REPUBLIQUE ALGERIENNE DEMOCRATIQUE ET POPULAIRE
MINISTERE DE L'ENSEIGNEMENT SUPERIEUR
ET DE LA RECHERCHE SCIENTIFIQUE

UNIVERSITE FRERES MENTOURI CONSTANTINE
FACULTE DES SCIENCES EXACTES
DEPARTEMENT DE PHYSIQUE

N° d'ordre :
Série :

THESE

POUR OBTENIR LE DIPLOME DE DOCTORAT 3ème CYCLE (LMD) EN PHYSIQUE

SPECIALITE

PHYSIQUE THEORIQUE

THEME

String Theories in Modified Formulations

Par

Hocine RANDJI

Soutenue le : 02/07/2020

Devant le jury:

Président:	Achour Benslama	Prof.	Univ. Mentouri Constantine
Rapporteur:	Nadir Belaloui	Prof.	Univ. Mentouri Constantine
Examinateurs:	Noureddine Mebarki	Prof.	Univ. Mentouri Constantine
	Ahmed Bouda	Prof.	Univ. de Béjaia
	Tahar Boudjedaa	Prof.	Univ. de Jijel
	Khireddine Nouicer	Prof.	Univ. de Jijel

PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA

MINISTRY OF HIGHER EDUCATION AND

SCIENTIFIC RESEARCH

FRERES MENTOURI CONSTANTINE 1 UNIVERSITY

FACULTY OF EXACT SCIENCES

DEPARTEMENT OF PHYSICS

DISSERTATION

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR

THE DEGREE OF DOCTOR OF PHYSICS

SPECIALITY: THEORETICAL PHYSICS

String Theories in Modified Formulations

By

Hocine RANDJI

Defended on: July 2, 2020

Jury:

President:	A. Benslama	Prof. Univ. Mentouri Constantine
Supervisor:	N. Belaloui	Prof. Univ. Mentouri Constantine
Examiners:	N. Mebarki	Prof. Univ. Mentouri Constantine
	A. Bouda	Prof. Univ. of Béjaia
	T. Boudjedaa	Prof. Univ. of Jijel
	K. Nouicer	Prof. Univ. of Jijel

To my family

Acknowledgements

With a great pleasure, I am grateful to my supervisor professor Nadir Belaloui for his support and guidance during the last years. I am also grateful to my teachers during my life. It is my joy and duty to thank the jury members, the professor Achour Benslama, the professor Noureddine Mebarki, the professor Ahmed Bouda, the professor Tahar Boudjedaa, and the professor Khireddine Nouicer. Finally but importantly, I thank all my family members.

Contents

General introduction

The quantum gravity and string theory are now being a very important fields of theoretical and mathematical physics. The standard model of elementary particles is very successful to describe the quantum theory of electromagnetism, weak and strong nuclear interactions. But the evolution of physics uses higher and higher energies, then this automatically leads to the quantum gravity. So we need to build a theory which manipulates the gravity with the laws of quantum mechanics.

The string theory was begun as an attempt to describe the strong interaction. The quantum string theory is presently one of the most quantum gravity candidates and the area of investigating the theory of unification. The ideas in this domain are lots of, but the experimental realization remains very hard and problematic. The elementary particles in string theory became a one dimensional extended object (the string). One of the features of quantum gravity theories, including string theory is to find a renormalized quantum gravity; without undesirable infinities, and in principle, string theory apparently can do that.

String theory is a theory of quantum gravity even if doesn't describe our world. But it excites several criticisms, one of them is the unverified extra dimensions. In this context several non-critical string theories were proposed, theses ones can be consistent in $D < 26$ for the bosonic string or $D < 10$ for the fermionic string. There are many ideas for constructing a consistent and complete non-critical quantum string theory, for example Igor Nikitin non-critical string theory [1], and the Polyakov version with linear-dilaton background [2]. In this issue of the possibility of non-critical string, J. Gegenberg and V. Husain insert a new term (which depends on a scalar field called dust) to the Polyakov action; which describes the ordinary string theory [3]. A similar term was used for solving some problems in Einstein Hilbert action, which are proposed by the Wheeler-Dewitt equation [4], and since the Polyakov action has the form of scalar fields in gravity with two dimensions, the same procedure can be possible for string theory quantization. The additional scalar field (The dust field) can be used to fix the world sheet dynamical parameter in the theory and gives a non vanishing physical Hamiltonian of the system.

Paraquantum string or the string theory in parastatistical formalisms [5] is started by [6] and developed in [7] [8], what is striking here is the space-time dimensions became

$D = \frac{24}{Q} + 2$ for the parabosonic string and $D = \frac{8}{Q} + 2$ for the parafermionic string. It is known from the supersymmetry that further symmetries in the world sheet reduce the numbers of the critical space-time dimensions of string theory. Unfortunately the $N = 2$ supersymmetry on the world sheet gives a two space-time dimensions as a critical dimension [9]. In the other hand, parastatistics can introduce some degrees of freedom which can lower the space-time dimension, with $D > 2$.

Canonical quantization is a successful program for reconciling the quantum mechanics (QM) (with the Planck constant $\hbar$) and the special relativity (SR) (with the velocity of light c as a universal constant), into the quantum field theory. The general relativity (GR) came with the gravity constant G, and therefore Planck length L_P, and the Planck energy E_P, etc can be constructed.

The universal smallest length makes the threshold between quantum and classical description of space-time, and the length contraction in special relativity is restricted by this length. These ideas may be imply a deformation in the conventional canonical quantization, for reconciling QM and GR [10, 11]. The candidate theories of quantum gravity can predict an energy dependence in the physical constants [12]. The relativistic relation between energy and momentum can encode some of these modifications. In this context, the deformed special relativity (DSR) can give an importance to the Planck length as much as the speed of light. So the DSR proposes a universal scale; like the Planck length, which is the same for all the inertial frames in the same manner as the speed of light. The first version of DSR is projected by Giovanni Almelino-Camelia [13, 14, 15], and the other one was given by Joào Magueijo and Lee Smolin [16], in this stage they have treated the bosonic string theories with deformed dispersion relations [17], which is inspired by the DSR models. The smallest length scale can also result a deformation in commutation relations called the generalized uncertainty principle (GUP), which leads to a deformation in the Heisenberg uncertainty relations [18, 19, 20].

Instead of modifying the Poisson brackets or the commutation relations, J. Magueijo and L. Smolin introduce a method based on a deformation of the usual bosonic string constraints by two functions f and g depend on total energy of the string, these constraints lead to deformed mass shell condition, which mirrors an analogy with the DSR and can give importance to a universal length scale; as the Planck length. These theories use the canonical quantization, with the opportunity to take advantage of the usual string theory successes [9, 21, 22]. The main results are the energy independent speed of light and, the possibility of non-tachyonic ground state.

The best candidates for the realistic verifications of doubly special relativity or the violation of Lorentz symmetry in the Planck scale are in the astrophysical observations, but there are other experimental ideas for investigating these phenomena for example in: the clock-comparison experiments, Doppler shift experiments, Penning traps [23].

The outline of this thesis is as follows:
First, in chapter 1 we give a brief introduction to the theory of special relativity and the DSR one.
In chapter 2, we study some aspects of the ordinary bosonic and fermionic classical string theories. Next, in chapter 3 we studied the open bosonic string theory with dust, principally the closed constraint algebra and the string stretched between two Dp-branes, we interested here on the non-critical string, non-tachyonic and non-ghost cases.

In chapter 4, we study bosonic string theories with deformed relativistic momentum energy relations which were developed in [17]. We show the possibility of such theories to describe a string with non deformed constraints in a non-commutative space-time . We examine also the paraquantum extension.

After that, in chapter 5 we have extended the work of Joào Magueijo and Lee Smolin on the bosonic string models with deformed energy momentum relations to the fermionic string with deformed dispersion relations [24]. We have used the square root method on the deformed bosonic string constraints [17] to obtain the deformed dispersion relations of the modified fermionic string theories. These constraints are also redefined to fit the fermionic model by providing the closure of the whole deformed constraints super-algebra. We performed the canonical quantization procedure, the obtained super-Virasoro algebra has energy dependent central charges. In a subset of these models we found that the ordinary fermionic string theory results are still realized; including the tachyonic ground state of the NS sector, and the non-tachyonic spectrum in the R sector, it is also possible to use the GSO projection to get something like a theory with space-time supersymmetry. In the end, we write a general conclusion.

Chapter 1

Special relativity and deformed special relativity

The special theory of relativity (SR) was proposed by Einstein for solving the inconsistency between Maxwell equations and the Galilean relativity. The main postulates of SR are: first, the laws of nature are the same for all observers and second, the speed of light is the maximal velocity in the universe; which is observer independent. In this chapter, we present the basic ideas of special relativity and deformed or doubly special relativity (DSR). We will see a deformed dispersion relation which is obeyed by a DSR model [25, 26, 27, 28], and we will also see the same dispersion relation, which is given by a κ-Minkowski phase space model; which describes non-commutative space-time model.

1.1 Galilean reference frame events and Poincaré group

The special relativity interests with the Galilean or inertial reference frames, which are at rest or moving with constant linear velocity. The point in space-time is known as an event. We take $c = 1$ as the speed of light. We can write

$$(x^\mu) \equiv (x^0, x^i) \tag{1.1}$$

where $d = 1, \ldots, D-1$. An observer through space-time sweeps a path called the world-line. The time in the observer's reference frame is known as the proper time of the observer, and can be obtained by

$$-d\tau^2 = ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \tag{1.2}$$

where $\eta_{\mu\nu} \equiv diag(-1, 1, \ldots, 1)$ is the Minkowski metric, and satisfies $\eta^{\mu\rho}\eta_{\rho\nu} = \delta^\mu_\nu$.

