N} S(j, j-1)) \tag{5.10}$$

the short time action in this case is

$$\begin{aligned} S(j,j-1) =& \frac{m_0}{2\tau_j}(\Delta r_j^2 + 2r_j r_{j-1}(1 - cos(\Delta\theta_j))) - \frac{k}{r_j}\tau_j - \frac{g}{m_0 r_j^2}\tau_j + \\ & - \frac{\hbar^2(\kappa(\kappa-1))}{2m_0 r_j^2 cos^2(\theta_j)}\tau_j - \frac{\hbar^2(\lambda(\lambda-1))}{2m_0 r_j^2 sin^2(\theta_j)}\tau_j \end{aligned} \tag{5.11}$$

we may therefore write at the mid-point

$$\begin{aligned} S(j,j-1) =& \frac{m_0}{2\tau_j}(\Delta r_j^2 + \tilde{r}_j^2\Delta\theta_j^2 - \frac{1}{4}\Delta r_j^2\Delta\theta_j^2 - \frac{1}{12}\tilde{r}_j^2\Delta\theta_j^4) - \frac{k}{\tilde{r}_j}\tau_j - \frac{g}{m_0 \tilde{r}_j^2}\tau_j + \\ & - \frac{\hbar^2(\kappa(\kappa-1))}{2m_0 \tilde{r}_j^2 cos^2(\tilde{\theta}_j)}\tau_j - \frac{\hbar^2(\lambda(\lambda-1))}{2m_0 \tilde{r}_j^2 sin^2(\tilde{\theta}_j)}\tau_j \end{aligned} \tag{5.12}$$

and the measure

$$\begin{aligned} \prod_{j=1}^{N-1} r_j dr_j d\theta_j &= \frac{1}{\sqrt{r_f r_i}} \prod_{j=1}^{N} (r_j r_{j-1})^{1/2} \prod_{j=1}^{N-1} dr_j d\theta_j \\ &= \frac{1}{\sqrt{r_f r_i}} \prod_{j=1}^{N} \tilde{r}_j \left(1 - \frac{\Delta r_j^2}{8\tilde{r}_j^2}\right) \prod_{j=1}^{N-1} dr_j d\theta_j \\ &= \frac{1}{\sqrt{r_f r_i}} \prod_{j=1}^{N} \tilde{r}_j e^{-\frac{\Delta r_j^2}{8\tilde{r}_j^2}} \prod_{j=1}^{N-1} dr_j d\theta_j \end{aligned} \tag{5.13}$$

We insert (5.12)and (5.13) in (5.10), to arrive at the correct time-sliced form of the propagator in polar coordinates

$$\begin{aligned} \mathbf{k}(\vec{r}_f, \vec{r}_i; T) = \lim_{N\to\infty} \prod_{j=0}^{N} \left(\frac{m}{2\pi i\hbar\tau_j} \right) \int \prod_{j=1}^{N-1} r_j dr_j d\theta_j (1 - \frac{1}{8} r_j^{-2}\Delta r_j^2 + \\ - \frac{i m_0}{8\hbar\tau_j}(\Delta r_j^2 \Delta\theta_j^2 + \frac{1}{3}\tilde{r}_j^2\Delta\theta_j^4)) exp(\frac{i}{\hbar}\sum_{j=1}^{N}(\frac{m_0}{2\tau_j}(\Delta r_j^2 + \tilde{r}_j^2\Delta\theta_j^2) + \\ - \frac{k}{\tilde{r}_j}\tau_j - \frac{g}{m_0\tilde{r}_j^2}\tau_j - \frac{\hbar^2(\kappa(\kappa-1))}{2m_0\tilde{r}_j^2 cos^2(\tilde{\theta}_j)}\tau_j - \frac{\hbar^2(\lambda(\lambda-1))}{2m_0\tilde{r}_j^2 sin^2(\tilde{\theta}_j)})\tau_j \end{aligned} \tag{5.14}$$

To make this propagator in a useful form we will follow McLaughlin Schulman procedure which leads to the replacement

$$\Delta r_j^2 \to i\hbar/\tau_j$$
$$\Delta\theta_j^2 \to\to (i\hbar\tau_j/m)\,\tilde{r}_j^{-2}$$
$$\Delta\theta_j^4 \to 3\,(i\hbar\tau_j/m)^2\,\tilde{r}_j^{-4} \tag{5.15}$$

then we get the polar form of the discretized propagator

$$\begin{aligned}\mathbf{k}(\vec{r}_f,\vec{r}_i;T) =& (r'r'')^{-1/2}\lim_{N\to\infty}\left(\frac{m_0}{2\pi i\hbar\tau_j}\right)^N\int\prod_{j=1}^{N-1}dr_jd\theta_j\prod_{j=1}^{N}\tilde{r}_j \\ & exp\Big(\frac{i}{\hbar}\sum_{j=1}^{N}\frac{m_0}{2\tau_j}(\Delta r_j^2+\tilde{r}_j^2\Delta\theta_j^2)+\frac{\hbar^2}{8m_0\tilde{r}_j^2}\tau_j-\frac{k}{\tilde{r}_j}\tau_j-\frac{g}{m_0\tilde{r}_j^2}\tau_j \\ & -\frac{\hbar^2(\kappa(\kappa-1))}{2m_0\tilde{r}_j^2cos^2(\tilde{\theta}_j)}\tau_j-\frac{\hbar^2(\lambda(\lambda-1))}{2m_0\tilde{r}_j^2sin^2(\tilde{\theta}_j)}\tau_j\Big)\end{aligned} \tag{5.16}$$

5.4 Green's function

The propagator (5.16) can not be evaluated directly since the radial and the angular part are not separated, to separate them we need to make time transformation, for that we need to make the energy appear in our expression. To do so we need to define the Green's function which is the Fourier transform of the propagator

$$G(\vec{r}_f,\vec{r}_i;E)=\int_0^\infty dSexp(\frac{i}{\hbar}ES)\mathbf{k}(\vec{r}_f,\vec{r}_i;S) \tag{5.17}$$

In order to be able to separate angular part from the radial we change the time from s to s'

$$ds'=r^{-2}ds \tag{5.18}$$

which is equivalent to

$$\tau_j=r_jr_{j-1}\tau_j' \text{ with } S'=\int_0^S r^{-2}ds \tag{5.19}$$

Then

$$\tau_j = \tau'_j \tilde{r}_j^2 \left(1 - \frac{\Delta r_j^2}{4\tilde{r}_j^2}\right) \tag{5.20}$$

if we insert the condition $r_f r_i \int_0^\infty dS' \delta(S - \int_0^{S'} r^2(s')ds') = 1$, with the above transformation in (5.17) one would find that the Green function would be written as follow

$$\begin{aligned} G(\vec{r}_f, \vec{r}_i; E) &= \int_0^\infty dS' \delta(S - \int_0^{S'} r^2(s')ds')\mathbf{p}(\vec{r}_f, \vec{r}_i; S) \\ &= \int_0^\infty dS' \mathbf{p}(\vec{r}_f, \vec{r}_i; \int_0^{S'} r^2(s')ds') \end{aligned} \tag{5.21}$$

with $\mathbf{p}(\vec{r}_f, \vec{r}_i; \int_0^{S'} r^2(s')ds')$ is given by

$$\begin{aligned} \mathbf{p}(\vec{r}_f, \vec{r}_i; S') =& (r_i r_f)^{1/2} \lim_{N\to\infty} \prod_{j=1}^{N} \left(\frac{m_0}{2\pi i\hbar\tau'_j}\right) \int \prod_{j=1}^{N-1} dr_j d\theta_j \prod_{j=1}^{N} \tilde{r}_j^{-1} \left(1 + \frac{\Delta r_j^2}{4\tilde{r}_j^2}\right) \\ & \times exp(\frac{i}{\hbar} \sum_{j=1}^{N} \frac{m_0}{2\tau'_j} \left(1 + \frac{\Delta r_j^2}{4\tilde{r}_j^2}\right) (\frac{\Delta r_j^2}{\tilde{r}_j^2} + \Delta\theta_j^2) + E\tilde{r}_j^2\tau'_j - k\tilde{r}_j\sigma_j + \\ & + \frac{\hbar^2/8 - g}{m_0}\tau'_j - \frac{\hbar^2(\kappa(\kappa-1))}{2m_0 cos^2(\tilde{\theta}_j)}\tau'_j - \frac{\hbar^2(\lambda(\lambda-1))}{2m_0 sin^2(\tilde{\theta}_j)}\tau'_j) \end{aligned} \tag{5.22}$$

using the corrections

$$\begin{aligned} \Delta r_j^2 &\to \tilde{r}_j^2 \left(\frac{i\hbar\tau'_j}{m_0}\right) \\ \Delta r_j^2 \Delta\theta_j^2 &\to \tilde{r}_j^2 \left(\frac{i\hbar\tau'_j}{m_0}\right)^2 \\ \Delta r_j^4 &\to 3\tilde{r}_j^4 \left(\frac{i\hbar\tau'_j}{m_0}\right)^2 \end{aligned} \tag{5.23}$$

that will make (5.22) have the following form

$$\begin{aligned}\mathbf{p}(\vec{r}_f,\vec{r}_i;S') =&(r_ir_f)^{1/2}\lim_{N\to\infty}\prod_{j=1}^{N}\left(\frac{m_0}{2\pi i\hbar\tau'_j}\right)\int\prod_{j=1}^{N-1}dr_jd\theta_j\prod_{j=1}^{N}\tilde{r}_j^{-1}\\ &\times exp(\frac{i}{\hbar}\sum_{j=1}^{N}\frac{m_0}{2\tau'_j}(\frac{\Delta r_j^2}{\tilde{r}_j^2}+\Delta\theta_j^2)+E\tilde{r}_j^2\tau'_j-k\tilde{r}_j\tau'_j-\frac{\hbar^2/8+g}{m_0}\tau'_j\\ &-\frac{\hbar^2(\kappa(\kappa-1))}{2m_0cos^2(\tilde{\theta}_j)}\tau'_j-\frac{\hbar^2(\lambda(\lambda-1))}{2m_0sin^2(\tilde{\theta}_j)}\tau'_j)\\ &=\mathbf{p}_r(r_f,r_i,S')\mathbf{p}_\theta(\theta_f,\theta_i,S')\end{aligned} \tag{5.24}$$

where

$$\begin{aligned}\mathbf{p}_r(r_f,r_i,S') =&(r_ir_f)^{1/2}\lim_{N\to\infty}\prod_{j=1}^{N}\left(\frac{m_0}{2\pi i\hbar\tau'_j}\right)^{1/2}\int\prod_{j=1}^{N-1}dr_j\prod_{j=1}^{N}\tilde{r}_j^{-1}\\ &\times exp(\frac{i}{\hbar}\sum_{j=1}^{N}\frac{m_0}{2\tau'_j}\frac{\Delta r_j^2}{\tilde{r}_j^2}+E\tilde{r}_j^2\tau'_j-k\tilde{r}_j\tau'_j-\frac{\hbar^2/8+g}{m_0}\tau'_j\end{aligned} \tag{5.25}$$

and

$$\begin{aligned}\mathbf{p}_\theta(\theta_f,\theta_i,S') =& \lim_{N\to\infty}\prod_{j=1}^{N}\left(\frac{m_0}{2\pi i\hbar\tau'_j}\right)^{1/2}\int\prod_{j=1}^{N-1}d\theta_jexp(\frac{i}{\hbar}\sum_{j=1}^{N}\frac{m_0}{2\tau'_j}\Delta\theta_j^2\\ &-\frac{\hbar^2(\kappa(\kappa-1))}{2m_0cos^2(\tilde{\theta}_j)}\tau'_j-\frac{\hbar^2(\lambda(\lambda-1))}{2m_0sin^2(\tilde{\theta}_j)}\tau'_j)\end{aligned} \tag{5.26}$$

The last expression is just the propagator of a particle subjected in Pöschel-Teller potential Refs. [22,23] which is exactly evaluated that has the form Ref. [22]

$$\begin{aligned}\mathbf{p}_\theta(\theta_f,\theta_i,S') =&\sum_{l=0}^{\infty}exp(-i\frac{\hbar}{2m_0}(2l+\kappa+\lambda)^2S')(2l+\kappa+\lambda)\\ &\times\frac{2l!\Gamma(\lambda+\kappa+l)}{\Gamma(\kappa+l+1/2)\Gamma(\lambda+l+1/2)}(sin(\theta_i)sin(\theta_f))^{\lambda}(cos(\theta_i)cos(\theta_f))^{\kappa}\\ &\times P_l^{\lambda-1/2,\kappa-1/2}(1-2sin^2(\theta_i))P_l^{\lambda-1/2,\kappa-1/2}(1-2sin^2(\theta_f))\end{aligned} \tag{5.27}$$

Using the same result that given in Ref. [22] where $\mathbf{p}_\theta(\theta_f,\theta_i,S')$ will be written as a summation

$$\mathbf{p}_\theta(\theta_f,\theta_i,S') = \sum_{n=0}^{\infty}\phi_n(\theta_f)\phi_n(\theta_i)e^{-\frac{i}{\hbar}E_{\theta,n}S} \tag{5.28}$$

using this relation one would find that the energy spectrum of this system will be given by

$$E_{\theta,n} = \frac{2\hbar^2}{m_0}(2l+\kappa+\lambda)^2 \tag{5.29}$$

The normalized wave functions corresponding to this case can be directly deduced

$$\phi_n(\theta) = \left(2(\lambda+\kappa+2l)\frac{l!\Gamma(\lambda+\kappa+l)}{\Gamma(\lambda+l+1/2)\Gamma(\kappa+l+1/2)}\right)^{1/2} \times (sin(\theta))^{\lambda}(cos(\theta))^{\kappa})P_l^{\lambda-1/2,\kappa-1/2}(1-2sin^2(\theta)) \tag{5.30}$$

To find the Green's function (5.17) we still have to find the exact expression of the radial propagator (5.25), and to do so we need to make the transformation $s \to t$ defined by

$$ds' = r^{-2}(s)ds \tag{5.31}$$

which means

$$\tau'_j = \frac{\tau_j}{r_j r_{j-1}} = \frac{\tau_j}{\tilde{r}_j^2}\left(1+\frac{\Delta r_j^2}{4\tilde{r}_j^2}\right) \tag{5.32}$$

then

$$\mathbf{p}_r(r_f, r_i, S) = (r_i r_f)^{1/2} e^{\frac{i}{\hbar}ES} \lim_{N\to\infty}\prod_{j=1}^{N}\left(\frac{m_0}{2\pi i\hbar\tau_j}\right)^{1/2}\int\prod_{j=1}^{N-1}dr_j \times exp(\frac{i}{\hbar}\sum_{j=1}^{N}\frac{m_0}{2\tau_j}\Delta r_j^2 - \frac{k}{\tilde{r}_j}\tau_j + \frac{\hbar^2/8 - g - m_0 E_{\theta,n}}{m_0\tilde{r}_j^2}\tau_j) \tag{5.33}$$

The last expression is the propagator of a particle subjected in Coulomb and an inverse quadratic potential in one dimension, to find the exact expression of it we will make a space-time transformation. The suitable space time transformation that can be chosen in this case is

$$r = u^2$$
$$\frac{ds}{dt'} = 4u^2(t') \tag{5.34}$$

which will simplify the problem to that of an Harmonic oscillator with an inverse quadratic potential.
The discrete version of this transformation can be given by

$$\begin{aligned} r_j &= u_j^2, \quad , r_{j-1} = u_{j-1}^2 \\ \tau_j &= 4\epsilon' u_j u_{j-1} = 4\epsilon' \tilde{u}_j^2 \left(1 - \frac{\Delta u_j^2}{4\tilde{u}_j^2}\right) \end{aligned} \tag{5.35}$$

this will leads to a quantum effective potential $\tilde{V} = \frac{3\hbar^2\epsilon'}{8m_0\tilde{u}_j^2}$. Then the Green's function will take the form

$$\begin{aligned} G(\vec{r}_f, \vec{r}_i; E) &= \frac{2}{(u_f u_i)^{1/2}} \sum_{l=0}^{\infty} \frac{2l!\Gamma(\lambda+\kappa+l)(2l+\kappa+\lambda)}{\Gamma(\kappa+l+1/2)\Gamma(\lambda+l+1/2)} (sin(\theta_i)sin(\theta_f))^{\lambda} \\ &\times (cos(\theta_i)cos(\theta_f))^{\kappa} P_l^{\lambda-1/2,\kappa-1/2}(1-2sin^2(\theta_i)) \\ &\times P_l^{\lambda-1/2,\kappa-1/2}(1-2sin^2(\theta_f)) \int_0^{\infty} dT' e^{-\frac{i}{\hbar}4kT'} \int D[u(t')] \\ &\times exp(\frac{i}{\hbar}\int(\frac{m_0}{2}\dot{u}^2 + \frac{\hbar^2}{2m_0u^2}(1 - 8g/\hbar^2 - 8m_0E_{\theta,n}/\hbar^2) + 4Eu^2)dt') \\ &= 2\sum_{l=0}^{\infty} \frac{2l!\Gamma(\lambda+\kappa+l)(2l+\kappa+\lambda)}{\Gamma(\kappa+l+1/2)\Gamma(\lambda+l+1/2)} (sin(\theta_i)sin(\theta_f))^{\lambda}(cos(\theta_i)cos(\theta_f))^{\kappa} \\ &\times P_l^{\lambda-1/2,\kappa-1/2}(1-2sin^2(\theta_i))P_l^{\lambda-1/2,\kappa-1/2}(1-2sin^2(\theta_f)) \int_0^{\infty} dT' e^{-\frac{i}{\hbar}4kT'} \\ &\times \left(\frac{m_0\omega}{i\hbar sin(\omega T')}\right) exp(\frac{im_0\omega}{2\hbar}(u_f^2+u_i^2)cot(\omega T'))I_{2\alpha}\left(\frac{m_0\omega u_f u_i}{i\hbar sin(\omega T')}\right) \end{aligned} \tag{5.36}$$

with $4E = -\frac{1}{2}m_0\omega^2$, $\alpha = \frac{2g}{\hbar^2} + \frac{2m_0E_{n,\theta}}{\hbar^2}$. In the next step we will make the following change of variables

$$y = \frac{m_0\omega}{\hbar}u_f^2 \tag{5.37}$$

$$x = \frac{m_0\omega}{\hbar}u_i^2 \tag{5.38}$$

$$z = exp(-2i\omega T') \tag{5.39}$$

with

$$sin(\omega T') = \frac{z^{-1/2}(1-z)}{2i}, \quad cos(\omega T') = \frac{z^{-1/2}(1+z)}{2} \tag{5.40}$$

taking this into account then

$$G(\vec{r}_f,\vec{r}_i;E) = 2\sum_{l=0}^{\infty}\frac{2l!\Gamma(\lambda+\kappa+l)(2l+\kappa+\lambda)}{\Gamma(\kappa+l+1/2)\Gamma(\lambda+l+1/2)}(sin(\theta_i)sin(\theta_f))^{\lambda}(cos(\theta_i)cos(\theta_f))^{\kappa}$$
$$\times P_l^{\lambda-1/2,\kappa-1/2}(1-2sin^2(\theta_i))P_l^{\lambda-1/2,\kappa-1/2}(1-2sin^2(\theta_f))\int_0^{\infty}dT'e^{-\frac{i}{\hbar}(4k+\hbar\omega)T'}$$
$$\times\left(\frac{2m_0\omega}{\hbar(1-z)}\right)exp\left(-\frac{1}{2}\frac{(1+z)}{(1-z)}(x+y)\right)I_{2\alpha}\left(\frac{-2\sqrt{xyz}}{1-z}\right) \quad (5.41)$$

Using Hille-Hardy [24]

$$\frac{1}{1-z}exp\left(-\frac{1}{2}\frac{(1_+z)}{(1-z)}(x+y)\right)I_{2\alpha}\left(2\frac{\sqrt{xyz}}{1-z}\right)$$
$$=\sum_{n=0}^{\infty}z^n\frac{n!}{\Gamma(2\alpha+n+1)}exp(-1/2(x+y))(xyz)^{\alpha}L_n^{2\alpha}(x)L_n^{2\alpha}(y) \quad (5.42)$$

with $|z|<1$, and $L_n^{2\alpha}(x)$ are Laguerre polynomial functions. After replacing all of this in the Green's function we will have

$$G(\vec{r}_f,\vec{r}_i;E) = 4\sum_n\sum_{l=0}^{\infty}\frac{2l!\Gamma(\lambda+\kappa+l)(2l+\kappa+\lambda)}{\Gamma(\kappa+l+1/2)\Gamma(\lambda+l+1/2)}\left(\frac{m_0\omega}{\hbar}\right)^{2\alpha+1}$$
$$\times(sin(\theta_i)sin(\theta_f))^{\lambda}(cos(\theta_i)cos(\theta_f))^{\kappa}P_l^{\lambda-1/2,\kappa-1/2}(1-2sin^2(\theta_i))$$
$$\times P_l^{\lambda-1/2,\kappa-1/2}(1-2sin^2(\theta_f))\int_0^{\infty}dT'e^{-\frac{i}{\hbar}(4k+(1+2\alpha+2n)\hbar\omega)T'}$$
$$\times exp(-\frac{m_0\omega}{2\hbar}(u_i^2+u_f^2))L_n^{2\alpha}(\frac{m_0\omega}{\hbar}u_i^2)L_n^{2\alpha}(\frac{m_0\omega}{\hbar}u_f^2) \quad (5.43)$$

5.5 The energy spectrum and the wave functions

To find the energy spectrum and the related wave functions, we need to look for the poles of our Green function and to do so we will perform the integration with respect to the variable T'

$$\int_0^{\infty}dT'e^{-\frac{i}{\hbar}(4k+(1+2\alpha+2n)\hbar\omega)T'} = i(4k+(1+2\alpha+2n)\hbar\omega)^{-1} \quad (5.44)$$

after some arrangement it is clear that

$$E_n = -\frac{2m_0k^2}{\hbar^2(1+2\alpha+2n)^2} \quad (5.45)$$

The wave functions can be deduced for the residue of the Green function then

$$\Psi_{n,l}(u_f,\theta_f)\Psi^*_{n,l}(u_i,\theta_i) = \lim_{E\to E_n}\frac{E-E_n}{i}G(\vec{r}_f,\vec{r}_i;E) \tag{5.46}$$

which means that

$$\begin{aligned}\Psi_{n,l}(u,\theta) =& 2^{3/2}\left(\frac{l!\Gamma(\lambda+\kappa+l)(2l+\kappa+\lambda)}{\Gamma(\kappa+l+1/2)\Gamma(\lambda+l+1/2)}\right)^{1/2}\left(\frac{m_0\omega}{\hbar}\right)^{\alpha+1/2}\\ &\times sin(\theta)^{\lambda}cos(\theta)^{\kappa}P_l^{\lambda-1/2,\kappa-1/2}(1-2sin^2(\theta))\\ &\times exp(-\frac{m_0\omega}{2\hbar}(u_i^2+u_f^2))L_n^{2\alpha}(\frac{m_0\omega}{\hbar}u^2)\end{aligned} \tag{5.47}$$

5.6 Conclusion

Then, using the path integral technique we were able to exactly solve the problem of a particle with the time dependent mass $m = m_0 exp(\alpha t)$, subjected to a Coulomb potential in two dimensions, by preforming suitable transformations. We have also obtained the corresponding eigenfunctions and energy spectrum. The problem can be evaluated in three dimensions following the same way done here, and an extra phase term will appear in the wave functions.

We remark that we can reach the stationary results by putting $\alpha = 0$, which is the same results found by authors. This is feel good about the way and the technique we have chosen which make us tray to generalize it to the problem with a time-position dependent mass in other works.

Through the formulation and the results given above and the obtained wave functions and energies we conclude that the path integral is a powerful technique to study quantum dynamics of particles in non-relativistic theory.

Chapter 6

Position-time-dependent mass

6.1 Introduction

In recent years, the study of quantum mechanical systems with position-dependent effective masses has received considerable attention[25-32]. They constitute interesting and useful models for the description of several physical problems in different areas of the material sciences and condensed matter physics, especially in the case of many- body problems[33], electronic properties of semi-conductors[34], quantum dots[35], quantum liquids[35] and metal clusters[37],...etc. This wide range of applications has led to the development of methods and techniques for studding such systems. Among them, we can cite the point-canonical transformation method[30,31,32], the algebraic methods[36,40] and the supersymmetric quantum mechanics[41]. Note that in all of theses methods, the common procedure is to convert the position-dependent mass problem into that of constant mass and the main aim is to get energy spectra and/or the wave functions for theses systems ones the position-dependent mass is given.