1.1.1 Lorentz group

The principles of special relativity are preserved by the Lorentz transformations; which are linear on the spacetime and keep the proper time interval (1.2) invariant. One can write these transformations in the space-time coordinates as follows

$$x^\mu \to x^{\mu\prime} = \Lambda^\mu{}_\nu x^\nu \tag{1.3}$$

or in the momentum space as

$$p^\mu \to p^{\mu\prime} = \Lambda^\mu{}_\nu p^\nu \tag{1.4}$$

while the space-time metric $\eta_{\mu\nu}$ satisfies

$$\eta_{\mu\nu} = \Lambda^\rho{}_\mu \eta_{\rho\sigma} \Lambda^\sigma{}_\nu \equiv \eta = \Lambda^t \eta \Lambda. \tag{1.5}$$

We can also get the infinitesimal transformation:

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \tag{1.6}$$

where $\omega^{\mu\nu}$ are parameters of infinitesimal transformation in the neighborhood of the identity. The substitution of (1.6) in (1.5), leads to

$$\omega^{\mu\nu} = -\omega^{\nu\mu} \tag{1.7}$$

The kinetic momentum is defined by

$$L_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu \tag{1.8}$$

Straightforward, one can obtain

$$\{L_{\mu\nu}, x^\rho\} = \delta^\rho_\nu x_\mu - \delta^\rho_\mu x_\nu \tag{1.9}$$

$$\{L_{\mu\nu}, x^\rho\} = \delta^\rho_\nu x_\mu - \delta^\rho_\mu x_\nu \tag{1.10}$$

$$\{p^\mu, x_\nu\} = \delta^\mu_\nu. \tag{1.11}$$

In four space-time dimensions, we can write:

$$(x^0, x^1, x^2, x^3) = (t, x, y, z) \tag{1.12}$$

$$(p^0, p^1, p^2, p^3) = (E, p_x, p_y, p_z) \tag{1.13}$$

The Lorentz transformations (where the movement is along the x direction) can be given by

$$\begin{cases} t' = \gamma(t - vx) \\ x' = \gamma(x - vt) \\ y' = y \\ z' = z \end{cases} \tag{1.14}$$

while for the momentum we have

$$\begin{cases} E' = \gamma(E - vp_x) \\ p'_x = \gamma(p_x - vE) \\ p'_y = p_y \\ p'_z = p_z \end{cases} \tag{1.15}$$

where v is the relative velocity of the initial reference frame. We use the following definitions

$$\tanh(\phi) = v \tag{1.16}$$

$$\cosh(\phi) = \frac{1}{\sqrt{1-\tanh^2(\phi)}} = \frac{1}{\sqrt{1-v^2}} = \gamma \tag{1.17}$$

then, we can rewrite (1.14) as follows

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \tag{1.18}$$

We have also the Lorentz algebra:

$$\{x^\mu, L^{\nu\rho}\} = \eta^{\nu\mu}x^\rho - \eta^{\rho\mu}p^\nu \tag{1.19}$$

$$\{p^\mu, L^{\nu\rho}\} = \eta^{\nu\mu}p^\rho - \eta^{\rho\mu}x^\nu \tag{1.20}$$

$$\{L^{\mu\nu}, L^{\rho\sigma}\} = \eta^{\nu\rho}L^{\mu\sigma} - \eta^{\mu\rho}L^{\nu\sigma} - \eta^{\nu\sigma}L^{\mu\rho} + \eta^{\mu\sigma}L^{\nu\rho} \tag{1.21}$$

In terms of commutators the Lorentz algebra reads

$$[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho}) \tag{1.22}$$

In this case where $D = 4$, we can distinguish 3 rotations and 3 boosts generators; which are

$$J^i = \epsilon^{ijk}L_{jk} \tag{1.23}$$

and

$$K_i \equiv L_{0i} \tag{1.24}$$

respectively, and they satisfy

$$[J^i, K^j] = \epsilon^{ijk}K_k \tag{1.25}$$

$$[K^i, K^j] = \epsilon^{ijk}J_k \tag{1.26}$$

$$[J^i, J^j] = \epsilon^{ijk}J_k. \tag{1.27}$$

Notice that, the relation (1.27) is the rotation algebra, which is generated by the subgroup $SO(3)$ of the homogeneous Lorentz group $SO(3,1)$.

1.1.2 Poincaré group

The Poincaré group also called the inhomogeneous Lorentz group, noted by $ISO(D-1,1)$. In $D=4$, this group has 10 parameters: rotations, boosts, parities, time reversions, and translations. The group $ISO(D-1,1)$ preserves the invariance of the distance between any two points in Minkowski space. The transformations preserve the elementary length in the space-time (1.2), satisfy (1.5) and are encoded in the following expression

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu} \tag{1.28}$$

where a^{μ} is a parameter of space-time translation. The generator of translation in space-time is the momentum p^{μ}.
The relations (1.22),

$$[p_{\mu}, p_{\nu}] = 0, \tag{1.29}$$

and

$$[L_{\mu\nu}, p_{\rho}] = i(\eta_{\mu\rho} p_{\nu} - \eta_{\nu\rho} p_{\mu}) \tag{1.30}$$

are called Poincaré algebra.

The Poincaré group has two invariant Casimir operators, in 4 Dimensions we can write them as follows

$$p_{\mu} p^{\mu} = m^2 \tag{1.31}$$

and

$$W_{\lambda} = \frac{1}{2} \epsilon_{\lambda\mu\nu\rho} p^{\mu} L^{\nu\rho} \tag{1.32}$$

These two operators are used in particle physics for classifying the particles. Where (1.31) and (1.32) are connected with the mass and the spin of the particle respectively.

1.2 Conformal group

The conformal group in Minkowski space is the group which preserves the light cones, this means that the light like trajectories [29, 30, 31, 32]:

$$ds^2 = \eta_{\mu\nu} dx^{\mu} dx^{\nu} = 0 \tag{1.33}$$

are invariants; so that the conformal map can change the lengths, but must keep the angles unchangeable.
In addition to the Poincaré transformations, the conformal ones, include:
(i) The transformation of dilation:

$$x^{\mu\prime} = \lambda x^{\mu} \tag{1.34}$$

where λ is real.
(ii) The spatial conformal transformations:
such transformations start with inversion element

$$x^{\mu} \to \frac{x^{\mu}}{x^2} \tag{1.35}$$

so

$$dx_\mu dx^\mu \to \frac{dx_\mu dx^\mu}{(x^2)^2} \tag{1.36}$$

after that, one makes translation ($x^\mu \to x^\mu + a^\mu$) and a conjugate inversion, this leads to the spatial conformal transformations

$$x^\mu \to \frac{x^\mu + a^\mu a^2}{1 + 2a^\mu x_\mu + a^2 x^2} \tag{1.37}$$

The conformal group reads the following relations:

$$[p_\mu, p_\nu] = 0 \tag{1.38}$$

$$[D, p_\mu] = -p_\mu \tag{1.39}$$

$$[D, k_\mu] = k_\mu \tag{1.40}$$

$$[D, L_{\mu\nu}] = 0 \tag{1.41}$$

$$[k_\mu, p_\nu] = 2(-\eta_{\mu\nu} D + L_{\mu\nu}) \tag{1.42}$$

$$[k_\rho, L_{\mu\nu}] = -(-\eta_{\rho\mu} k_\nu + \eta_{\rho\nu} k_\mu) \tag{1.43}$$

$$[p_\rho, L_{\mu\nu}] = -(-\eta_{\rho\mu} p_\nu + \eta_{\rho\nu} p_\mu) \tag{1.44}$$

$$[L_{\mu\nu}, L_{\rho\sigma}] = -(\eta_{\nu\rho} L_{\mu\sigma} + \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho}) \tag{1.45}$$

Where the generators in the position space are

$$p_\mu = \partial_\mu \tag{1.46}$$

$$L_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu \tag{1.47}$$

while

$$D = x^\mu \partial_\mu \tag{1.48}$$

and

$$k_\mu = (\eta_{\mu\nu} x^2 - 2x_\mu x_\nu)\partial^\nu \tag{1.49}$$

are the dilation and the spatial conformal transformations generators respectively.

1.3 Deformed special relativity

The deformed special relativity introduces a minimum invariant length, which is the same for all observers. This implies:
(i) The covariance principle in the inertial frames.
(ii) The Planck energy L_p^{-1} is the same for all inertial frame.
(iii) The speed of light in the vacuum is a universal constant, represents the maximum velocity in the universe.

1.3.1 Non-linear action on the momentum space and the deformed relativistic energy momentum relations (The dispersion relations)

The usual Lorentz generators in the momentum space [16, 33] can be given by

$$L_{\mu\nu} = p_\mu \frac{\partial}{\partial p^\nu} - p_\nu \frac{\partial}{\partial p^\mu} \tag{1.50}$$

while the generator of dilation is

$$D = p_\mu \frac{\partial}{\partial p^\mu} \tag{1.51}$$

with the introduction of the operator [16]

$$U(p_0) \equiv e^{L_P p_0 D} \tag{1.52}$$

where $L_P = E_p^{-1}$, and the use of Champbell-Hausdorff formula

$$e^{-A} B e^{A} = B + [B, A] + \frac{1}{2!}\Big[[B, A], A\Big] + \frac{1}{3!}\Big[[[B, A], A], A\Big] + \dots \tag{1.53}$$

one can take the following deformed boost generator,

$$\begin{aligned} K^i \equiv e^{-L_P p_0 D} L_0^i e^{L_P p_0 D} &= L_0^i + [L_0^i, L_P p_0 D] \\ &= L_0^i + L_P p^i D \end{aligned} \tag{1.54}$$

where we use the relation (1.44).

Let us take $B = p^\mu$ and $A = L_p p^i D$, with the same method, one can see that

$$e^{-L_P p_0 D} p_\mu e^{L_P p_0 D} = \frac{p_\mu}{1 - L_P p_0} \tag{1.55}$$

this happens when we insert $A = L_P p_0 D$ and $B = p_\mu$ in (1.53), so we have

$$[B, A] = L_P p_0 p_\mu \tag{1.56}$$

$$\Big[[B, A], A\Big] = 2L_P^2 p_0^2 p_\mu \tag{1.57}$$

$$\Big[\Big[[B, A], A\Big], A\Big] = 6L_P^3 p_0^3 p^\mu \tag{1.58}$$

and with the relation

$$(1 - x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \dots \tag{1.59}$$

we get (1.55).

In the same way, the non-linear action of the Lorentz group on the momentum space can lead to the following expression

$$||p||^2 = \frac{\eta^{\mu\nu} p_\mu p_\nu}{(1 - L_p p_0)^2} \tag{1.60}$$

which is not unitary equivalence, in general. The most general invariant quantity is

$$||p||^2 \equiv \eta^{\mu\nu} U(p_\mu) U(p_\nu). \tag{1.61}$$

Notice that the map

$$\tilde{E} = \frac{E}{1 - L_P E} \tag{1.62}$$

and

$$\tilde{p}_i = \frac{p_i}{1 - L_P E} \tag{1.63}$$

or

$$E = \frac{\tilde{E}}{1 + L_P \tilde{E}} \tag{1.64}$$

and

$$p_i = \frac{\tilde{p}_i}{1 + L_P \tilde{E}} \tag{1.65}$$

lead to the usual dispersion relation

$$\tilde{E}^2 - \tilde{p}^2 = m^2 \tag{1.66}$$

The energy momentum relation encodes the experimental results in different energy scales, then one can calculate U from these modifications of the dispersion relation. Generally, one can write the deformed dispersion relation as follows [34]

$$E^2 f_1^2(E, L_P) - p^2 f_2^2(E, L_P) = m^2. \tag{1.67}$$

The U-map can act as follows

$$U \circ (E, \mathbf{p}) = (E f_1, \mathbf{p} f_2) \tag{1.68}$$

Notice that the map U is not always invertible, but the equation (1.55) needs an invertible U-map, then one can define the physical space as the subspace of the energy momentum space where U is invertible.