The problem of variable mass can also be formulated by the path integral approach. Some examples have been treated in configuration space [42,43,44] where in [40] the Green's function of position-dependent mass has been related to that of constant mass according to a direct calculation.

In this chapter we are interested in developing a systematic procedure to study one-dimensional path integral in phase space for a class of position-time dependent masses and time-dependent potentiates. This later can provide not only many exact results known in the literatures but also a various new ones.

By using an explicitly time-dependent canonical transformation as well as a time transformation, we were able to absorb the time dependence of the path integral. Then by shifting the momentum and performing sn other judicious time transformation, we reduced the problem with position-time dependent mass to that relating to a constant mass and stationary potential.

As application, we have considered two different mass distributions each of which being relative to a chosen potential so that the corresponding path integral be exactly resolved.

6.2 Hamiltonian and path integral

There is an ambiguity in writing the quantum Hamiltonian for systems with position-dependent mass. This ambiguity arising from the fact that $\hat{x}$ and $\hat{p}$ do not commute. There are several forms for the hermitian Hamiltonian with a position dependent mass, all of them have the same classical limit but they differ in the quantum level. In general we can write

$$\hat{H} = \frac{1}{4}(m^{\alpha}\hat{p}m^{\beta}\hat{p}m^{\gamma} + m^{\gamma}\hat{p}m^{\beta}\hat{p}m^{\alpha} + \hat{V}(\hat{x},t), \quad \text{with } \alpha+\beta+\gamma=-1 \tag{6.1}$$

This formulas is the most general one that can save the hermiticity of Hamiltonian. The parameters α, β and γ will be chosen such that the condition given in Eq.(6.1) is holds. In our case we will choose the Hamiltonian with parameters $\alpha=-1$, and $\beta=\gamma=0$, because it has many applications. Also we will be interesting with the time dependent potentials of the form $\hat{V}(\hat{x},t) = f^2(t)\hat{V}(f(t)\hat{x})$ and our chosen time dependent mass has the form $m(x,t) = m(f(t)x)$, where $f(t)$ is an arbitrary time-dependent function. By reordering the Hamiltonian this will produces an effective potential term (See Ref. [45])

$$\hat{H} = \frac{1}{4}(\frac{1}{m}\hat{p}^2 + \hat{p}^2\frac{1}{m}) - f^2\frac{m''}{8m^2} + \frac{9f^2}{32}\frac{m'^2}{m^3} + f^2\hat{V}(f\hat{x}), \tag{6.2}$$

here the primes denote the derivatives with respect to the coordinate x.
The propagator related to the system Eq.(6.1) can be given in phase space by the following relation

$$\begin{aligned} \mathrm{K}(x'', t''; x', t') =& (m(f''x'')m(f'x'))^{-1/4} \int D[x(t)]D[p(t)] \\ & \times exp(i \int dt(p\dot{x} - \frac{1}{2}\frac{p^2}{m} + f^2\frac{m''}{8m^2} - \frac{9f^2}{32}\frac{m'^2}{m^3} - f^2V(fx)). \end{aligned} \tag{6.3}$$

This is the propagator of a position-time dependent mass particle subjected to the time-dependent potential $f^2(t)V(f(t)x)$. Th problem is time-dependent and it may not be easy to be evaluated directly unless we find procedure to transform it to a time-independent problem which is more easier to be evaluated. To do so we preform the following canonical transformation

$$\begin{aligned} x &= g(t)Q \\ p &= \frac{P}{g(t)}, \end{aligned} \tag{6.4}$$

with the generating function

$$F(x, P, t) = \frac{xP}{g(t)}, \tag{6.5}$$

where $g(t)$ is a real function.
The propagator (6.3) will be after this transformations

$$\begin{aligned} K(Q'', P'', t''; Q', P', t') =& \frac{(m(f''g''Q'')m(f'g'Q'))^{-1/4}}{\sqrt{g(t'')g(t')}} \int \frac{D[Q(t)]D[P(t)]}{2\pi} \\ & \times exp(i \int_{t'}^{t''} dt(P\dot{Q} - \frac{1}{2}\frac{P^2}{mg^2} + f^2\frac{m''}{8m^2} - \frac{9f^2}{32}\frac{m'^2}{m^3} + \\ & - f^2V(fgQ) + \frac{\dot{g}}{g}PQ)). \end{aligned} \tag{6.6}$$

Since $g(t)$ is an arbitrary function we will choose it such that $g(t)f(t) = 1$, to make the potential $V(x, t)$ and the mass functions $m(x, t)$ time-independent.
At this level, we notice that the kinetic term does not have the standard form. For this reason we carry out the time-transformation $dt = g^2(t)d\tau$ in order to

eliminate $g^{-2}(t)$ in the kinetic term and to absorb the explicit time-dependence of the path integral. The time rescaled path integral is

$$K(Q'', P'', \tau''; Q', P', \tau') = \frac{(m(f''g''Q'')m(f'g'Q'))^{-1/4}}{\sqrt{g(t'')g(t'')}} \int \frac{D[Q(\tau)]D[P(\tau)]}{2\pi} \times exp(i \int_{\tau'}^{\tau''} d\tau (P\dot{Q} - \frac{1}{2}\frac{P^2}{m} + \frac{m''}{8m^2} - \frac{9}{32}\frac{m'^2}{m^3} - V(Q) - g\dot{g}PQ)). \tag{6.7}$$

The propagator (6.7) is time independent except for the term $g\dot{g}PQ$. Then the function $f(t)$ is chosen to satisfy the condition $\dot{f}f^{-3}(t) = g\dot{g} = \kappa$. Moreover, shifting the momentum by $m\kappa Q$, or $P = P - m\kappa Q$, will cancel the PQ's term

$$K(Q'', P'', t''; Q', P', t') = \frac{(m(Q'')m(Q'))^{-1/4}}{\sqrt{g(t'')g(t')}} e^{-\kappa i(G(Q'')-G(Q'))} \int \frac{D[Q(\tau)]D[P(\tau)]}{2\pi} \times exp(i \int_{\tau'}^{\tau''} d\tau (P\dot{Q} - \frac{1}{2}\frac{P^2}{m} + \frac{m''}{8m^2} - \frac{9}{32}\frac{m'^2}{m^3} - V(Q) + - \frac{\kappa^2}{2} mQ^2)), \tag{6.8}$$

where the function $G(z)$ obeys the relation

$$G_z(z) = \frac{\partial G(z)}{\partial z} = m(z)z. \tag{6.9}$$

The term $\frac{P^2}{2m}$ is nasty since it's related factor $m(Q)$ is position-dependent function. To achieve a more convenient form of the path integral (6.3) we will perform the following time transformation defined as

$$\xi(Q(\tau))d\tau = ds, \tag{6.10}$$

This will lead to a new expression of the propagator (6.3)

$$K(Q'',P'',s'';Q',P',s') = \frac{(m(Q'')\xi(Q'')^2 m(Q')\xi(Q')^2)^{-1/4}}{\sqrt{g(t'')g(t')}} e^{-\kappa i(G(Q'')-G(Q'))} \times \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-iET} \int_0^{\infty} dS \int D[Q(s)]D[P(s)] \times exp(i \int ds(P\dot{Q} - \frac{1}{2}\frac{P^2}{m\xi} - \frac{\kappa^2}{2}\frac{m}{\xi}Q^2 - \frac{\xi'^2}{8m\xi^3} + - \frac{m'\xi'}{4m^2\xi^2} - \frac{1}{\xi}(-\frac{m''}{8m^2} + \frac{9}{32}\frac{m'^2}{m^3} + V(Q)) + \frac{E}{\xi}). \quad (6.11)$$

Since ξ is an arbitrary function we will choose it such that $\xi m = 1$ to obtain the following standard expression of the propagator

$$K(Q'',P'',s'';Q',P',s') = \frac{(m(Q'')m(Q'))^{1/4}}{\sqrt{g(t'')g(t'')}} e^{-\kappa i(G(Q'')-G(Q'))} \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-iET} \times \int_0^{\infty} dS \int D[Q(s)]D[P(s)] exp(i \int ds(P\dot{Q} - \frac{1}{2}P^2 + - \frac{\kappa^2}{2}m^2Q^2 - \frac{5}{32}\frac{m'^2}{m^2} + \frac{1}{8}\frac{m''}{m} - mV(Q) + mE). \quad (6.12)$$

Then the problem is transformed to that of constant mass but with different potential.

There are other choices of ξ such that the problem can be evaluated, for example $\xi m = \frac{\alpha}{Q}$, $\xi m = \frac{\alpha}{Q^2}$, the choice depends on the system that we have.

6.3 Applications

6.3.1 Example 1

We will be interested into the system of the mass $m(x) = x^\sigma$ and the potential $V(x) = V_0 + \frac{\beta}{x^\sigma} + \frac{\gamma}{x^{2\sigma}}$, $\sigma = \pm 2$.

This system is stationary, it is the case where $\kappa = 0$. The function $f(t)$ is chosen to be 1. Then after the transformations given above one can find that the propagator related to this problem [23]

$$K(Q'',P'',s'';Q',P',s') = (x'^{\sigma}x''^{\sigma})^{1/4}\int_{-\infty}^{\infty}\frac{dE}{2\pi}e^{-iET}\int_0^{\infty}dSe^{i\beta S}\int D[x(s)]D[p(s)]$$
$$\times\, exp(i\int ds(p\dot{x}-\frac{1}{2}p^2-\frac{1}{32}\frac{\sigma(\sigma+4)}{x^2}+(E-V_0)x^{\sigma}-\frac{\gamma}{x^{\sigma}})$$
$$=\frac{\omega}{i}(x'^{\sigma+2}x''^{\sigma+2})^{1/4}\int_{-\infty}^{\infty}\frac{dE}{2\pi}e^{-iET}\int_0^{\infty}dS\frac{e^{i\beta S}}{sin(\omega S)}$$
$$\times\, e^{i\frac{\omega}{2}(x''^2+x'^2)cot(\omega S)}I_{2\nu}\left(\omega\frac{x''x'}{isin(\omega S)}\right) \tag{6.13}$$

with $\nu=\sqrt{\frac{1}{8}\sigma(\sigma+4)+\frac{\gamma}{2}+\frac{1}{16}}$ and $\omega^2=-2(E-V_0)$ for $\sigma=2$, and $\nu=\sqrt{\frac{1}{8}\sigma(\sigma+4)-\frac{E-V_0}{2}+\frac{1}{16}}$ and $\omega^2=2\gamma$, for $\sigma=-2$. The integral over S in (6.13) can be performed using the formula

$$\int_0^{\infty}dx\ exp(2px-\beta coth(x))cosech(x)J_{2\gamma}(\alpha cosech(x))$$
$$=\alpha^{-1}\frac{\Gamma(1/2-p+\gamma)}{\Gamma(2\gamma+1)}M_{-p,\gamma}(\sqrt{\alpha^2+\beta^2}-\beta)W_{p,\gamma}(\sqrt{\alpha^2+\beta^2}-\beta) \tag{6.14}$$