1.3.2 The κ-Minkowski space-time

In the covariant form, the classical κ-Minkowski phase space is defined by the following deformed Poisson brackets

$$\{x^\mu, x^\nu\} = \frac{1}{\kappa}(x^\mu \eta^{\nu 0} - x^\nu \eta^{\mu 0}) \tag{1.69}$$

$$\{x^\mu, p^\nu\} = -\eta^{\mu\nu} + \frac{1}{\kappa}\eta^{\mu 0} p^\nu \tag{1.70}$$

$$\{p^\mu, p^\nu\} = 0 \tag{1.71}$$

which encode the non-commutativity of space time, where $\kappa = L_P^{-1}$ can be characterized by the Planck length. So if $E_p >> p^0$, then the phase space became as usual.
We can calculate the following Poisson brackets

$$\{x^\mu, L^{\nu\rho}\} = (\eta^{\mu\nu}x^\rho - \eta^{\mu\rho}x^\nu) + \frac{x^\mu}{\kappa}(\eta^{\nu 0}p^\rho - \eta^{\rho 0}p^\nu) \tag{1.72}$$

$$\{p^\mu, L^{\nu\rho}\} = (\eta^{\nu\mu}p^\rho - \eta^{\rho\mu}p^\nu) + \frac{p^\mu}{\kappa}(\eta^{\rho 0}p^\nu - \eta^{\nu 0}p^\rho) \tag{1.73}$$

$$\{L^{\mu\nu}, L^{\rho\sigma}\} = \eta^{\nu\rho}L^{\mu\sigma} - \eta^{\mu\rho}L^{\nu\sigma} - \eta^{\nu\sigma}L^{\mu\rho} + \eta^{\mu\sigma}L^{\nu\rho} \tag{1.74}$$

The equations (1.72), (1.73), and (1.74) represent the deformed Lorentz algebra. The κ-Lorentz deformed transformations in $D = 4$ read; for the space-time coordinates, as

$$\begin{cases} t' = \alpha_\kappa \gamma(t - vx) \\ x' = \alpha_\kappa \gamma(x - vt) \\ y' = \alpha_\kappa y \\ z' = \alpha_\kappa z \end{cases} \tag{1.75}$$

and for the energy and momentum as:

$$\begin{cases} E' = \frac{1}{\alpha_\kappa}\gamma(E - vp_x) \\ p'_x = \frac{1}{\alpha_\kappa}\gamma(p_x - vE) \\ p'_y = \frac{p_y}{\alpha_\kappa} \\ p'_z = \frac{p_z}{\alpha_\kappa} \end{cases} \tag{1.76}$$

where

$$\alpha_\kappa \equiv 1 + \frac{1}{\kappa}\big((\gamma - 1)E - \gamma v p_x\big) \tag{1.77}$$

The Planck energy $E_p = \kappa$ is invariant in respect of the above transformations. Notice that [25]

$$\begin{aligned} E' &= \frac{\gamma(\kappa - vp_x)}{1 + \frac{1}{\kappa}\big((\gamma - 1)\kappa - \gamma v p_x\big)} \\ &= \frac{\gamma(\kappa - vp_x)}{\frac{1}{\kappa}(\gamma(\kappa - vp_x))} \\ &= \kappa \end{aligned} \tag{1.78}$$

which reflects the invariance of E_p.

Relativistic invariants:

First, one can use (1.76) and (1.77) to write [25]

$$1 - \frac{p'_t}{\kappa} = \frac{1}{\alpha_\kappa}(1 - \frac{p_t}{\kappa}) \tag{1.79}$$

which implies that

$$\alpha_\kappa = \frac{1 - \frac{p_t}{\kappa}}{1 - \frac{p'_t}{\kappa}} \tag{1.80}$$

In order to use (1.76) and the metric $\eta_{\mu\nu}$, we can write

$$\eta_{\mu\nu} p'^{\mu} p'^{\nu} = \frac{1}{\alpha_\kappa^2} \eta_{\mu\nu} p^{\mu} p^{\nu} \tag{1.81}$$

and in the same way, we can find that

$$\eta_{\mu\nu} x'^{\mu} x'^{\nu} = \alpha_\kappa^2 \eta_{\mu\nu} x^{\mu} x^{\nu} \tag{1.82}$$

Note that one can define the following two invariant quantities

$$(1 - \frac{p'_t}{\kappa})^{-2} p'_\mu p'^\mu = (1 - \frac{p_t}{\kappa})^{-2} p_\mu p^\mu = I_p \tag{1.83}$$

and

$$(1 - \frac{p'_t}{\kappa})^{2} x'_\mu x'^\mu = (1 - \frac{p_t}{\kappa})^{2} x_\mu x^\mu = I_x \tag{1.84}$$

Let us define [25]

$$I_p \equiv M^2 \tag{1.85}$$

then we can write

$$E^2 - p^2 = (1 - \frac{E}{E_p})^2 M^2 \tag{1.86}$$

and

$$(E_p^2 - M^2)E^2 + 2M^2 E_p E - E_p^2(p^2 + M^2) = 0 \tag{1.87}$$

so

$$E^2 - p^2 = m_0^2 (1 - \frac{E}{E_p})^2 \tag{1.88}$$

where $m_0 = M$ is the rest mass. Notice that the above equation represents a deformed relativistic relation of energy and mass, which is identical with (1.60). In the other hand, this relation keeps the speed of light; where $m_0^2 = 0$, energy independent.

1.4 Conclusion

In this chapter we have given elements of special relativity and its extension to deformed special relativity. We did this deformation by modifying the Lorentz boosts and transformations. A map which depends on dilation generator is used. Then we derived the deformed dispersion relation, after that we arrived to the same results by using a κ-Minkowski space which is a non-commutative space-time. In the next chapter, we will give a classical description of the bosonic and fermionic strings by writing the actions and Hamiltonians.

Chapter 2

Classical bosonic and fermionic string theories

This chapter is a short introduction to the classical bosonic and fermionic string theories.

2.1 Relativistic particle

The free relativistic particle is useful to understand some interesting arguments concerning the description of the bosonic string theory. The particle propagates in space-time along a world-line, where the relativistic action is proportional to the length of, this world-line, in D space-time dimensions [35, 36]:

$$S = -m \int d\tau \sqrt{-\frac{dX^{\mu}}{d\tau}\frac{dX_{\mu}}{d\tau}} \tag{2.1}$$

where m is the mass of particle, and τ is a parameter on the world-line. The action (2.1) is invariant under the reparametrization: $\tau \to \tilde{\tau}(\tau)$, and Lorentz transformation: $X^{\mu} \to \Lambda^{\mu}_{\nu} X^{\nu}$ where $\Lambda^{T}\eta\Lambda = \eta$.
The canonical momentum of X^{μ} is given by

$$p_{\mu} = \frac{\delta \mathcal{L}}{\delta \dot{X}^{\mu}} = m\frac{\dot{X}_{\mu}}{\sqrt{-\dot{X}^{2}}} \tag{2.2}$$

and satisfies the mass shell condition

$$\phi \equiv p_{\mu}p^{\mu} + m^{2} = 0 \tag{2.3}$$

The above equation represents a primary constraints on this system, because it takes its definition without using equations of motion, and as

$$\{\phi_{a}, \phi_{b}\}_{PB} = 0 \tag{2.4}$$

is a closed algebra, ϕ is also a first class constraint.

In the other hand, there is an equivalent quadratic form of the action (2.1) with the expense of introducing an auxiliary field on the world-line $e(\tau)$, and the advantage to describe also the massive particles, so

$$S = \frac{1}{2}\int_{\tau_0}^{\tau_1}\left(\frac{1}{e}\frac{dX^\mu}{d\tau}\frac{dX_\mu}{d\tau} - em^2\right) \tag{2.5}$$

The equations of motion inspired by (2.5) are:

$$\frac{\delta S}{\delta X^\mu} = \frac{d}{d\tau}\left(\frac{1}{e}\dot{X}^\mu\right) = 0 \tag{2.6}$$

$$\frac{\delta S}{\delta e} = \dot{X}^2 + m^2e^2 = 0 \tag{2.7}$$

then

$$e^2 = -\frac{1}{m^2}\dot{X}^2 \tag{2.8}$$

which gives

$$\frac{d}{d\tau}\left(\frac{m}{\sqrt{-\dot{X}^\rho\dot{X}_\rho}}\dot{X}^\mu\right) = 0 \tag{2.9}$$

and leads to the action (2.1). In this form the constraint (2.3) became secondary.

2.2 Nambu-Goto action

The string theory studies a one dimension extended object, which sweeps out a two dimensional surface in D-dimensional Minkowski space-time, called *the world-sheet*. The bosonic string action is proportional to the area of this world-sheet,

$$S = -T\int_0^l d\sigma^2\sqrt{-det\gamma} \tag{2.10}$$

where T is the string tension, $l = \pi$ (2π) for the open (closed) string. The relations between the characteristic length of the string l_s, the Regge slope parameter α', and the tension of the string T are

$$T = \frac{1}{2\pi\alpha'}, \quad l_s^2 = 2\alpha'. \tag{2.11}$$

While

$$\gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}\eta_{\mu\nu} \tag{2.12}$$

is the induced metric; and represents the pull-back of the flat metric on the space-time. The Nambu-Goto action satisfies two symmetries:

- The Poincaré symmetry is the combination of Lorentz symmetry and the symmetry under translation.
- The reparametrization symmetry reflects the redundancy in the description of this system.