Thus (6.13) will be

$$K(Q'',P'',s'';Q',P',s') = (-1)^{\nu+1}(x'^{\sigma}x''^{\sigma})^{1/4}\int_{-\infty}^{\infty}\frac{dE}{2\pi}e^{-iET}\frac{\Gamma(1/2-p+\nu)}{\Gamma(2\nu+1)\omega}$$
$$\times\, M_{-p,\nu}(\frac{\omega}{2}x'^2)W_{p,\nu}(\frac{\omega}{2}x''^2), \tag{6.15}$$

with $p=\frac{\beta}{2\omega}$. The Energies related to the bound state of this system are given by the relation

$$1/2-p+\nu=-n,\quad n\in\mathbb{N}. \tag{6.16}$$

By inserting the value of p in (6.16) we will find that the energies are

- For $\sigma=2$

$$E_n=-\frac{\beta^2/8}{(1/2+n+\nu)^2}+V_0, \tag{6.17}$$

- For $\sigma=-2$

$$E_n=-2(-n+\frac{\beta}{4\gamma}-1/2)^2-7/8+V_0. \tag{6.18}$$

6.3.2 Example 2

We will be interested to the system of The mass $m(x) = e^{\lambda x}$ and the potential $V(x) = V_0 + \beta e^{-\lambda x} + \gamma e^{\lambda x}$.
For this system the mass grows exponentially $m = e^{\lambda x}$, where λ is a real constant. Following the steps given above one can find that

$$\begin{aligned} K(Q'', P'', s''; Q', P', s') =& e^{\frac{\lambda}{4}(x''+x')} \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-iET} \int_0^{\infty} dS e^{-i\frac{\lambda^2+32\beta}{32}S} \\ & \times \int D[x(s)]D[p(s)]exp(i \int ds(p\dot{x} - \frac{1}{2}p^2 \\ & + (E - V_0)e^{\lambda x} - \gamma e^{2\lambda x}). \end{aligned} \tag{6.19}$$

Thus we see the solution reduced to the path integral of Morse potential[23]. Following [46,47] one can find

$$\begin{aligned} K(Q'', P'', s''; Q', P', s') =& e^{\frac{\lambda}{4}(x''+x')} \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-iET} \int_0^{\infty} dS e^{-i\frac{\lambda^2+32\beta}{32}S} \\ & \times \int \frac{dE_M}{2\pi} e^{-iE_M S} \frac{(-1)^{\lambda'+1}}{i\lambda\omega exp(\frac{\lambda}{2}(x''+x'))} \frac{\Gamma(\lambda' - p + 1/2)}{\Gamma(2\lambda' + 1)} \\ & \times M_{-p,\lambda'}(-\omega e^{\frac{\lambda}{2}x'}) W_{-p,\lambda'}(-\omega e^{\frac{\lambda}{2}x''}), \end{aligned} \tag{6.20}$$

where $p = 2\frac{E-V_0}{\lambda^2\omega}$, $\omega = \sqrt{8\gamma/\lambda^2}$ and $\lambda' = \sqrt{-2E_M/\lambda^2}$.
Preforming the S and E_M integrals we find that $E_M = -\frac{\lambda^2+32\beta}{32}$, then

$$\begin{aligned} K(Q'', P'', s''; Q', P', s') =& \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-iET} \frac{(-1)^{\lambda'+1}}{i\lambda\omega exp(\frac{\lambda}{4}(x''+x'))} \frac{\Gamma(\lambda' - p + 1/2)}{\Gamma(2\lambda' + 1)} \\ & \times M_{-p,\lambda'}(-\omega e^{\frac{\lambda}{2}x'}) W_{-p,\lambda'}(-\omega e^{\frac{\lambda}{2}x''}). \end{aligned} \tag{6.21}$$

The discrete energy levels can be found from the poles of the Γ function in the numerator

$$\lambda' - p + 1/2 = -n, \quad n \in \mathbb{N}. \tag{6.22}$$

Replacing λ', and p by their values one can find that

$$E_n = \frac{\lambda^2\omega}{2}(\sqrt{1/16 + 2\beta/\lambda^2} + 1/2 + n) + V_0. \tag{6.23}$$

We have presented two examples here, but using the method that have been presented in this chapter one can find many systems that can have exact solutions, for example the system with the mass $m(x) = m_0/x$, and the potential $V(x) = \alpha/x + \beta x$ where the problem will be reduced to the system of a free particle in Coulomb and an inverse quadratic potentials in one dimension.

6.4 Conclusion

In the present work we reduce the phase space path integral with position-dependent mass and time-dependent potential to that with constant mass and stationary potential, simply by using explicitly time-dependent canonical transformation and appropriate time transformations. The general form of the propagator is given and closed expressions are deduced for tow specific mass functions particles moving in familiar physical potentials, together with their energy spectra and corresponding wave functions.

We should point out that that our result can provide solutions for systems with different mass functions and typical potentials frequently used in the literatures and can also be extended to get solutions for systems with more complicated time-dependent mass distributions combined with other potentials to model interesting physical phenomena.

Chapter 7

Path integral for a particle in an infinite square well

The problem of a particle in an infinite square well potential is a simple problem in quantum mechanics and is known as the simplest bound-state problem. The system has been solved by Schroedinger equation exactly but via path integral the problem was one of the great puzzles for a long time. After by introducing the image point method equivalent to the sum over all classical paths[48] the author was able to solve it. Then Sokman [46] using a point canonical transformation on the coordinate, he could find the exact solution of such problem.

The propagator is not invariant under any change of variables, which means that the change should taken such that it will lead to the same quantum theory as the original one. In [46] a general method to compute the exact propagator under a point canonical transformation accompanied by a new time-transformation, where the problem reduced to that of a particle in a Rosen-Morse potential.

We aims in this work to find the exact propagator of a particle in infinite square well, in which the will be constrained in the interval $0 < x < 1$ with some potentials.

7.1 The propagator

Let us first present the Lagrangian of this problem which has the following form

$$L(\dot{x}, x, t) = \frac{m}{2}\dot{x}^2 - V(x) - U(x) \tag{7.1}$$

where the potential $V(x)$ is defined by

$$V(x) = \begin{cases} 0 & \text{where } \; 0 < x < 1 \\ \infty & \text{elsewhere,} \end{cases} \tag{7.2}$$

and $U(x)$ will be given latter(It will be chosen).
The propagator expressed in phase space as

$$K(x'', p'', t''; x', p', t') = \int \frac{D[x(t)]D[p(t)]}{2\pi\hbar} e^{\frac{i}{\hbar}\int_{t'}^{t''} dt(p\dot{x} - \frac{1}{2m}\dot{p}^2 - V(x) - U(x))}. \tag{7.3}$$

This propagator may not be evaluated directly because that we will transform it to another one that has a known exact solution. To deal with such problem we need to make a point canonical transformation, a transformation that saves the system quantum mechanically. The chosen transformation will be

$$\begin{aligned} x &= f(q) = \text{arctanh}(-\cos(\pi q)) \\ p &= \pi^{-1}\sin(\pi q)P, \end{aligned} \tag{7.4}$$

where P is the conjugate momentum of the variable q. As shown in [45] that an effective potential will be created and giving by the following expression

$$V_e = \frac{9}{31}\frac{g'^2}{g^3} - \frac{g''}{8g^2}, \tag{7.5}$$

with the function $g(q)$ is given by

$$g(q)^{1/2} = f'(q) = \frac{\pi}{sin(\pi q)}, \tag{7.6}$$

which means that

$$V_e(q) = \frac{5}{8}cos^2(\pi q) - \frac{1}{4}, \tag{7.7}$$

Then the propagator related to this system will be

$$K(p'', x'', t''; p', x', t') = (g(q'')g(q'))^{-1/4} \int \frac{D[q(t)]D[P(t)]}{2\pi\hbar} e^{\frac{i}{\hbar}\int_{t'}^{t''} dt(P\dot{q} - \frac{m}{2g(q)}P^2 - U(q) - V_e(q))} \tag{7.8}$$

The factor of kinetic part is some how nasty, because it is position-dependent and we may not be able to find the path integral of this problem easily in this case. We will make a time transformation [45] $t \to s$

$$\frac{ds}{dt} = 1/f'(q)^2 \tag{7.9}$$

which means that the propagator after this transformation is

$$\begin{aligned} K(p'', x'', t''; p', x', t') =& (g(q'')g(q'))^{1/4} \int \frac{dE}{2\pi} e^{-iET} \int dS \int \frac{D[q(s)]D[P(s)]}{2\pi\hbar} \\ & e^{\frac{i}{\hbar}\int_{s'}^{s''} ds(P\dot{q} - \frac{m}{2}P^2 - g(q)U(q) - g(q)V_e(q)) - Eg(q) + \frac{g'^2}{8g^2})} \\ =& \int \frac{dE}{2\pi} e^{-iET} G(q'', q'; E) \end{aligned} \tag{7.10}$$

With $G(q'', q'; E)$ is the Green's function related to this system, which has the expression

$$G(q'', q'; E) = \int dS \int \frac{D[q(s)]D[P(s)]}{2\pi\hbar} e^{\frac{i}{\hbar}\int_{s'}^{s''} ds(P\dot{q} - \frac{m}{2}P^2 - \frac{\pi^2}{sin^2(\pi q)}U(q) + \frac{\pi^2}{8}\frac{cos^2(\pi q) - 2 - 8E}{sin^2(\pi q)})} \tag{7.11}$$

At this stage we see that the problem taking a form that we can deal with for some chosen potentials. Using the transformation we were able to transform the problem of a particle in an infinite square wall subjected in the potential U to that of a particle subjected in the potentials $V \propto \frac{1}{sin^2(\pi q)}$ and $\frac{\pi^2}{sin^2(\pi q)}U(q)$.

7.2 Examples

7.2.1 $U(x) = U_0 = constant$

In this case the particle will be subjected in a constant potential U_0, here the Green's function we will be

$$G(q'', q'; E) = \int dS e^{-\frac{i}{\hbar}\frac{\pi^2}{8}S} \int \frac{D[q(s)]D[P(s)]}{2\pi\hbar} e^{\frac{i}{\hbar}\int_{s'}^{s''} ds(P\dot{q} - \frac{m}{2}P^2 - \frac{\pi^2}{8}\frac{1+8E+8U_0}{sin^2(\pi q)})} \tag{7.12}$$

7.2.2 $U(x) = U_0 tanh^2(x)$

In this case the particle will be subjected in the Rosen-Morse potential $U(x) = U_0 tanh^2(x)$, after the point canonical transformation that has been chosen it is clear that the expression of this potential will be changed to take the form

$$U(q) = U_0 cos^2(\pi q) \tag{7.13}$$

Then inserting this in the Green's function given above one would find that

$$G(q'', q'; E) = \int dS e^{\frac{i}{\hbar}\pi^2(U_0 - 1/8)S} \int \frac{D[q(s)]D[P(s)]}{2\pi\hbar} e^{\frac{i}{\hbar}\int_{s'}^{s''} ds(P\dot{q} - \frac{m}{2}P^2 - \frac{\pi^2}{8}\frac{1+8E+8U_0}{sin^2(\pi q)})} \tag{7.14}$$

7.2.3 $U(x) = U_0 tanh(x)$

The potential $U(x)$ will be

$$U(q) = U_0 cos(\pi q) \tag{7.15}$$

Then the Green's function will be

$$G(q'', q'; E) = \int dS e^{-\frac{i}{\hbar}\frac{\pi^2}{8}S} \int \frac{D[q(s)]D[P(s)]}{2\pi\hbar} e^{\frac{i}{\hbar}\int_{s'}^{s''} ds(P\dot{q} - \frac{m}{2}P^2 - \frac{\pi^2}{8}\frac{1+8E+8U_0 cos(\pi q)}{sin^2(\pi q)})} \tag{7.16}$$

Using the relations

$$\frac{1}{sin^2(2x)} = \frac{1}{4}\left(\frac{1}{sin^2(x)} + \frac{1}{cos^2(x)}\right) \tag{7.17}$$

and

$$\frac{cos(2x)}{sin^2(2x)} = \frac{1}{4}\left(\frac{1}{sin^2(x)} - \frac{1}{cos^2(x)}\right) \tag{7.18}$$

Then one will find that the Green's function will be

$$G(q'', q'; E) = \int dS e^{-\frac{i}{\hbar}\frac{\pi^2}{8}S} \int \frac{D[q(s)]D[P(s)]}{2\pi\hbar} e^{\frac{i}{\hbar}\int_{s'}^{s''} ds(P\dot{q} - \frac{m}{2}P^2 - \frac{\pi^2}{32}\frac{1+8E+8U_0}{sin^2(\pi/2q)} - \frac{\pi^2}{32}\frac{1+8E-8U_0}{cos^2(\pi/2q)})} \tag{7.19}$$

7.2.4 $U(x) = \frac{U_0}{tanh^2(x)}$

This is the case of a particle on a infinite square wall with an inverse Rosen-Morse potential. Under the transformation that has been taken this potential will take the form

$$U = \frac{U_0}{cos^2(\pi q)} \tag{7.20}$$

Inserting this in Green's function it will be

$$G(q'', q'; E) = \int dS e^{-\frac{i}{\hbar}\frac{\pi^2}{8}S} \int \frac{D[q(s)]D[P(s)]}{2\pi\hbar} e^{\frac{i}{\hbar}\int_{s'}^{s''} ds(P\dot{q} - \frac{m}{2}P^2 - \frac{\pi^2}{8}\frac{1+8E+8U_0}{sin^2(\pi q)} - \frac{\pi^2 U_0}{cos^2(\pi q)})} \tag{7.21}$$

7.3 Conclusion

In this chapter we have seen the problem of a particle in an infinite square wall with some chosen potentials, where we used a point canonical transformation to relate the problem to another one that has an exact solution, for many cases. Using the Schrödinger equation it may be difficult to find the solution for each case, but via path integral technique we were able to find the exact propagators.