2.3 Polyakov action

The square-root in (2.10) makes the path-integral quantization very difficult, so we need an equivalent quadratic form of this action, but it introduces a new field $g^{\alpha\beta}$

$$S = -\frac{T}{2}\int_0^l d^2\sigma\sqrt{-g}g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu} \tag{2.13}$$

We can see that the Nambu-Goto action (2.10) and the Polyakov action (2.13) coincide under some conditions. The Polyakov action is D scalar fields in two-dimensional gravity. The field X^0 has a negative sign, which is incompatible with the Poincaré symmetry $SO(D)$.
The variation of the action (2.13) plus the cosmological constant with respect to the metric $g_{\alpha\beta}$ is

$$\frac{\delta S}{\delta g_{\alpha\beta}} = \frac{1}{2}\sqrt{-g}T^{\alpha\beta} + \frac{\Lambda}{2}\sqrt{-g}g^{\alpha\beta} = 0 \tag{2.14}$$

where Λ is the cosmological constant, and the energy-momentum tensor is given by,

$$T_{\alpha\beta} = -T(\frac{1}{2}g_{\alpha\beta}g^{\rho\sigma}\partial_\rho X^\mu\partial_\sigma X_\mu - \partial_\alpha X^\mu\partial_\beta X_\mu) \tag{2.15}$$

The Weyl invariance implies that $\Lambda = 0$, so

$$T_{\alpha\beta} = 0 \tag{2.16}$$

In flat world-sheet Eq. (2.16) satisfies

$$Tr(T_{\alpha\beta}) = 0 \tag{2.17}$$

$$\partial^\alpha T_{\alpha\beta} = 0 \tag{2.18}$$

which leads to,

$$g_{\alpha\beta} = k\partial_\alpha X^\mu\partial_\beta X_\mu \tag{2.19}$$

2.4 p-brane action

The p-brane is a p-dimensional extended object, its motion in space-time is described by the action

$$S_p = T_p\int d\mu_p \tag{2.20}$$

where T_p is the p-brane tension and

$$d\mu_p = \sqrt{-\det G_{\alpha\beta}}d^{p+1}\sigma \tag{2.21}$$

the induced metric is

$$G_{\alpha\beta} = \partial_\alpha X^\mu\partial_\beta X_\mu \tag{2.22}$$

so in the Nambu-Goto form the action became

$$S_{NG} = m^{2-D} \int d^D\sigma \sqrt{\det(\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu})} \tag{2.23}$$

while in the Howe-Tucker form can be written as follows

$$S_{HT} = -\frac{1}{2} \int d^D\sigma \sqrt{-h}\big(h^{ij}\partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu} + m^2(D-2)\big) \tag{2.24}$$

The above action can be seen as a D scalar field in p dimensional gravity, plus a cosmological constant $\Lambda_p = m^2(D-2)$.

2.5 Hamiltonian formalism

The canonical form of the bosonic string action can be written as [9]

$$S = \int_0^\pi d\sigma d\tau (\dot{X}^\mu \mathcal{P}_\mu - N\mathcal{H}^{bosonic} - N^\sigma \mathcal{H}_\sigma^{bosonic}) \tag{2.25}$$

where, N and N^σ are the lapse and shift functions respectively, which play the role of Lagrange multipliers. $\mathcal{H}^{bosonic}$ and $\mathcal{H}_\sigma^{bosonic}$ are the Hamiltonian and the spatial diffeomorphism constraints respectively, while $\mathcal{P}^\mu$ is the canonical conjugate of X^μ. Let us start this study with the bosonic string constraints:

$$\begin{cases} \mathcal{H}^{bosonic} = \frac{1}{2T}\mathcal{P}_\mu \mathcal{P}^\mu + \frac{T}{2} X'_\mu X'^\mu \\ \mathcal{H}_\sigma^{bosonic} = \mathcal{P}^\mu X'_\mu \end{cases} \tag{2.26}$$

So the classical Virasoro algebra is

$$[\mathcal{H}^{bosonic}(\sigma), \mathcal{H}^{bosonic}(\sigma')]_{PB} = (\mathcal{H}_\sigma^{bosonic}(\sigma) + \mathcal{H}_\sigma^{bosonic}(\sigma'))\delta'(\sigma - \sigma') \tag{2.27}$$

$$[\mathcal{H}^{bosonic}(\sigma), \mathcal{H}_\sigma^{bosonic}(\sigma')]_{PB} = (\mathcal{H}^{bosonic}(\sigma) + \mathcal{H}^{bosonic}(\sigma'))\delta'(\sigma - \sigma') \tag{2.28}$$

$$[\mathcal{H}_\sigma^{bosonic}(\sigma), \mathcal{H}_\sigma^{bosonic}(\sigma')]_{PB} = (\mathcal{H}_\sigma^{bosonic}(\sigma) + \mathcal{H}_\sigma^{bosonic}(\sigma'))\delta'(\sigma - \sigma') \tag{2.29}$$

the constraint $\mathcal{H}_\sigma^{bosonic}$ generates the reparametrization of σ, the physics is invariant under the coordinate transformations

$$\sigma \to \sigma' = f(\sigma) \tag{2.30}$$

then

$$[X^\mu(\sigma), \int d\sigma' \zeta^1(\sigma')\mathcal{H}_\sigma^{bosonic}]_{PB} = \zeta^1(\sigma)X'^\mu(\sigma) = L_\zeta X^\mu \tag{2.31}$$

and

$$[\mathcal{P}^\mu(\sigma), \int d\sigma' \zeta^1(\sigma')\mathcal{H}_\sigma^{bosonic}]_{PB} = (\zeta^1(\sigma)\mathcal{P}^\mu(\sigma))' = L_\zeta \mathcal{P}^\mu \tag{2.32}$$

where L_ζ represents Lie derivative.

If one use the following definitions

$$Q^+(\sigma) = 2\pi(\mathcal{H}^{bosonic}(\sigma) + \mathcal{H}^{bosonic}_\sigma(\sigma)) \tag{2.33}$$

$$Q^-(\sigma) = 2\pi(\mathcal{H}^{bosonic}(\sigma) - \mathcal{H}^{bosonic}_\sigma(\sigma)) \tag{2.34}$$

then we find that

$$[Q^+(\sigma), Q^+(\sigma')]_{PB} = 4\pi(Q^+(\sigma) + Q^+(\sigma')\delta'(\sigma - \sigma') \tag{2.35}$$

$$[Q^-(\sigma), Q^-(\sigma')]_{PB} = -4\pi(Q^-(\sigma) + Q^-(\sigma')\delta'(\sigma - \sigma') \tag{2.36}$$

$$[Q^-(\sigma), Q^+(\sigma')]_{PB} = 0 \tag{2.37}$$

which represents the classical Virasoro closed algebra.

2.6 Supergravity in one dimension (the spining particle)

The study of the spining particle is a simple way to understand many aspects of the fermionic string computations,

2.6.1 Hamiltonian method

P.A.M. Dirac has introduced his famous equation after some square root of the Klein-Gorden one, so let us take advantage of the results of the relativistic point particle as a system of constraints, for constructing the fermionic constraint as the square root of the bosonic particle one [21, 37, 38]. The constraints in this system are:

$$S = \theta_\mu P^\mu + \theta_5 m \approx 0 \tag{2.38}$$

and

$$\mathcal{H} = SS = P_\mu P^\mu + m^2 \approx 0 \tag{2.39}$$

where $\theta_\mu = \frac{i}{\sqrt{2}}\gamma_5\gamma_\mu$ and $\theta_5 = \frac{1}{\sqrt{2}}\gamma_5$. The symbol $\approx$ denotes the Dirac weak equality [39], which suggests that, the expression is not vanishing when we compute the Poisson brackets. The algebra of S and $\mathcal{H}$ is closed in the sense that:

$$\{\mathcal{H}, \mathcal{H}\}_- = 0 \tag{2.40}$$

$$\{S, S\}_+ = -\mathcal{H} \tag{2.41}$$

$$\{S, \mathcal{H}\}_- = 0 \tag{2.42}$$

The canonical quantization Dirac algorithm is:

$$\{,\} \longrightarrow -i[,]$$

where any physical state $|\psi\rangle_{ph}$ must be annihilated by the constraints:

$$\mathcal{H}|\psi\rangle_{ph} = 0 \tag{2.43}$$

$$S|\psi\rangle_{ph} = 0 \tag{2.44}$$

one can notice that (2.43) and (2.44) give the Klein-Gordon and Dirac equations respectively. The mass-shell condition (2.43) results from the reparametrization invariance, while the condition (2.44) generates the locale supersymmetry transformations [1]:

$$\begin{aligned} \delta X^\mu &= [X^\mu, \epsilon(\theta_\mu P^\mu + \theta_5 m)] \\ &= \epsilon\theta^\mu \end{aligned} \tag{2.45}$$

where $\epsilon(\tau)$ is an odd Grassmann function with τ is a parameter in the world line. The same procedure leads to,

$$\delta P^\mu = 0 \tag{2.46}$$

$$\delta\theta_\mu = -i\epsilon P_\mu \tag{2.47}$$

$$\delta\theta_5 = -i\epsilon m \tag{2.48}$$

where P^μ is the canonical conjugate of $X^\mu(\tau)$, which is a scalar fields in the world line and characterizes the coordinates in the space-time, and m is the mass of the particle.

2.7 Fermionic string as a 2-D supergravity

Let us try to understand the two dimensional local supersymmetry, which is needed to construct the supersymmetry in the world-sheet, this method gives fermionic degrees of freedom to our original bosonic string action *i.e. the Polyakov action*, then the resulting theory is called, the Ramond-Neveu-Schwarz (RNS) fermionic string.

Without using the equations of motion i.e. off-shell, the matter multiplet is

$$(X^\mu,\ \psi^\mu,\ F^\mu) \tag{2.49}$$

where X^μ is a bosonic field in the world sheet, ψ^μ is a real fermionic field in the world sheet, and F^μ is a auxiliary real scalar field, which allows the equality between the fermionic and the bosonic degrees of freedom. The supergravity in two dimensions has the following multiplet:

$$(e^a_\alpha, \chi_\alpha, A) \tag{2.50}$$

where χ_α represents the gravitino in two dimensions, A is an auxiliary field which preserves the equality between the fermionic and the bosonic degrees of freedom in the off-shell world sheet action. We also define

$$e \equiv |det e^a_\alpha| = \sqrt{h} \tag{2.51}$$

[1]let us use $[A, B] = [A, B]_- + (-1)^{\epsilon_A}[A, B]_+$ with $\epsilon_A = 1(0)$ for A odd (even) Grassmann variable.