Chapter 8

Charged particle in a field of Dayon

The problem of a particle with an electric charge $-e$ interacting with an electromagnetic field of Dirac monopole with a positive electric charge q and a magnetic g has been considered via path integral. The Green function and the discrete energy spectrum and its correspond eigenfunctions have been calculated exactly

A great deal of attention has been paid to the subject of existence of monopole and dyons [49,50] and the problem has become a challenging new frontier and the object of more interest in high energy physics. Dirac proved [49] that the quantum mechanics of an electrically charged particle of charge e and a magnetic charge g is consistent only if $\frac{eg}{\hbar c} = 2\pi n$, where n being an integer. Then a generalization has been made by Schwinger-Zwanziger [50] shows that for two particles of electric and magnetic charges (e, g) and (e', g') the relations $\frac{eg-e'g'}{\hbar c} = 2\pi n$ for the consistence of quantum mechanics. Then a great attention and generalization in the context of quantum field theory has been mad.

In this job we will find the propagator and the eigenfunctions for the non relativistic problem of a charged particle $-e$ with mass m interacting with electromagnetic field of a Dirac monopole of charge q and magnetic g
A magnetic monopole of charge g at the origin produces a radial field $\vec{B} = g\frac{\vec{r}}{r^2}$.
One possible vector potential can be chosen in spherical coordinate is

$$\vec{A} = g\frac{\alpha + \beta cos(\theta) + \gamma cos^2(\frac{\theta}{2})}{rsin(\theta)}\vec{e_\phi} \tag{8.1}$$

where α, β and γ are constants, with $\beta + \gamma/2 = -1$, in the simple case where $\alpha = 1$, $\beta = -1$ and $\gamma = 0$,we will be in the case given in [51,52]. We have chosen the potential $\vec{A}$ to have this form for the sake of being in a general case. The Lagrangian related to our problem in general can take the form

$$L(\vec{r}, \dot{\vec{r}}) = \frac{m}{2}\dot{\vec{r}}^2 - \frac{e}{c}\vec{A}\dot{\vec{r}} + \frac{eq}{r} \tag{8.2}$$

Where $\vec{A}$ is the vector potential, which in this case will be considered to have the following form

8.1 Green's Function

The Lagrangian describing the system is given in spherical coordinates (r, θ, ϕ) by

$$L(\vec{r}, \dot{\vec{r}}) = \frac{m}{2}\dot{\vec{r}}^2 - \frac{e}{c}\vec{A}\dot{\vec{r}} + \frac{eQ}{r}. \tag{8.3}$$

Following the path integral approach, the discrete expression of the propagator is explicitly defined in the post-point prescription by

$$\begin{aligned} K(\vec{r}_f, \vec{r}_i; T) = \lim_{N\to\infty} (\frac{\mu}{2\pi i\hbar\epsilon})^{3N/2} \int \prod_{j=1}^{N-1} r_j^2 sin\theta_j dr_j d\theta_j d\phi_j \\ exp\frac{i}{\hbar}\sum_{j=1}^{N}\Big(\frac{m}{2\epsilon}(r_j^2 + r_{j-1}^2(cos\theta_j cos\theta_{j-1} + sin\theta_j sin\theta_{j-1} cos\Delta\phi_j) + \\ -\frac{eg}{c}(\alpha + \beta cos\theta_j + \gamma cos^2\frac{\theta_j}{2})\Delta\phi_j + \frac{eQ}{r_j}\epsilon\Big), \end{aligned} \tag{8.4}$$

with the standard notation:
$\epsilon = t_j - t_{j-1}, T = N\epsilon = t_f - t_i, \vec{r}_f = \vec{r}(t_N = T), \vec{r}_i = \vec{r}(r_0 = 0)$.
We note that adopting the post-point prescription, a simplification arises in the computation of the elementary action. However, owing to the Coulomb's attractive term, this action presents a singularity at the origin. It is therefore essential to stabilize the path integral(8.4) by first introducing the energy E by means of Green's function, which is the Fourier transform of the propagator.

$$G(\vec{r}_f,\vec{r}_i;E) = \int_0^\infty dT e^{\frac{i}{\hbar}ET} K(\vec{r}_f,\vec{r}_i;T) \tag{8.5}$$

Then we use a time transformation $t \to s$ with a regulating function $f(r)$[53]

$$dt = f(r)ds = f_R(r)f_L(r)ds, \tag{8.6}$$

its discrete version is

$$\epsilon = \epsilon_s f_R(r_j) f_L(r_{j-1}),\ \epsilon_s = S_j - S_{j-1},\quad S = N\epsilon_s. \tag{8.7}$$

Taking into account the constrain $dT = dS_R(r_f)f_L(r_i)$, the Green's function (8.5) is rewriting following Ref.[19] as

$$G(\vec{r}_f,\vec{r}_i;E) = \int_0^\infty dS P_E^N(\vec{r}_f,\vec{r}_i;S), \tag{8.8}$$

where the promoter $P_E^N(\vec{r}_f,\vec{r}_i;S)$ is given formally by the path integral

$$\begin{aligned} P_E^N(\vec{r}_f,\vec{r}_i;S) =& f_R(r_f)f_L(r_i) lim_{N\to\infty} \int \prod_{j=1}^{N} \left(\frac{\mu}{2\pi i\hbar\epsilon_s f_R(r_j) f_L(r_{j-1})} \right)^{3/2} \\ & \prod_{j=1}^{N-1} r_j^2 sin(\theta_j) dr_j d\theta_j d\phi_j \prod_{j=1}^{N} exp(\frac{i}{\hbar} A_E^N(j,j-1)), \end{aligned} \tag{8.9}$$

with the pseudo-time sliced action,

$$\begin{aligned} A_E^N(j,j-1) =& \sum_{j=1}^{N} \Big(\frac{\mu}{2\epsilon_s f_R(r_j) f_L(r_{j-1})} (r_j^2 + r_{j-1}^2 (cos\theta_j cos\theta_{j-1} + \\ & sin\theta_j sin\theta_{j-1} cos\Delta\phi_j)) - \frac{eg}{c}(\alpha + \beta cos\tilde{\theta}_j + \gamma cos^2\frac{\tilde{\theta}_j}{2})\Delta\phi_j + \\ & - (\frac{eq}{r_j} - E)\epsilon_s f_R(r_j) f_L(r_{j-1}) \Big). \end{aligned} \tag{8.10}$$

The path integral(8.9) is too complicated for explicit calculation. Then in order to make it manageable, we use the approximation

$$cos\Delta\phi \simeq cos\Delta\phi + c\epsilon + c\epsilon\Delta\phi + 1/2c^2\epsilon^2, \tag{8.11}$$

and the Fourier expansion[24]

$$exp(zcos\Delta\phi) = \sum_{k=-\infty}^{\infty}(\frac{1}{2\pi z})^{1/2}exp(z - \frac{1}{2z}(k^2 - 1/4))e^{ik\Delta\phi}. \tag{8.12}$$

Then we can find that

$$\begin{aligned} exp(\frac{i}{\hbar}\sum_{j=1}^{N}A_E^N(j,j-1)) =& \prod_{j=1}^{N}\sum_{k_j=-\infty}^{+\infty}\sqrt{\frac{\epsilon_s f_R(r_j)f_L(r_{j-1})}{2\pi u_j}}exp\Big(\frac{i}{\hbar}(\frac{\mu}{2\epsilon}(r_j^2 + r_{j-1}^2 + \\ & - 2r_jr_{j-1}cos\Delta\theta_j) - (\frac{eq}{r_j} - E)\epsilon_s f_R(r_j)f_L(r_{j-1}) + \\ & - \frac{\epsilon}{2u_j}((k_j - iu_jv_j)^2 - \frac{1}{4}))\Big)e^{ik_j\Delta\phi_j}, \end{aligned} \tag{8.13}$$

where

$$\begin{aligned} u_j &= \frac{m}{i\hbar\epsilon_s f_R(r_j)f_L(r_{j-1})}r_jr_{j-1}sin\theta_j sin\theta_{j-1} \\ v_j &= -\frac{eg}{mc}\frac{\alpha + \beta cos\tilde{\theta}_j + \gamma cos^2\frac{\tilde{\theta}_j}{2}}{r_jr_{j-1}, sin\tilde{\theta}_j sin\tilde{\theta}_{j-1}}, \end{aligned} \tag{8.14}$$

which leads to the following expression of the promoter

$$\begin{aligned} P_E^N(\vec{r}_f,\vec{r}_i;S) =& f_R(r_f)f_L(r_i)lim_{N\to\infty}\int\prod_{j=1}^{N}\Big(\frac{\mu}{2\pi i\hbar\epsilon_s f_R(r_j)f_L(r_{j-1})}\Big)^{3/2} \\ & \prod_{j=1}^{N-1}r_j^2 sin(\theta_j)dr_jd\theta_jd\phi_j\prod_{j=1}^{N}exp(\frac{i}{\hbar}A_E^N(j,j-1)) \\ & \prod_{j=1}^{N}\sum_{k_j=-\infty}^{+\infty}\sqrt{\frac{\epsilon_s f_R(r_j)f_L(r_{j-1})}{2\pi u_j}}exp\Big(\frac{i}{\hbar}(\frac{\mu}{2\epsilon}(r_j^2 + r_{j-1}^2 - 2r_jr_{j-1}cos(\Delta\theta_j)) \\ & - (\frac{eq}{r_j} - E)\epsilon_s f_R(r_j)f_L(r_{j-1}) - \frac{\epsilon}{2u_j}((k_j - iu_jv_j)^2 - 1/4))\Big)e^{ik_j\Delta\phi_j}. \end{aligned} \tag{8.15}$$

Then we will perform the integral over ϕ, where we have

$$\int_0^{2\pi}e^{i(k'-k)\phi}d\phi = 2\pi\delta_{k',k}, \tag{8.16}$$

then

$$\int_0^{2\pi}\prod_{j=1}^{N-1} d\phi_j e^{i\sum_{j=1}^{N} k_j\Delta\Phi_j} = (2\pi)^{N-1}e^{ik(\phi_N-\phi_0)}, \tag{8.17}$$

which means that promotor (8.15) will have the following expression

$$\begin{aligned}
P_E^N(\vec{r}_f,\vec{r}_i;S) =& f_R(r_f)f_L(r_i)lim_{N\to\infty}\int\prod_{j=1}^{N}\left(\frac{\mu}{2\pi i\hbar\epsilon_s f_R(r_j)f_L(r_{j-1})}\right)^{3/2} \\
&\prod_{j=1}^{N-1} r_j^2 sin(\theta_j)dr_j d\theta_j d\phi_j \prod_{j=1}^{N} exp(\frac{i}{\hbar}A_E^N(j,j-1)) \\
&= f_R(r_f)f_L(r_i)lim_{N\to\infty}\sum_{k_j=-\infty}^{+\infty}\frac{e^{ik(\phi_N-\phi_0)}}{2\pi}(2\pi)^N \\
&\int\prod_{j=1}^{N}\left(\frac{\mu}{2\pi i\hbar\epsilon_s f_R(r_j)f_L(r_{j-1})}\right)^{3/2}\prod_{j=1}^{N-1} r_j^2 sin\theta_j dr_j d\theta_j \\
&\prod_{j=1}^{N}\sqrt{\frac{\epsilon_s f_R(r_j)f_L(r_{j-1})}{2\pi u_j}}exp\Big(\frac{i}{\hbar}(\frac{\mu}{2\epsilon}(r_j^2+r_{j-1}^2-2r_jr_{j-1}cos\Delta\theta_j) \\
&-(\frac{eq}{r_j}-E)\epsilon_s f_R(r_j)f_L(r_{j-1})-\frac{\epsilon_s f_R(r_j)f_L(r_{j-1})}{2u_j}((k-iu_jv_j)^2-1/4))\Big) \\
&= f_R(r_f)f_L(r_i)lim_{N\to\infty}\sum_{k_j=-\infty}^{+\infty}\frac{e^{ik(\phi_N-\phi_0)}}{2\pi}\int\prod_{j=1}^{N}\left(\frac{\mu}{2\pi i\hbar\epsilon_s f_R(r_j)f_L(r_{j-1})}\right) \\
&\prod_{j=1}^{N-1} r_j^2 sin\theta_j dr_j d\theta_j\prod_{j=1}^{N}\sqrt{\frac{1}{r_jr_{j-1}sin\theta_j sin\theta_{j-1}}}exp\Big(\frac{i}{\hbar}(\frac{\mu}{2\epsilon}(r_j^2+r_{j-1}^2+ \\
&-2r_jr_{j-1}cos\Delta\theta_j)-(\frac{eq}{r_j}-E)\epsilon_s f_R(r_j)f_L(r_{j-1})+ \\
&-\frac{\epsilon_s f_R(r_j)f_L(r_{j-1})}{2u_j}((k-iu_jv_j)^2-1/4))\Big).
\end{aligned} \tag{8.18}$$