The local supersymmetric action action is

$$S = -\frac{1}{8\pi}\int d^2\sigma\left(\frac{2}{\alpha'}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu + 2i\bar{\psi}^\mu\rho^\alpha\partial_\alpha\psi_\mu - i\bar{\chi}_\alpha\rho^\beta\rho^\alpha\psi^\mu\left(\sqrt{\frac{2}{\alpha'}}\partial_\beta X_\mu - \frac{i}{4}\bar{\chi}_\beta\psi_\mu\right)\right) \tag{2.52}$$

where $h_{\alpha\beta} = e^a_\alpha e^b_\beta \eta_{ab}$, and ρ^α are 2×2 Dirac matrices with

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{2.53}$$

which satisfy the 2-D Clifford algebra, with the following anti-commuting relation

$$[\rho^\alpha, \rho^\beta]_+ = 2\eta^{\alpha\beta} \tag{2.54}$$

the world-sheet spinor ψ^μ is the super-partner of X^μ, and gravitino χ_α is the super-partner of the zwei-bein e^a_α (for more details, one can see Appendix A for a general formulation in D dimensions).
The action (2.52) satisfies five symmetries: the diffeomorphisms, the local Lorentz symmetry, the local supersymmetry in the world sheet, Weyl invariance, and the super-Weyl invariance.

The Lorentz transformations, the diffeomorphism, and the supersymmetry can help us to simplify the action (2.52) by fixing the gauge. Like the conformal gauge in the bosonic string theory, we can use the super-conformal gauge in the fermionic string theory (for more details see chapter 7 in [22]):

$$e^a_\alpha = e^\phi \delta^a_\alpha \tag{2.55}$$

$$\chi_\alpha = \rho_\alpha \lambda \tag{2.56}$$

where ϕ and λ are parameters, and characterize Weyl rescaling and super-Weyl transformation respectively. So we can use the gauge choice where

$$e^a_\alpha = \delta^a_\alpha \tag{2.57}$$

and

$$\chi_\alpha = 0 \tag{2.58}$$

to simplify the action (2.52).

2.8 RNS action as supersymmetric free field action

With the use of the super conformal gauge (2.57) and (2.58) in (2.52), we can read the RNS action as

$$S = \frac{T}{2}\int d\tau d\sigma\left(\partial_\alpha X^\mu\partial^\alpha X_\mu - i\bar{\psi}^\mu\rho^\alpha\partial_\alpha\psi_\mu\right) \tag{2.59}$$

where $\psi^\mu = \begin{pmatrix} \psi^\mu_+ \\ \psi^\mu_- \end{pmatrix}$ is a two dimensional Majorana spinor in the world-sheet.

We consider the infinitesimal Majorana spinor

$$\epsilon^\mu = \begin{pmatrix} \epsilon^\mu_+ \\ \epsilon^\mu_- \end{pmatrix} \tag{2.60}$$

The action (2.59) has the following supersymmetric transformations on the world-sheet

$$\delta X^\mu = \bar{\epsilon}\psi^\mu \tag{2.61}$$

$$\delta\psi^\mu = -i\rho^\alpha \partial_\alpha X^\mu \epsilon \tag{2.62}$$

These transformations are supersymmetric because they mix the bosonic and world-sheet field variables.
The classical equations of motion are obtained from the variation of the action (2.59), which leads to

$$\Box X^\mu = 0 \tag{2.63}$$

and

$$\partial\!\!\!/ \psi^\mu = 0. \tag{2.64}$$

with $\Box \equiv \partial_\alpha\partial^\alpha$, and $\partial\!\!\!/ \equiv \rho^\alpha\partial_\alpha$, and also we define

$$\sigma^\pm = \tau \pm \sigma \tag{2.65}$$

then we write

$$\partial_\tau = \partial_+ + \partial_-. \tag{2.66}$$

So we can read the equation (2.64) as follows

$$\partial_\pm \psi_\mp = 0 \tag{2.67}$$

Straightforward one can rewrite the action (2.59):

$$S = T \int d\tau\sigma(2\partial_+ X \cdot \partial_- X + i\psi_- \cdot \partial_+\psi_- + i\psi_+ \cdot \partial_-\psi_+) \tag{2.68}$$

The equations of motion here are obviously (2.63) and (2.67).

2.8.1 Conserved supercurrents and super-energy-momentum tensor

In order to examen the supercurrents, we start with the action (2.59), which varies under local supersymmetry, as follows

$$\delta S \sim \int d\tau d\sigma \partial_\alpha \bar{\epsilon} J^\alpha \tag{2.69}$$

then the conserved supercurrents are

$$J_\alpha^\mu = \frac{1}{2}\rho^\beta\rho_\alpha\psi \cdot \partial_\beta X \tag{2.70}$$

or

$$J_+ = \psi_+ \cdot \partial_+ X = 0 \tag{2.71}$$

$$J_- = \psi_- \cdot \partial_- X = 0 \tag{2.72}$$

Let us now turn to the energy-momentum tensor; which is the conserved current of the translation in the world-sheet

$$T_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X + \frac{1}{4}\bar{\psi} \cdot \rho_\alpha \partial_\beta \psi + \frac{1}{4}\bar{\psi} \cdot \rho_\beta \partial_\alpha \psi - (trac) \tag{2.73}$$

then

$$T_{++} = \partial_+ X \cdot \partial_+ X + \frac{i}{2}\psi_+ \partial_+ \cdot \psi_+ \tag{2.74}$$

$$T_{--} = \partial_- X \cdot \partial_- X + \frac{i}{2}\psi_- \partial_- \cdot \psi_- \tag{2.75}$$

$$T_{+-} = T_{-+} = 0 \tag{2.76}$$

This energy-momentum tensor has the properties:

$$\partial_- T_{++} = \partial_+ T_{--} = 0 \tag{2.77}$$

2.8.2 Boundary conditions

To define the boundary conditions, we impose the vanishing of the surface term:

$$[\psi_- \cdot \delta\psi_- - \psi_+ \cdot \delta\psi_+]_0^\pi = 0 \tag{2.78}$$

If we chose: $\psi_+(\tau, 0) = \psi_-(\tau, 0)$ at $\sigma = 0$ boundary, we can distinguish two sectors:

$$\psi_+(\tau, \pi) = \psi_-(\tau, \pi) \tag{2.79}$$

for the Ramond sector or

$$\psi_+(\tau, \pi) = -\psi_-(\tau, \pi) \tag{2.80}$$

for the Neveu-Shwarz sector. These boundary conditions give the following solutions of the equations of motion (2.67):

$$\psi_\pm^\mu = \sqrt{2}\sum_{n=-\infty}^{+\infty} d_n^\mu e^{-in(\tau\pm\sigma)} \tag{2.81}$$

and

$$\psi_\pm^\mu = \sqrt{2}\sum_{n=-\infty}^{+\infty} b_r^\mu e^{-ir(\tau\pm\sigma)} \tag{2.82}$$

for the Ramond and Neveu-Shwarz sectors respectively, where n is an integer number and r is a half-integer one.

2.8.3 Super-Virasoro generators

The super-Virasoro conditions are:

$$J_+ = J_- = T_{++} = T_{--} = 0. \tag{2.83}$$

The super-Virasoro generators are the following Fourier modes:

$$L_n = T \int_0^\pi d\sigma (e^{in\sigma} T_{++} + e^{-in\sigma} T_{--}) \tag{2.84}$$

$$F_n = \frac{\sqrt{2}}{\pi} \int_0^\pi d\sigma (e^{in\sigma} J_+ + e^{-in\sigma} J_-) \tag{2.85}$$

for the Ramond sector and

$$G_r = \frac{\sqrt{2}}{\pi} \int_0^\pi d\sigma (e^{ir\sigma} J_+ + e^{-ir\sigma} J_-) \tag{2.86}$$

for the Neveu-Schwarz sector.

2.8.4 Hamiltonian formulation

The conformal algebra is described by direct product of two diffeomorphism groups in one dimension. This property manifest itself in terms of

$$T_{++}^{bosonic}(\sigma) = P_\mu P^\mu = 2\pi(\mathcal{H} + \mathcal{H}_1) \tag{2.87}$$

$$T_{--}^{bosonic}(\sigma) = S_\mu S^\mu = 2\pi(\mathcal{H} - \mathcal{H}_1) \tag{2.88}$$

where, we take

$$P^\mu(\sigma) = \pi\sqrt{2\alpha'}\mathcal{P}^\mu(\sigma) + \frac{1}{\sqrt{2\alpha'}} X^{\mu\prime}(\sigma) \tag{2.89}$$

and

$$S^\mu(\sigma) = \pi\sqrt{2\alpha'}\mathcal{P}^\mu(\sigma) - \frac{1}{\sqrt{2\alpha'}} X^{\mu\prime}(\sigma) \tag{2.90}$$

Let us introduce fermionic degrees of freedom, which are represented by the Majorana spinor in two dimensions $\psi_a^\mu(\sigma)$; with the spinor index $a = 1, 2$, and suppose the constraints $\mathcal{S}_a(\sigma)$, which can generate the transformations in this degrees of freedom. The anti-commuting Poisson brackets in the extended phase space are

$$[\psi_a^\mu(\sigma), \psi_b^\nu(\sigma')]_{PB} = -4\pi i \eta^{\mu\nu} \delta_{ab} \delta(\sigma, \sigma'). \tag{2.91}$$

We can define

$$\mathcal{S}_1(\sigma) = J_+ = \psi_1^\mu(\sigma) P_\mu(\sigma) \tag{2.92}$$

and

$$\mathcal{S}_2(\sigma) = J_- = \psi_2^\mu(\sigma) S_\mu(\sigma) \tag{2.93}$$

then, it is easy to find:

$$[\mathcal{S}_1(\sigma), \mathcal{S}_1(\sigma')]_{PB} = -4\pi i(P_\mu P^\mu + \frac{i}{2}\psi_1^\mu \frac{d\psi_{1\mu}}{d\sigma})\delta(\sigma - \sigma') \tag{2.94}$$

$$[\mathcal{S}_2(\sigma), \mathcal{S}_2(\sigma')]_{PB} = -4\pi i(P_\mu P^\mu - \frac{i}{2}\psi_2^\mu \frac{d\psi_{1\mu}}{d\sigma})\delta(\sigma - \sigma') \tag{2.95}$$

$$[\mathcal{S}_1(\sigma), \mathcal{S}_2(\sigma')]_{PB} = 0 \tag{2.96}$$

for consistency, we use the redefinitions

$$T_{++}^{fermionic}(\sigma) = P_\mu P^\mu + \frac{i}{2}\psi_1^\mu \frac{d\psi_{1\mu}}{d\sigma} \tag{2.97}$$

$$T_{--}^{ferminic}(\sigma) = S_\mu S^\mu - \frac{i}{2}\psi_2^\mu \frac{d\psi_{2\mu}}{d\sigma} \tag{2.98}$$

Notice that, if one takes $\mathcal{P}^\mu = \frac{1}{T}\dot{X}^\mu$, the generators (2.92), (2.93), (2.97), and (2.98) became identical to (2.71), (2.72), (2.74), and (2.75).

The constraint (2.92) and (2.93) can be used to generate the 2-D supergravity in the world-sheet:

$$\begin{aligned} \delta\psi_1^\mu(\sigma) &= [\psi_1^\mu, \int \epsilon^1(\sigma')\psi^\nu P_\nu(\sigma')d\sigma'] \\ &= 4\pi i\epsilon^1(\sigma)P^\mu \end{aligned} \tag{2.99}$$

and with the same method, one can find that

$$\delta\psi_2^\mu(\sigma) = 4\pi\epsilon^2(\sigma)S^\mu(\sigma) \tag{2.100}$$

$$\delta X^\mu(\sigma) = \pi\sqrt{2}\alpha'\epsilon^i(\sigma)\psi_a^\mu(\sigma) \tag{2.101}$$

$$\delta P_\mu = 2\pi(-\epsilon^1\psi_{\mu 1} + \epsilon^2\psi_{\mu 2})' \tag{2.102}$$

which are the same infinitesimal local supersymmetry transformations, satisfied by the action.

2.9 Conclusion

In this chapter we have given a short overview on the theory of classical bosonic and fermionic strings. We used the bosonic and spinning particles as the starting points for understanding the basic ideas behind actions of the (bosonic and fermionic) string theories. We can also write the constraints and Hamiltonians of these theories. In the next Chapter, we will study a modified string theory represented by the open bosonic string with dust field, which may became a non-critical string theory.

Chapter 3

Open bosonic string with dust

The ordinary string theory needs unverified extra-dimensions to became complete and consistent. Jack Gegenberg and Viqar Husain suppose a new scalar field in the word-sheet [3], which breaks conformal invariance, and gives a consistent closed bosonic string model with a space-time dimension $D \leq 26$.

In this chapter we follow the same steps; used in [3], in the context of the open string. The action of this model includes an additional term, which breaks the conformal symmetry at the classical level, so this symmetry is not a precondition for a consistent quantum theory. We mention that the path integral ghost or the BRST method of quantization leads to restricting the space-time dimension ($D = 26$), for preserving the conformal symmetry after the quantization of the bosonic string, therefore, the non-critical string theory is possible, while the idea here is different from the Igor Nikitin non-critical string (the redefinition of the light cone axes), or the Polyakov non-critical string (with a linear dilaton background). After the use of the dust gauge, we find a consistent algebra (with no anomaly term) of the remaining diffeomorphism symmetry. And unlike the closed string with dust where the equality of the anomaly terms of the Virasoro algebras must be used for consistent diffeomorphism algebra, the open string case can eliminate automatically the anomaly of the algebra. In this stage, we also study the bosonic string with dust field between two parallel Dp-branes.

3.1 Classical action

The classical action is written as the sum of two terms [3]:

$$S = -\frac{1}{4\pi\alpha'}\int_{\Sigma} d^2x\sqrt{-g}g^{\alpha\beta}X^{\mu}_{,\alpha}X^{\nu}_{,\beta}\eta_{\mu\nu} - \int_{\Sigma} d^2x\sqrt{-g}\mathcal{M}(g^{\alpha\beta}T_{,\alpha}T_{,\beta}+1) \tag{3.1}$$

where $\Sigma = \mathbb{R} \times [0,\pi]$ for the open string and $\Sigma = \mathbb{R} \times S^1$ for the closed one, and here α, β are the world sheet indices, while μ and ν are the space-time indices. T is a world sheet scalar field i.e. the dust field, and $\mathcal{M}$ is its mass density.
The action of this theory is not conformal invariant; especially the last term in the dust

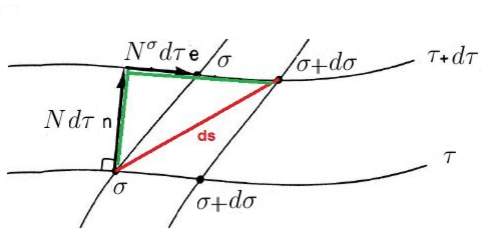

Figure 3.1: An infinitesimal piece of the world sheet showing the relation between the metric $h_{\alpha\beta}$ and the functions N and N^σ.

field action; while the first term is the Polyakov action (2.13). The world sheet metric can be defined by the lapse and shift functions as follows

$$ds^2 = e^{2\rho} h_{\alpha\beta}(N, N^\sigma) dx^\alpha dx^\beta = e^{2\rho}[-N^2 d\tau^2 + (d\sigma + N^\sigma d\tau)^2] \tag{3.2}$$

where N and N^σ are Lagrange multiplier (the lapse and shift functions respectively), and we can choose $\rho = 0$, which means that $g_{11} = det(g_{11}) = 1$ (see Figure 3.1).

In the open string case, the boundary conditions are important for eliminating the surface term in the variation of the action.
The surface term of the action S_B remains the same as the ordinary bosonic string one, while after variation the dusty string action gives,

$$\begin{aligned}\delta S_D =& \\ &-\Big[\int_0^\pi d\sigma \sqrt{-g}\mathcal{M} g^{\alpha 0}\partial_\alpha T \delta T\Big]_{\tau_1}^{\tau_2} \\ &-\Big[\int_{\tau_1}^{\tau_2} d\tau \sqrt{-g}\mathcal{M} g^{1\alpha}\partial_\alpha T \delta T\Big]_0^\pi \\ &+\int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \Big[2\nabla_\alpha(\sqrt{-g}g^{\alpha\beta}\mathcal{M}\partial_\beta T)\delta T + \sqrt{-g}(g^{\alpha\beta}T_{,\alpha}T_{,\beta}+1)\delta\mathcal{M} + \sqrt{-g}T_D^{\alpha\beta}\delta g_{\alpha\beta}\Big]\end{aligned}$$

where $T_D^{\alpha\beta}$ is the contribution of the dust field in the stress energy tensor. So the boundary term of the scalar field $\mathcal{M}T$ satisfies the boundary conditions of the scalar $X^\mu(\tau, \sigma)$.
The suitable boundary conditions are:

$$n^\alpha \partial_\alpha X^\mu \delta X_\mu \Big|_{\sigma=0,\pi} = 0 \tag{3.3}$$

After that, through the variation of the action, the equation of motion for the fields T, $\mathcal{M}$, X^μ can be found respectively, as [3]

$$\Box_h X^\mu = \frac{1}{\sqrt{-h}}\partial_\alpha(\sqrt{-h}h^{\alpha\beta}\partial_\beta X^\mu) = 0 \tag{3.4}$$

$$h^{\alpha\beta}T_{,\alpha}T_{,\beta}+1=0 \tag{3.5}$$

$$\nabla_\alpha(\sqrt{-h}h^{\alpha\beta}\mathcal{M}T_\beta)=0 \tag{3.6}$$

where ∇_α represents the covariate derivative with the Christoffel connection $\Gamma^i_{jk} = \frac{1}{2}h^{il}(\partial_j h_{lk}+\partial_k h_{il}-\partial_l h_{jk})$, then $\Box_h := \nabla^\alpha\nabla_\alpha$.

3.2 Hamiltonian formulation and constraints

The Legendre transformation gives the relation between the Lagrangian formalism and the Hamiltonian formalism, and with the use of the equation of motion of $\mathcal{M}$ (3.6) we can write [3]:

$$S=\int dtd\sigma(\mathcal{P}_T\dot{T}+\mathcal{P}_\mu\dot{X}^\mu-N\mathcal{H}-N^\sigma\mathcal{H}_1) \tag{3.7}$$

where

$$\mathcal{H}_1=T'\mathcal{P}_T-X^{\mu\prime}\mathcal{P}_\mu \tag{3.8}$$

and

$$\mathcal{H}=sgn(\mathcal{M})\mathcal{P}_T\sqrt{1+(T')^2}+\frac{1}{2}\Big[(2\pi\alpha')(\mathcal{P}_\mu)^2+(2\pi\alpha')^{-1}(X^{\mu\prime})^2\Big]=0 \tag{3.9}$$

The dust time gauge is given by [3]: $T(t,\sigma)=t$, so clearly $\dot{T}=1$ and $T'=0$ then the Hamiltonian constraint becomes:

$$sgn(\mathcal{M})\mathcal{P}_T=-\frac{1}{2}\Big[(2\pi\alpha')(\mathcal{P}_\mu)^2+(2\pi\alpha')^{-1}(X^{\mu'})^2\Big] \tag{3.10}$$

after some simplifications, one can identify the physical Hamiltonian with $\mathcal{P}_T$:

$$\mathcal{H}_P:=-\mathcal{P}_T=\frac{1}{2}\Big[(2\pi\alpha')(\mathcal{P}_\mu)^2+(2\pi\alpha')^{-1}(X^{\mu\prime})^2\Big]. \tag{3.11}$$

which is not vanishing in general. For more details one can refer to [3].
In fact, the diffeomorphism constraints remain in this formalism (the redundancy in the description of the σ parameter is not fixed), and it may appear when one can simply distinguish $(d-1)$ independent dynamical variables, in the other hand the Poincaré symmetry is preserved in this choice of the gauge.