After some simplification this will be

$$P_E^N(\vec{r}_f, \vec{r}_i; S) = \frac{f_R(r_f) f_L(r_0)}{\sqrt{r_0 r_N sin\theta_0 sin\theta_N}} lim_{N\to\infty} \sum_{k_j=-\infty}^{+\infty} \frac{e^{ik(\phi_N-\phi_0)}}{2\pi}$$
$$\int \prod_{j=1}^{N} \left(\frac{\mu}{2\pi i\hbar\epsilon_s f_R(r_j) f_L(r_{j-1})} \right) \prod_{j=1}^{N-1} r_j dr_j d\theta_j exp\Big(\frac{i}{\hbar} (\frac{\mu}{2\epsilon} (r_j^2 +$$
$$+ r_{j-1}^2 - 2 r_j r_{j-1} cos\Delta\theta_j) - (\frac{eq}{r_j} - E) \epsilon_s f_R(r_j) f_L(r_{j-1}) +$$
$$- \frac{\hbar^2 \epsilon_s f_R(r_j) f_L(r_{j-1})}{2\mu r_j r_{j-1} sin\theta_j sin\theta_{j-1}} ((k + \frac{eg}{\hbar c} (\alpha + \beta cos\theta_j +$$
$$+ \gamma cos^2\theta_j))^2 - 1/4)) \Big). \tag{8.19}$$

In order to separate the variables r and θ, we first turn to throw the singularity at $r = 0$ to the infinity with the spacial transformation $r \to q$ defined by the equation:

$$r = e^{\sigma q}, \quad -\infty < q < \infty, \quad and \quad \sigma > 0. \tag{8.20}$$

In parallel we have to take the following choice for the regulating functions[53]

$$f(r) = e^{2\sigma q} \text{ and } f_R = 1. \tag{8.21}$$

With these new variables, the promotor (8.18) is then rewritten as

$$P_E^N(\vec{r}_f, \vec{r}_i; S) = \frac{e^{2\sigma q_0}}{\sqrt{e^{\sigma(q_N+q_0)} sin\theta_0 sin\theta_N}} lim_{N\to\infty} \sum_{k_j=-\infty}^{+\infty} \frac{e^{ik(\phi_N-\phi_0)}}{2\pi}$$
$$\int \prod_{j=1}^{N} \left(\frac{\mu}{2\pi i\hbar\epsilon_s e^{2\sigma q_j}} \right) \prod_{j=1}^{N-1} \sigma e^{2\sigma q_j} dq_j d\theta_j exp\Big(\frac{i}{\hbar} (\frac{\mu}{2\epsilon_s e^{2\sigma q_j}} (e^{2\sigma q_j} +$$
$$+ e^{2\sigma q_{j-1}} - 2 e^{\sigma(q_j+q_{j-1})} cos\Delta\theta_j) - (\frac{eq}{e^{\sigma q_j}} - E) \epsilon_s e^{2\sigma q_j}$$
$$- \frac{\hbar^2 \epsilon_s e^{2\sigma q_j}}{2\mu e^{\sigma(q_j+q_{j-1})} sin\theta_j sin\theta_{j-1}} ((k + \frac{eg}{\hbar c} (\alpha + \beta cos\theta_j +$$
$$+ \gamma cos^2\theta_j))^2 - 1/4)) \Big). \tag{8.22}$$

In addition, it is useful to replace the integration over the upper-position q_j and θ_j by one over intervals Δq_j and $\Delta\theta_j$, thanks to the identity

$$\int\prod_{j=1}^{N-1} dq_j d\theta_j = \int\prod_{j=2}^{N} d\Delta q_j d\Delta\theta_j, \tag{8.23}$$

with the result that the measure also changes as

$$\prod_{j=1}^{N-1} \sigma e^{2\sigma q_j} dq_j d\theta_j = \prod_{j=2}^{N} e^{2\sigma q_{j-1}} d\sigma\Delta q_j d\Delta\theta_j, \tag{8.24}$$

which will lead to

$$\begin{aligned} P_E^N(\vec{r}_f,\vec{r}_i;S) =& \frac{1}{\sqrt{sin\theta_0 sin\theta_N}} lim_{N\to\infty}\sum_{k_j=-\infty}^{+\infty}\frac{e^{ik(\phi_N-\phi_0)}}{2\pi}\int\prod_{j=1}^{N}\Big(\frac{\mu}{2\pi i\hbar\epsilon_s}\Big) \\ & \prod_{j=1}^{N} e^{-2\sigma\Delta q_j}\prod_{j=2}^{N} d\Delta\sigma q_j d\Delta\theta_j exp\Big(\frac{i}{\hbar}\frac{\mu}{2\epsilon_s}(1+e^{-2\sigma\Delta q_j}+ \\ & -2e^{-\sigma\Delta q_j}cos\Delta\theta_j - (\frac{eq}{e^{\sigma q_j}} - E)\epsilon_s e^{2\sigma q_j} \\ & -\frac{\hbar^2\epsilon_s e^{2\sigma q_j}}{2\mu e^{\sigma(q_j+q_{j-1})} sin\theta_j sin\theta_{j-1}}((k+\frac{eg}{\hbar c}(\alpha+\beta cos\theta_j+ \\ & +\gamma cos^2\theta_j))^2 - 1/4))\Big). \end{aligned} \tag{8.25}$$

From the other part we have at the post-point

$$e^{-2\sigma\Delta q_j} \approx 1 - 2\sigma\Delta q_j + 2\sigma^2\Delta q_j^2 \tag{8.26}$$

and

$$cos\Delta\theta_j \approx 1 - \frac{1}{2}\Delta\theta_j^2 + \frac{1}{24}\Delta\theta_j^4, \tag{8.27}$$

then inserting this in (8.25) one find that

$$P_E^N(\vec{r}_f,\vec{r}_i;S) = \frac{1}{\sqrt{sin\theta_0 sin\theta_N}} lim_{N\to\infty} \sum_{k_j=-\infty}^{+\infty} \frac{e^{ik(\phi_N-\phi_0)}}{2\pi} \int \prod_{j=1}^{N}\left(\frac{\mu}{2\pi i\hbar\epsilon_s}\right)$$
$$\prod_{j=1}^{N}(1+C_{mes})\prod_{j=2}^{N} d\Delta\sigma q_j d\Delta\theta_j exp\Big(\frac{i}{\hbar}\frac{\mu}{2\epsilon_s}(1+(1-2\sigma\Delta q_j+2\sigma^2\Delta q_j^2+$$
$$-\frac{4}{3}\sigma^3\Delta q_j^3+\frac{2}{3}\sigma^4\Delta q_j^4)-2(1-\sigma\Delta q_j+\frac{1}{2}\sigma^2\Delta q_j^2+$$
$$-\frac{1}{6}\sigma^3\Delta q_j^3+\frac{1}{24}\sigma^4\Delta q_j^4)(1-\frac{1}{2}\Delta\theta_j^2+\frac{1}{24}\Delta\theta_j^4)+$$
$$-(\frac{eq}{e^{\sigma q_j}}-E)\epsilon_s e^{2\sigma q_j}-\frac{\hbar^2\epsilon_s}{2\mu sin\theta_j sin\theta_{j-1}}((k+\frac{eg}{\hbar c}(\alpha+\beta cos\theta_j+$$
$$+\gamma cos^2\theta_j))^2-1/4))\Big), \tag{8.28}$$

where

$$C_{mes} = -2\sigma\Delta q_j + 2\sigma^2\delta q_j^2. \tag{8.29}$$

Taking into account all contributions up to first order in ϵ_s, the promotor (8.22) can be put in the form

$$P_E^N(\vec{r}_f,\vec{r}_i;S) = \frac{1}{\sqrt{sin(\theta_0)sin\theta_N}} lim_{N\to\infty} \sum_{k_j=-\infty}^{+\infty} \frac{e^{ik(\phi_N-\phi_0)}}{2\pi} \int \prod_{j=1}^{N}\left(\frac{\mu}{2\pi i\hbar\epsilon_s}\right)$$
$$\prod_{j=1}^{N}(1+C_{mes})\prod_{j=2}^{N} d\Delta\sigma q_j d\Delta\theta_j exp\Big(\frac{i}{\hbar}(\frac{\mu}{2\epsilon_s}(\sigma^2\Delta q_j^2+\Delta\theta_j^2)+\Delta A_E^N+$$
$$-(\frac{eq}{e^{\sigma q_j}}-E)\epsilon_s e^{2\sigma q_j}-\frac{\hbar^2\epsilon_s}{2\mu sin\theta_j sin\theta_{j-1}}((k+\frac{eg}{\hbar c}(\alpha+\beta cos\theta_j+$$
$$+\gamma cos^2\theta_j))^2-1/4))\Big), \tag{8.30}$$

where ΔA_E^N represent the correction terms which is

$$\Delta A_E^N = \frac{\mu}{2\epsilon_s}\left(-\sigma^3\Delta q_j^3-\sigma\Delta q_j\Delta\theta_j^2-\frac{1}{12}\Delta\theta_j^4+\frac{7}{12}\sigma^4\Delta q_j^4+\frac{1}{2}\sigma^2\Delta q_j^2\Delta\theta_j^2\right), \tag{8.31}$$

and the exponent of it will be

$$exp(\frac{i}{\hbar}\Delta A_E^N) \approx 1+\frac{i}{\hbar}\Delta A_E^N-\frac{1}{2\hbar^2}(\Delta A_E^N)^2, \tag{8.32}$$

which leads

$$\begin{aligned}P_E^N(\vec{r}_f,\vec{r}_i;S)=&\frac{1}{\sqrt{sin\theta_0 sin\theta_N}}lim_{N\to\infty}\sum_{k_j=-\infty}^{+\infty}\frac{e^{ik(\phi_N-\phi_0)}}{2\pi}\int\prod_{j=1}^{N}\left(\frac{\mu}{2\pi i\hbar\epsilon_s}\right)\\&\prod_{j=1}^{N}(1+C_T)\prod_{j=2}^{N}d\Delta\sigma q_j d\Delta\theta_j exp\Big(\frac{i}{\hbar}(\frac{\mu}{2\epsilon_s}(\sigma^2\Delta q_j^2+\Delta\theta_j^2)+\\&-(\frac{eq}{e^{\sigma q_j}}-E)\epsilon_s e^{2\sigma q_j}-\frac{\hbar^2\epsilon_s}{2\mu sin\theta_j sin\theta_{j-1}}((k+\frac{eg}{\hbar c}(\alpha+\beta cos\theta_j+\\&+\gamma cos^2\theta_j))^2-1/4))\Big),\end{aligned}\tag{8.33}$$

where C_T is the total correction given by

$$\begin{aligned}C_T=&\frac{i}{\hbar}\Delta A_E^N-\frac{1}{2\hbar^2}(\Delta A_E^N)^2+C_{mes}(1+\frac{i}{\hbar}\Delta A_E^N)\\=&\frac{i}{\hbar}\Delta A_E^N-\frac{\mu^2}{8\epsilon_s^2\hbar^2}(\sigma^6\Delta q_j^6+2\sigma^4\Delta q_j^4\Delta\theta_j^2+\sigma^2\Delta q_j^2\Delta\theta_j^4)+C_{mes}+\\&+2\frac{\mu^2}{2\epsilon_s}(\sigma^4\Delta q_j^4+\sigma^2\Delta q_j^2\Delta\theta_j^2).\end{aligned}\tag{8.34}$$