3.2.1 Use of the ordinary bosonic string constraints

The treatment of the open bosonic string with dust is not very different from the closed one, but the differences in the boundary condition lead to different results, so the Virasoro generators are given by

$$L_n:=\frac{1}{2}\int_{-\pi}^{\pi}d\sigma e^{in\sigma}\Big(\mathcal{H}_p+\mathcal{H}_1\Big), \tag{3.12}$$

the symmetry property of the open string constraints [9] is

$$\mathcal{H}_p(-\sigma) = \mathcal{H}_p(\sigma), \quad \mathcal{H}_1(-\sigma) = -\mathcal{H}_1(\sigma) \tag{3.13}$$

then the Virasoro generators can be written as,

$$L_n := \frac{1}{2}\int_{-\pi}^{\pi} d\sigma e^{-in\sigma}\Big(\mathcal{H}_p - \mathcal{H}_1\Big) \tag{3.14}$$

therefore, the physical Hamiltonian is,

$$\int_0^{\pi} d\sigma \mathcal{H}_p = L_0 \tag{3.15}$$

while the vanishing diffeomorphisme constraints can be solved by:

$$L_n = L_{-n} \tag{3.16}$$

thereby the check of the constraints leads to the definition of the diffeomorphism generators,

$$l_n := \frac{1}{2}\int_{-\pi}^{\pi} d\sigma e^{in\sigma}\mathcal{H}_1 = \frac{1}{2}\int_{-\pi}^{\pi} d\sigma e^{in\sigma}\mathcal{P}_\mu X^{\mu\prime} = 0 \tag{3.17}$$

which verify the Poisson bracket algebra:

$$[l_n, l_m]_{PB} = -i(n-m)l_{n+m} + i(n+m)l_{n-m}. \tag{3.18}$$

The Fourier expansion of the canonical variables is:

$$X^\mu(\tau,\sigma) = x^\mu + \frac{p^\mu}{\pi T}\tau + \frac{i}{\sqrt{\pi T}}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-in\tau}cos(n\sigma) \tag{3.19}$$

so the momenta conjugate reads

$$\mathcal{P}^\mu(\tau,\sigma) = \frac{p^\mu}{\pi} + \sqrt{\frac{T}{\pi}}\sum_{n\neq 0}\alpha_n^\mu e^{-in\tau}cos(n\sigma) \tag{3.20}$$

with the following Poisson bracket of the Fourier modes:

$$[x^\mu, p_\mu]_{PB} = \delta^\mu_\nu, \qquad [\alpha^\mu_m, \alpha^{\nu *}_n]_{PB} = -i\eta^{\mu\nu}\delta_{mn}. \tag{3.21}$$

Thus the physical Hamiltonian in terms of Fourier modes can be written as:

$$H_p = \alpha' p^2 + \sum_{n>0}\alpha_{-n}\cdot\alpha_n. \tag{3.22}$$

The Hamiltanian method is elegant for the canonical quantization of this classical theory, and allows to take advantage of the results of ordinary bosonic string theory, the difference here is that H_p is not a constraint so it does not necessarily vanish.

3.3 Quantization

In general, there are two ways to quantize the ordinary string theory, the first one is to solve the two constraints at the classical level, after that one can pass to the quantum theory. The other approach is the quantization of the classical string theory without fixing the constraints and then apply those constraints as operators which annihilate the physical states, this way is known as the covariant quantization of string theory. The quantum vacuum of the bosonic string with dust satisfies:

$$\hat{\alpha}^{\mu}_{n}|0,0\rangle = 0, \qquad \hat{p}|0,0\rangle = 0 \tag{3.23}$$

where

$$\left[\hat{\alpha}^{\mu}_{m}, \hat{\alpha}^{\nu\dagger}_{n}\right] = \eta^{\mu\nu}\delta_{mn}, \qquad \left[\hat{x}^{\mu}, \hat{p}^{\dagger}_{\nu}\right] = i\delta^{\mu}_{\nu} \tag{3.24}$$

In this case the physical Hamiltonian verifies

$$\hat{H}_P|\psi\rangle = \left[\alpha' p^2 + \sum_{n>0} \hat{\alpha}^{\mu}_{-n}\hat{\alpha}_{n\mu} - \alpha_0\right]|\psi\rangle = E_{open}|\psi\rangle. \tag{3.25}$$

The value $E = 0$ corresponds to the ordinary bosonic string theory case where H_P is a constraint. The mass shell condition allows to write

$$\alpha' M^2 = N - (\alpha_0 + E_{open}) \tag{3.26}$$

For a simple consistent quantization one can impose the following weak conditions on the physical states,

$$\langle\psi|l_n|\psi\rangle = \langle\psi|(\hat{L}_n - \hat{L}_{-n})|\psi\rangle \tag{3.27}$$

this last equation mirrors the diffeomorphism at the quantum level, and can be solved by taking the strong conditions:

$$\hat{L}_n|\psi\rangle = 0, \ \ for\ n > 0 \tag{3.28}$$

These conditions are strong because the diffeomorphism condition has just the form:

$$(\hat{L}_n - \hat{L}_{-n})|\psi\rangle = 0 \tag{3.29}$$

the advantage of (3.28) is the possibility of using the results of the ordinary bosonic string, so the Virasoro generators satisfy the following algebra,

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{D}{12}n(n^2-1)\delta_{n+m} \tag{3.30}$$

then the quantum spatial diffeomorphism algebra becomes anomaly free,

$$[l_n, l_m] = (n-m)l_{n+m} - (n+m)l_{n-m} \tag{3.31}$$

In the closed string, the diffeomorphism anomaly can be eliminated by using the equality of the anomalies in both the $\hat{L}_n$ and $\hat{\bar{L}}_n$ Virasoro algebras [3], while in the open string the anomaly is eliminated naturally in the diffeomophism algebra. Notice however that in this algebra, both l_{n+m} and l_{n-m} are present.

The theory with dust preserves the Poincaré invariance; and does not need to preserve the conformal symmetry for a consistent quantum theory; where $E + \alpha_0 < 1$ *and* $D \leq 26$. In general we have no vanishing Hamiltonian thus we can in principal use the Heisenberg equation of motion to study the dynamics of this system [3].

3.4 Open string with dust between two parallel Dp-branes

The X^a where $a \in \{p+1, .., D-1\}$ specify the normal coordinates to the brane world-volume and verify Dirichlet boundary conditions,

$$X^a(\tau,\sigma)\Big|_{\sigma=0} = X^a(\tau,\sigma)\Big|_{\sigma=\pi} = \bar{x}^a \tag{3.32}$$

So if $\bar{x}^a_1$ and $\bar{x}^a_2$ denote the values of the normal coordinates which specify the position of the first and the second Dp-branes respectively, then the solution can be obtained as

$$X^a(\tau,\sigma) = \bar{x}^a_1 + (\bar{x}^a_2 - \bar{x}^a_1)\frac{\sigma}{\pi} + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha^a_n e^{-in\tau} sin(n\sigma) \tag{3.33}$$

While X^i where $i \in \{0, 1, ..., p\}$ specify the tangential coordinates and satisfy Neumann boundary conditions,

$$X'^i(\tau,\sigma)\Big|_{\sigma=0} = X'^i(\tau,\sigma)\Big|_{\sigma=\pi} = 0 \tag{3.34}$$

So the Lorentz background space-time symmetry group $SO(1, D-1)$ becomes $SO(1,p)\times SO(D-p-1)$, the first part represents the Lorentz symmetry which is satisfied by the tangential Dp-branes coordinates, and the second part is the rotation symmetry in the transverse coordinates of the Dp-branes without translation symmetry. The Virasoro generators are given by

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} :\alpha^i_m \alpha^j_{n-m}: \eta_{ij} + \frac{1}{2}\sum_{m\in\mathbb{Z}} :\alpha^a_m \alpha^b_{n-m}: \delta_{ab} \tag{3.35}$$

and

$$L_0 = -\alpha' M^2 + \alpha'\lambda^2 + \frac{1}{2}\sum_{m>0} :\alpha^i_{-m}\alpha^j_m: \eta_{ij} + \frac{1}{2}\sum_{m>0} :\alpha^a_{-m}\alpha^b_m: \delta_{ab} \tag{3.36}$$

where $\lambda^a = \left(\frac{\bar{x}^a_2 - \bar{x}^a_1}{2\pi\alpha'}\right)$ and $\lambda^2 = \lambda^a\lambda_a$.

As we know above, L_0 is the Hamiltonian of the open string with dust, and its eigenvalues are not null in general, this Hamiltonian helps us to set up a concrete quantization to our system. So the mass squared expression can be written as follows

$$M^2 = \lambda^2 + \frac{1}{\alpha'}(N - [\alpha_0 + E]) \tag{3.37}$$

According to Brower-Goddard-Thorn theorem or No-ghost theorem, the Hilbert space of this model is without negative norm states if $\alpha_0 + E = 1$ and $D = 26$ for the critical string, or $\alpha_0 + E \leq 1$ and $D \leq 25$ for non-critical one, otherwise, the space of states has not a positive norm definition. Due to the presence of X_0 which has Neumann-Neumann boundary conditions; so that, there is no Dp-branes effect on the space-time dimension and the zero modes remain in the theory [40]. The Lorentzian index lives in the tangential part of the coordinates with the symmetry group $SO(1, p)$.

According to the no ghost theorem, the Hamiltonian eigenvalues E allow to give a non-critical string theories. While the string stretched between two parallel Dp-branes has effects of extra energy term on the spectrum. So let us see the first two levels in the spectrum:

- $N = 0$:

$$|k, \lambda\rangle \tag{3.38}$$

this state has the mass squared

$$M^2 = \lambda^2 - \frac{1}{\alpha'}(\alpha_0 + E) \tag{3.39}$$

If one supposes non-critical string i.e. $D < 26$ and $\alpha_0 + E < 1$, so the theory has no tachyons if

$$0 \leq \alpha'\lambda^2 - (\alpha_0 + E) \tag{3.40}$$

and no ghosts if

$$\alpha'\lambda^2 - 1 < \alpha'\lambda^2 - (\alpha_0 + E) \tag{3.41}$$

so from the values of λ^2; one can distinguish two cases, (i) first if $\alpha'\lambda^2 > 1$ then: (3.40) is satisfied (theory free of tachyons), imply that (3.41) is satisfied too (theory free of ghosts). (ii) if $\alpha'\lambda^2 < 1$ then if the condition (3.41) is satisfied (theory with no ghosts) implies that also (3.40) is satisfied (theory with no tachyons).

- N=1

$$\zeta_\mu \alpha^\mu_{-1}|k, \lambda\rangle \tag{3.42}$$

This level represents a vector state, and in the presence of two parallel Dp-branes becomes:

$$|\zeta, k\rangle_\lambda = (\zeta_i(k)\alpha^i_{-1} + \zeta_a(k)\alpha^a_{-1})|k\rangle \otimes |\lambda\rangle \tag{3.43}$$

so if the separation between the Dp-branes is non-zero, then a massive scalar state emerges in the spectrum [40],

$$\zeta_a \alpha^a_{-1}|k, \lambda\rangle, \; M^2 = \lambda^2 + \frac{1}{\alpha'}(1 - [\alpha_0 + E]) \tag{3.44}$$

the no-ghost theorem implies that $[\alpha_0 + E] \leq 1$, this property gives $(d-p)$ massive states Lorentz scalars in the above equation i.e. the index a doesn't represent a Lorentz index. In the other hand, one have the state

$$\zeta_i \alpha^i_{-1}|k, \lambda\rangle, \; M^2 = \lambda^2 + \frac{1}{\alpha'}(1 - [\alpha_0 + E]) \tag{3.45}$$

Let us define the following important state: $|\psi\rangle := L_{-1}|k, \lambda\rangle$. This is a spurious state since it is orthogonal to all physical states i.e. ${}_{ph}\langle\phi|\psi\rangle = \langle k, \lambda|L_1|\phi\rangle_{ph} = 0$ where $|\phi\rangle_{ph}$ is any physical state, also one can remark that $L_1|\psi\rangle = 2L_0|k, \lambda\rangle = 2\alpha'(k^2+\lambda^2)|k, \lambda\rangle$. Let us distinguish these three different values of $a := [\alpha_0 + E_{op}]$:

(i) $a < 1$ and $\lambda^2 = 0$: the momentum respects $k^2 \neq 0$ because $M^2 > 0$, and (according to the no ghost theorem) also gives a non-critical string with no ghosts.