To simplify this expression of path integration we will use the procedure given by McLaughlin-Shulman [21]. Using the integral

$$\int_0^\infty x^{2n}exp(-\alpha x^2)dx=\frac{(2n-1)!!}{2^{n+1}\alpha^n}\sqrt{\frac{\pi}{\alpha}}.\tag{8.35}$$

This will lead to a pure quantum effective potential by simply making the substitutions:

$$\begin{aligned}&\Delta q_j^2\to\frac{i\hbar\epsilon_s}{\sigma^2\mu},\ \Delta\theta_j^2\to\frac{i\hbar\epsilon_s}{\mu},\ \Delta q_j^4\to 3\left(\frac{i\hbar\epsilon_s}{\sigma^2\mu}\right)^2,\ \Delta\theta_j^4\to 3\left(\frac{i\hbar\epsilon_s}{\mu}\right)^2\\&\Delta q_j^2\Delta\theta_j^2\to\left(\frac{i\hbar\epsilon_s}{\sigma\mu}\right)^2,\ \Delta q_j^2\Delta\theta_j^4\to\frac{3}{\sigma^2}\left(\frac{i\hbar\epsilon_s}{\mu}\right)^3,\ \Delta q_j^4\Delta\theta_j^2\to\frac{3}{\sigma^4}\left(\frac{i\hbar\epsilon_s}{\mu}\right)^3,\\&\Delta q_j^6\to 15\left(\frac{i\hbar\epsilon_s}{\sigma^2\mu}\right)^3.\end{aligned}\tag{8.36}$$

and the contribution of the impair terms is 0. By replacing that in (8.33) one would find that $C_{tot} = 0$, which means that $P_E^N(\vec{r}_f, \vec{r}_i; S)$ will be

$$\begin{aligned}
P_E^N(\vec{r}_f, \vec{r}_i; S) =& \frac{1}{\sqrt{sin\theta_0 sin\theta_N}} lim_{N\to\infty} \sum_{k=-\infty}^{+\infty} \frac{e^{ik(\phi_N-\phi_0)}}{2\pi} \int \prod_{j=1}^{N} \left(\frac{\mu}{2\pi i\hbar\epsilon_s}\right) \\
& \prod_{j=2}^{N} d\Delta\sigma q_j d\Delta\theta_j exp\Big(\frac{i}{\hbar}(\frac{\mu}{2\epsilon_s}(\sigma^2\Delta q_j^2 + \Delta\theta_j^2) - (\frac{eq}{e^{\sigma q_j}} - E)\epsilon_s e^{2\sigma q_j} + \\
& - \frac{\hbar^2\epsilon_s}{2\mu sin^2\theta_j}((k + \frac{eg}{\hbar c}(\alpha + \beta cos\theta_j + \gamma cos^2\theta_j))^2 - 1/4))\Big) \\
=& \frac{1}{\sigma} \frac{1}{\sqrt{sin\theta_0 sin\theta_N}} lim_{N\to\infty} \sum_{k=-\infty}^{+\infty} \frac{e^{ik(\phi_N-\phi_0)}}{2\pi} \int \prod_{j=1}^{N} \left(\frac{\mu\sigma}{2\pi i\hbar\epsilon_s}\right) \\
& \prod_{j=1}^{N-1} dq_j d\theta_j exp\Big(\frac{i}{\hbar}(\frac{\mu}{2\epsilon_s}(\sigma^2\Delta q_j^2 + \Delta\theta_j^2) - (\frac{eq}{e^{\sigma q_j}} - E)\epsilon_s e^{2\sigma q_j} + \\
& - \frac{\hbar^2\epsilon_s}{2\mu sin^2\theta_j}((k + \frac{eg}{\hbar c}(\alpha + \beta cos\theta_j + \gamma cos^2\theta_j))^2 - 1/4))\Big) \\
=& lim_{N\to\infty} \sum_{k=-\infty}^{+\infty} \frac{e^{ik(\phi_N-\phi_0)}}{2\pi} P_E^N(q_f, q_i; S) P_E^N(\theta_f, \theta_i; S)
\end{aligned} \tag{8.37}$$

with

$$P_E^N(q_f, q_i; S) = \frac{1}{\sigma} \int \prod_{j=1}^{N} \left(\frac{\mu\sigma^2}{2\pi i\hbar\epsilon_s}\right)^{1/2} \prod_{j=1}^{N-1} dq_j exp\Big(\frac{i}{\hbar}(\frac{\mu\sigma^2}{2\epsilon_s}\Delta q_j^2 - (\frac{eq}{e^{\sigma q_j}} - E)\epsilon_s e^{2\sigma q_j}\Big) \tag{8.38}$$

and

$$\begin{aligned}
P_E^N(\theta_f, \theta_i; S) =& \frac{1}{\sqrt{sin\theta_0 sin\theta_N}} \int \prod_{j=1}^{N} \left(\frac{\mu}{2\pi i\hbar\epsilon_s}\right)^{1/2} \prod_{j=1}^{N-1} d\theta_j exp\Big(\frac{i}{\hbar}(\frac{\mu}{2\epsilon_s}\Delta\theta_j^2 + \\
& - \frac{\hbar^2\epsilon_s}{2\mu sin^2\theta_j}((k + \frac{eg}{\hbar c}(\alpha + \beta cos\theta_j + \gamma cos^2\theta_j))^2 - 1/4))\Big) \\
=& \frac{1}{\sqrt{sin\theta_0 sin\theta_N}} \int \prod_{j=1}^{N} \left(\frac{\mu}{2\pi i\hbar\epsilon_s}\right)^{1/2} \prod_{j=1}^{N-1} d\theta_j exp\Big(\frac{i}{\hbar}(\frac{\mu}{2\epsilon_s}\Delta\theta_j^2 + \\
& - \frac{\hbar^2\epsilon}{8m\tilde{r}_j^2}(\frac{\nu}{sin^2(\frac{\tilde{\theta}_j}{2})} + \frac{\kappa}{cos^2(\frac{\tilde{\theta}_j}{2})}))\Big)
\end{aligned} \tag{8.39}$$

With $\nu = k^2 - \frac{1}{4} + \frac{2keg}{\hbar c}(\alpha+\beta+\gamma) + \left(\frac{eg}{\hbar c}\right)^2(\alpha+\beta+\gamma)^2$, and $\kappa = k^2 - \frac{1}{4} + \frac{2keg}{\hbar c}(\alpha - \beta) + \left(\frac{eg}{\hbar c}\right)^2(\alpha-\beta)^2$.

It is obvious that the kernel $P_E^N(q_f, q_i; S)$ describes the motion of a particle with

mass $\mu\sigma^2$ subjected to Morse like potential(MP) which has been solved by different methods[23,53].
Then, after taking the Fourier transform of the kernel (8.38)

$$P_E^N(q_f, q_i; S) = \int \frac{dE_M}{2\pi} exp(-\frac{i}{\hbar} E_M S) G_{MP}, \tag{8.40}$$

we obtain the familiar result expressed in term of the standard Whittaker functions

$$G_{MP} = \left(\frac{4}{\sigma^2\hbar^2\omega^2 e^{\sigma(q_f+q_i)}}\right)^{1/2} \frac{\Gamma(\frac{\lambda-p+1}{2})}{\Gamma(\lambda+1)} M_{p/2,\lambda/2}(\mu\sigma^2\omega\hbar^{-1}e^{\sigma q_f/2})$$
$$W_{p/2,\lambda/2}(\mu\sigma^2\omega\hbar^{-1}e^{\sigma q_i/2}), \tag{8.41}$$

where $\lambda^2 = -(8\mu/\hbar^2)E_M$, $\omega = [8(-E)/\mu\sigma^4]^{1/2}$, $p = -4eq/\sigma^2\hbar\omega$.
On the other hand, the path integral(8.39) is recognized as the propagator for particle of mass μ in Pöschl-Teller potential which has spectral representation in terms of the associated Legendre polynomials $P_{n_\theta}(cos\theta)$ [22]

$$P_E^N(\theta_f, \theta_i; S) = \sum_{n_\theta} e^{-\frac{i}{\hbar}E_{n_\theta}S} \varphi^*_{k,n_\theta}(\theta_i)\varphi_{k,n_\theta}(\theta_f), \tag{8.42}$$

where $\varphi_{k,n_\theta}(\theta)$ is the angular wave function and are of the form

$$\varphi_{k,n_\theta}(\theta) = C_{k,n_\theta} sin(\theta)^{\frac{1}{4}\sqrt{1+4\nu}} cos(\theta)^{\frac{1}{4}\sqrt{1+4\kappa}+1/2} P_{n_\theta}^{(\frac{1}{4}\sqrt{1+4\nu},\frac{1}{4}\sqrt{1+4\kappa})}(cos(\theta)), \tag{8.43}$$

with the normalization constants

$$C_{k,n_\theta} = \sqrt{2^{3/2}\frac{n_\theta \frac{\sqrt{mE_{n_\theta}}}{\hbar}\Gamma(\frac{\sqrt{mE_{n_\theta}}}{\hbar} - n_\theta)}{\Gamma(\frac{\sqrt{mE_{n_\theta}}}{\hbar} - n_\theta - \frac{1}{4}\sqrt{1+4\nu})\Gamma(\frac{\sqrt{mE_{n_\theta}}}{\hbar} - n_\theta - \frac{1}{4}\sqrt{1+4\kappa})}}, \tag{8.44}$$

and the following formula for energy spectrum

$$E_{n_\theta} = \frac{\hbar^2}{4\mu}(8n_\theta + 2\sqrt{4\kappa+1} + \sqrt{4\nu+1} + 4)^2. \tag{8.45}$$

After (8.40) and (8.42) are substituted in (8.37) the Green's function (8.8) relative to out problem is found to be

$$G(\vec{r}_f, \vec{r}_i; E) = \sum_{k=-\infty}^{+\infty} \frac{e^{ik(\phi_N - \phi_0)}}{2\pi} \int_0^\infty dS \int \frac{dE_M}{2\pi} exp(-\frac{i}{\hbar}(E_M + E_{n_\theta})S)$$
$$\left(\frac{4}{\sigma^2\hbar^2\omega^2 e^{\sigma(q_f+q_i)}}\right)^{1/2} \frac{\Gamma(\frac{\lambda-p+1}{2})}{\Gamma(\lambda+1)} M_{p/2,\lambda/2}(\mu\sigma^2\omega\hbar^{-1}e^{\sigma q_f/2})$$
$$W_{p/2,\lambda/2}(\mu\sigma^2\omega\hbar^{-1}e^{\sigma q_i/2})\varphi^*_{k,n_\theta}(\theta_i)\varphi_{k,n_\theta}(\theta_f). \tag{8.46}$$

8.2 Energy spectrum and wave functions

In order to determine the bound states energy levels of the system and its corresponding wave functions, we perform the integration with respect to S and $E_M$successively. After some straightforward calculation, we easily get the final expression for the Green's function.

$$G(\vec{r}_f, \vec{r}_i; E) = \left(\frac{4}{\sigma^2\hbar^2\omega^2 e^{\sigma(q_f+q_i)}}\right)^{1/2} \frac{\Gamma(\frac{\lambda-p+1}{2})}{\Gamma(\lambda+1)} \sum_{k=-\infty}^{+\infty} \frac{e^{ik(\phi_N - \phi_0)}}{2\pi}$$
$$M_{p/2,\lambda/2}(\mu\sigma^2\omega\hbar^{-1}e^{\sigma q_f/2})W_{p/2,\lambda/2}(\mu\sigma^2\omega\hbar^{-1}e^{\sigma q_i/2})\varphi^*_{k,n_\theta}(\theta_i)\varphi_{k,n_\theta}(\theta_f), \tag{8.47}$$

with $\lambda^2 = 2(8n_\theta + \sqrt{4\kappa+1} + \sqrt{4\nu+1} + 4)^2$.
The energy spectrum and the wave functions can be obtained from the poles and from the residues of the Green's function (8.46),respectively. These poles occur when the argument of the Gamma function in the numerator in negative integer n or

$$\frac{\lambda - p + 1}{2} = -n \qquad n = 0, 1, 2.... \tag{8.48}$$

after solving this equation with respect to E we find the following formula for the energy levels

$$E_n = \frac{-2\mu e^2 q^2/\hbar^2}{(2n+1+\lambda)^2}. \tag{8.49}$$

From the residues of the function $G(\vec{r}_f, \vec{r}_i; E)$ at $E = E_n$, we can write the normalized wave functions as

$$\psi_n(\vec{r}) = \left(\frac{\mu\sigma^2\omega}{2\pi\hbar}\right)^{1/2} \frac{\sqrt{\Gamma(p-n)}}{\sqrt{n!}\Gamma(p-2n)} e^{ik\phi}\varphi_{k,n_\theta}(\theta) M_{\frac{p}{2},\frac{p-1}{2}-1}(\frac{\mu\sigma^2}{\hbar}\omega e^{\sigma q}). \tag{8.50}$$

We notice that the wave function $\psi_n(\vec{r})$ is single valued. Moreover, the vector potential $\vec{A} = g\frac{\alpha+\beta cos(\theta)+\gamma cos^2(\theta/2)}{rsin(\theta)}$ has a singularity at $\theta = \pi$. This can be avoided by making the change $\theta = \theta + \pi$. Then we will have another vector potential $\tilde{\vec{A}}$ which is singular at $\theta = 0$, such that

$$\tilde{\vec{A}} = -g\frac{\alpha - \beta cos(\theta) + \gamma sin^2(\theta/2)}{rsin(\theta)} \tag{8.51}$$

The corresponding Green's function $\tilde{G}(\vec{r}_f, \vec{r}_i; E)$ will differ form the obtained one (8.46) just by the phase $exp(i\nabla(\vec{A} - \tilde{\vec{A}})) = exp(ie\frac{g}{c\hbar}(2\alpha+\gamma)(\phi_f - \phi_i))$. In other words, the wave functions will differ from the old ones (8.50) by the phase factor $exp(\frac{ieq}{c\hbar}(2\alpha+\gamma)\ \phi)$. The wave function will be single valued before and after the transformation if and only if the following condition

$$e\frac{g}{c\hbar}(2\alpha+\gamma) = \text{integer}, \tag{8.52}$$

is satisfied.

Since $2e\frac{g}{c\hbar}$ is integer [49], $\alpha + \gamma/2$ should be an integer or half-integer.

8.3 Conclusion

In this research we calculated the path integral for an electrically charged particle in orbit around a dyon by connecting it to Morse potential and Pöschl-Teller potentials problems.

We have given a parametric form to the vector potential associated with the magnetic charge.

We have shown that the Green's function can be simply and naturally constructed in spherical coordinates with the post-point consideration.

As we have seen, after we have applied a coordinate transformation to Lagrangian path integral we have suitably chosen regulating functions so that the Green's function has become entirely defined by a stable promotor. Then we have exactly

extracted the bound states energies and the corresponding wave function.
Compared to the methods given in Refs.[51,52], it seems to us that our method leads most directly to exact solution without any need for cumbersome artifices and the advantage to significantly simplify the calculation and avoid the proliferation of unnecessary detail.
For the set of parameters ($\alpha = 1, \beta = -1$ and $\gamma = 0$), the obtained energy spectrum agrees with the result of the Refs. [51,52] which proves that our extended result is without doubt the correct propagator for a charged particle moving in the field od dyon.

Conclusion

We have presented a new description of non-relativistic quantum systems according to the Feynman path integral formalism. And we have shown that the path integral can be put under two explicit notions. The first called the Hamiltonian form (the integral path in the phase space), and the second is the Lagrangian form.

The study of harmonic oscillators with time-dependent mass has assumed in the second chapter and we have used space-time transformations in the phase and configuration spaces to treat the problem and find the exact propagators of new generalized examples.

We have studied a general model of explicitly time-dependent quantum problems by path integrals using some time-dependent transformations. The problem treated in both configuration and phase space, we used space-time transformations in configuration space and point canonical transformations in phase space, that leads to considerable simplification in computation and gives unambiguous results in comparison with already existing methods.

Using the space-time transformations to path integral we were able to exactly solve the problem of a particle with the exponentially time-dependent mass subjected to a Coulomb potential in two dimensions. We have also obtained the corresponding eigenfunctions and energy spectrum. The problem can be evaluated in three dimensions following the same way done here, and an extra phase term will appear in the wave functions.

We considered in the phase space the path integral with position-dependent mass and time-dependent potential and we reduced it to that of constant mass and stationary potential, simply by using explicitly time-dependent canonical transformation and appropriate time transformations. The general form of the propagator is given and closed expressions are deduced for tow specific mass functions of particles moving in familiar physical potentials, together with their energy spectra and corresponding wave functions.

The problem of a particle in an infinite square well with some chosen potentials was solved also, where we used a point canonical transformation to relate the problem to another one that has an exact solution, for many cases. Using the Schrödinger equation it may be difficult to find the solution for each case, but via path integral technique we were able to find the exact propagators.

Lastly, we calculated the path integral for an electrically charged particle in orbit around a dyon by connecting it to Morse and Pöschl-Teller potentials problems. We have given a parametric form to the vector potential associated with the magnetic charge. We have shown that the Green's function can be simply and naturally constructed in spherical coordinates with the post-point consideration. We have applied a coordinate transformation to Lagrangian path integral where we have suitably chosen regulating functions so that the Green's function has become entirely defined by a stable promotor. Then we have exactly extracted the energies of the bound states and the corresponding wave functions.

In future work, we will use path integral technique to solve more problems and even use it to build some artificial intelligence models.

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استخدام تكامل المسالك لدراسة الحركات الكمومية

ملخص

لقد قدمنا وصفًا جديدًا للأنظمة الكمومية غير النسبية وفقًا لتقنية تكامل المسالك لفيانمان. اهتممنا بدراسة الأنظمة المعتمدة على الزمن وكمثال على ذلك المتذبذبات التوافقية ذات الكتلة المعتمدة على الزمن في كلى الفضاءين فضاء الاجداثيات وفضاء الطور ؛ أيضًا باستخدام نفس التقنية مع التحولات في الزمان والمكان ، تمكنا من حل مشكلة الجسيم مع الكتلة المعتمدة على الزمن ، التي تخضع لكمون كولومب في بعدين

بفضل التحول القانوني ، تمكنا من حل مشاكل الجسيمات ذات الكتلة المعتمدة على الموضع والزمن عبر تقنية تكامل المسالك وكذا جملة الجسيم المحصور في بئر كموني وبعض الكمونات المختارة

أخيرًا ، قمنا بحساب سعة احتمال الانتقال او الناشر لجسيم مشحون كهربائيًا في مدار حول ديون عن طريق تحويل النظام الى ذلك الخاص بجسيم موجود في كموني مورس وبوشل تيلر

كلمات مفتاحية: سعة احتمال الانتقال؛ فضاء الاحداثيات ؛ فضاء الطور ؛ الأنظمة المعتمدة على الزمن ؛ المتذبذب التوافقي المعتمد على الزمن ؛ تكامل المسالك ؛كمون كولومب ؛ الكتلة المعتمدة على الزمن ؛ طيف الطاقة ودوال الموجة ؛ الكتلة المتعلقة بالموضع ؛ أحادي القطب المغناطيسي

Utilisation de l’intégrale de chemin dans l’étude des mouvements quantiques

RÉSUM É :

Cette thèse est consacrée à l'étude des systèmes quantiques non relativistes avec des coefficients dépendant explicitement du temps et aussi dépendant de la position et du temps simultanément dans le cadre du formalisme des intégrales de chemin de Feynman.

Nous avons présenté une méthode systématique pour construire le propagateur de systèmes dépendant du temps dans les espaces de configuration et de phase. Comme application, nous avons considéré le problème de l'oscillateur harmonique dont la masse et la fréquence sont des fonctions arbitraires du temps. Le traitement a été basé sur l'utilisation des transformations spatiales explicitement dépendantes du temps ainsi que des transformations temporelles, qui permettent de réduire le propagateur à celui dont la masse et la fréquence sont constantes. Nous avons illustré le résultat général en choisissant des modèles de masse et de fréquence variables.

D'autre part, nous avons étendu la technique des transformations spatio-temporelles pour ramener le problème d'une particule avec masse dépendante du temps se déplaçant dans un espace bidimensionnel et soumise au potentiel de Coulomb plus un potentiel quadratique inverse à un problème stationnaire. Ensuite, les coordonnées polaires étaient adéquates pour évaluer la fonction de Green et déduire exactement les niveaux d'énergie du spectre discret et les fonctions d'onde associées.

Nous nous sommes également intéressés au développement d'une procédure systématique pour étudier l'intégrale du chemin unidimensionnel dans l'espace des phases pour une classe de masses dépendant de la position et du temps et des potentiels dépendant du temps. Grâce à une transformation canonique explicitement dépendante du temps, nous avons pu absorber la dépendance temporelle de l'hamiltonien. Comme application, nous avons considéré deux distributions de masse différentes, chacune associée à un potentiel choisi de sorte que l'intégrale de chemin correspondant soit exactement résolue

Nous avons également obtenu des propagateurs exacts pour une particule confinée dans un puits carré infini et soumise en outre à certains potentiels. La fonction de Green a été construite pour chaque situation grâce à une transformation canonique ponctuelle appropriée.

Enfin, nous avons évalué l’intégrale de chemin pour une particule chargée électriquement sur une orbite autour d'un dyon. Des fonctions régulatrices judicieuses ont permis d'exprimer le promoteur comme un produit de deux noyaux partiels qui sont les problèmes des potentiels de Morse et de Pöschl-Teller.

Mots-clés: Intégral de chemin; Propagateur; Espace de configuration; Espace des phases; Transformations canoniques; Transformations temporelles; Systèmes dépendant du temps; Masse dépendante du temps; Masse dépendante de la position; Oscillateur harmonique dépendant du temps; Potentiel de Coulomb; Monopôle magnétique; Spectre d’énergie; Fonctions d'onde.

ABSTRACT:

This thesis is devoted to the study of non-relativistic quantum systems with explicitly time and position-time dependent coefficients in the framework of the Feynman's path integrals formalism.

We have presented a systematic method for constructing the propagator of time-dependent systems in both configuration and phase spaces. As application, we have considered the problem of harmonic oscillator with both mass and frequency being arbitrary functions of time. The treatment has been based on the use of explicitly time-dependent coordinate transformations as well as of time transformations, which permit to reduce the propagator to that with constant mass and frequency. We have illustrated the general result by choosing some models of varying mass and frequency.

On the other hand, we have extended the space-time transformations technique to bring the problem of a particle with time-dependent mass moving in two- dimensional space and subjected to Coulomb plus inverse quadratic potential to a stationary problem. Then, polar coordinates were adequate for evaluating the Green's function and exactly deducing the discrete spectrum energy levels and the relating wave functions.

We have been also interested in developing a systematic procedure to study one-dimensional path integral in phase space for a class of position-time dependent masses and time dependent potentials. Thanks to an explicitly time dependent canonical transformation, we have been able to absorb the time dependence of the Hamiltonian. As application, we have considered two different mass distributions each associated with a chosen potential so that the corresponding path integral have been exactly solved.

We have also obtained exact propagators for a particle confined in infinite square well and further subjected to some potentials. The Green's function have been constructed for each situation thanks to an appropiate point canonical transformation.

Finally, we have found the path integral solution for an electrically charged particle in orbit around a dyon. Judicious regulating functions have permitted to express the promotor as a product of two partial kernels that are the problems of Morse and Pöschl-Teller potentials.

Keywords: Path integral; Propagator; Configuration space; Phase space; Canonical transformations; Time transformations; Time-dependent systems; Time-dependent mass; Position-dependent mass; Time-dependent harmonic oscillator; Coulomb potential; Magnetic monopole; Energy spectrum; Wave functions.