(ii) $a > 1$: the theory contains ghosts, also k is spacelike ($k^2 < \lambda^2 < 0$), then the spacelike polarization is removed by the condition: $k_i\zeta^i + \lambda_a\zeta^a = 0$, which leaves p components. The spurious state is not *physical* since $k^2 + \lambda^2 \neq 0$.

(iii) $a = 1$: this means that $M^2 = \lambda^2$, also $k^2 + \lambda^2 = 0$ then the spurious state is physical since it is null, thus two degrees of freedom are removed from the polarization states.

3.5 Regge trajectories

Let us do this short comments concerning the Regge trajectories [41]. It has been known since the early time of hadrons physics that there is a relation between the mass and angular momentum of many hadronic states

$$J = \alpha' M^2 + \alpha_0 \tag{3.46}$$

The string view of the quark–antiquark potential has been proposed for describing such phenomenology. In fact, Regge's trajectories are approximative linear functions that arise when tacking the angular momentum as a function of the energy squared for hadronic excitations. When string theory was investigated as a theory of strong interactions, the slope parameter $\alpha' \approx 1\ \mathrm{Ge}V^{-2}$ was an experimentally measured quantity. In the string theory with dust $\alpha_0 = E + a$, E can be used to characterize the trajectory of meson. While the slope α' almost the same for all trajectories.

3.6 Conclusion

In this chapter, we gave a study on the open bosonic string with dust. Using the usual canonical quantization, we defined the physical Hamiltonian, wrote down the algebra, and applied the mass-shell condition. As a result, we find the possibility of a non-critical string theory. Furthermore, we studied the open bosonic string with dust stretched between parallel two Dp-branes and discussed the relation between the presence of ghosts and tachyons, and the separation between the Dp-branes and the eigenvalues of the physical Hamiltonian. We noted that this model can be used to describe Regge trajectories. In the next chapter we will give the bosonic string theories with deformed energy momentum relation.

Chapter 4

Bosonic string theories with deformed dispersion relations

The scope of this chapter is the study of bosonic string theories with deformed energy momentum relations which is originally given in [17]. First we write the bosonic string action with deformed constraints, while their algebra is first class in Dirac terminology. These constraints generate the gauge transformations of the relativistic bosonic string. We can also use the Hamiltonian constraint to generate the dynamics of the string, and so the equations of motion which have a nonlinear term in its form, this term can be eliminated if one proposes that $f = g$ or $f' = f$ and $g' = g$ (where f and g are total energy functions of the string, which encode the deformations), or we can suppose that the nonlinear term is an interaction term, and follow the study with the free part. Then we can write the solution of our system.

4.1 Constraints algebra

The canonical form of the bosonic string action can be written as

$$S = \int_0^\pi d\sigma d\tau (\dot{X}^\mu \mathcal{P}_\mu - N\mathcal{H}^{bosonic} - N^\sigma \mathcal{H}_\sigma^{bosonic}) \tag{4.1}$$

where, N and N^σ are the lapse and shift functions respectively, which play the role of Lagrange multipliers. $\mathcal{H}^{bosonic}$ and $\mathcal{H}_\sigma^{bosonic}$ are the Hamiltonian and the spatial diffeomorphism constraints respectively, while $\mathcal{P}^\mu$ is the canonical conjugate of X^μ. Let us start this study with the following redefinition of the bosonic string constraints, as those given in [17]:

$$\begin{cases} \mathcal{H}^{bosonic} = \frac{f(E)}{2T}\mathcal{P}_\mu \mathcal{P}^\mu + \frac{Tg(E)}{2} X'_\mu X'^\mu \\ \mathcal{H}_\sigma^{bosonic} = \sqrt{f(E)g(E)}\mathcal{P}^\mu X'_\mu \end{cases} \tag{4.2}$$

where $E = \int d\sigma \mathcal{P}^0(\sigma, 0)$. And let us use the following definition of canonical Poisson brackets for $X^\mu(\sigma)$ and $\mathcal{P}^\mu(\sigma)$:

$$\left[F, G\right]_{PB} = \int_0^l d\sigma' \Big(\frac{\delta F}{\delta X(\sigma')} \frac{\delta G}{\delta \mathcal{P}(\sigma')} - \frac{\delta F}{\delta \mathcal{P}(\sigma')} \frac{\delta G}{\delta X(\sigma')} \Big) \tag{4.3}$$

particularly, the above expression leads to

$$\big[X_\mu(\sigma), \mathcal{P}_\nu(\sigma')\big]_{PB} = \delta(\sigma-\sigma')\eta_{\mu\nu} \tag{4.4}$$

so

$$\big[X'_\mu(\sigma), \mathcal{P}_\nu(\sigma')\big]_{PB} = \delta'(\sigma-\sigma')\eta_{\mu\nu}. \tag{4.5}$$

Let us take

$$F(E) := \sqrt{f\Big(\int_0^l da\mathcal{P}^0(a)\Big)} \tag{4.6}$$

In this stage one may obtain the following Poisson bracket,

$$\begin{aligned}\big[F(E), X^0(\sigma)\big]_{PB} &= \int_0^l d\sigma'\bigg(-\frac{\delta F}{\delta\mathcal{P}^0(\sigma')}\delta(\sigma-\sigma')\bigg)\\ &= -\frac{\delta F[\mathcal{P}^0(\sigma)]}{\delta\mathcal{P}^0(\sigma)}\end{aligned} \tag{4.7}$$

the function is the map which turns a number to another one, while the functional maps a function to a number. Thus with the use of

$$F[\mathcal{P}^0(\sigma)] = F\Big(\int_0^l \mathcal{P}^0(\sigma)d\sigma\Big) \tag{4.8}$$

and the functional definition [42, 43], one can proves that

$$\begin{aligned}\frac{\delta F[\mathcal{P}^0]}{\delta\mathcal{P}^0(\sigma)} &= \lim_{\epsilon\to0}\frac{F[\mathcal{P}^0(\sigma')+\epsilon\delta(\sigma-\sigma')]-F[\mathcal{P}^0(\sigma')]}{\epsilon}\\ &= \lim_{\epsilon\to0}\frac{F\big(\int_0^l\mathcal{P}^0(\sigma')+\epsilon\delta(\sigma-\sigma')d\sigma'\big)-F\big(\int_0^l\mathcal{P}^0(\sigma')d\sigma'\big)}{\epsilon}\\ &= \lim_{\epsilon\to0}\frac{F(E+\epsilon)-F(E)}{\epsilon}\\ &= \frac{dF(E)}{dE}\end{aligned}$$

Next, with the use of the relations

$$\frac{\delta F[\mathcal{P}^0]}{\delta\mathcal{P}^0(\sigma)} = \frac{dF(E)}{dE} \tag{4.9}$$

$$\frac{\partial}{\partial\sigma'}\delta(\sigma-\sigma') = -\frac{\partial}{\partial\sigma}\delta(\sigma-\sigma') \tag{4.10}$$

and

$$F(\sigma')\delta'(\sigma-\sigma') = F'(\sigma)\delta(\sigma-\sigma') + F(\sigma)\delta'(\sigma-\sigma') \tag{4.11}$$

one can prove that the constraints (4.2) obey

$$
\begin{aligned}
[\mathcal{H}^{bosonic}(\sigma), \mathcal{H}^{bosonic}(\sigma')]_{PB} &= fg\mathcal{P}(\sigma)\cdot X'(\sigma')\frac{\partial}{\partial\sigma}\delta(\sigma-\sigma') \\
&\quad + fg\mathcal{P}(\sigma')\cdot X'(\sigma)\frac{\partial}{\partial\sigma}\delta(\sigma-\sigma') \\
&= \sqrt{fg}\Big(\mathcal{P}(\sigma)\cdot X''(\sigma)\delta(\sigma-\sigma') + \mathcal{P}(\sigma)\cdot X'(\sigma)\delta'(\sigma-\sigma') \\
&\quad + \mathcal{P}'(\sigma)\cdot X'(\sigma)\delta(\sigma-\sigma') + \mathcal{P}(\sigma)\cdot X'(\sigma)\delta'(\sigma-\sigma')\Big) \\
&= \sqrt{fg}\Big(\frac{\partial}{\partial\sigma}\mathcal{H}^{bosonic}_{\sigma}(\sigma)\delta(\sigma-\sigma') + 2\mathcal{H}^{bosonic}_{\sigma}(\sigma)\delta'(\sigma-\sigma')\Big) \\
&= \sqrt{fg}(\mathcal{H}^{bosonic}_{\sigma}(\sigma') + \mathcal{H}^{bosonic}_{\sigma}(\sigma))\delta'(\sigma-\sigma')
\end{aligned}
$$

then

$$[\mathcal{H}^{bosonic}(\sigma), \mathcal{H}^{bosonic}(\sigma')]_{PB} = \sqrt{fg}(\mathcal{H}^{bosonic}_{\sigma}(\sigma) + \mathcal{H}^{bosonic}_{\sigma}(\sigma'))\delta'(\sigma-\sigma') \tag{4.12}$$

$$[\mathcal{H}^{bosonic}(\sigma), \mathcal{H}^{bosonic}_{\sigma}(\sigma')]_{PB} = \sqrt{fg}(\mathcal{H}^{bosonic}(\sigma) + \mathcal{H}^{bosonic}(\sigma'))\delta'(\sigma-\sigma') \tag{4.13}$$

$$[\mathcal{H}^{bosonic}_{\sigma}(\sigma), \mathcal{H}^{bosonic}_{\sigma}(\sigma')]_{PB} = \sqrt{fg}(\mathcal{H}^{bosonic}_{\sigma}(\sigma) + \mathcal{H}^{bosonic}_{\sigma}(\sigma'))\delta'(\sigma-\sigma') \tag{4.14}$$

the above relations tell us that the algebra of constraints is first class; which means that they generate gauge transformations on the world-sheet.

One can verify that the constraints (4.2) generate parametrizations on the world-sheet, indeed we find that

$$[X^{\mu}(\sigma), \int d\sigma'\mathcal{H}^{bosonic}_{\sigma}(\sigma')\zeta^{\sigma}(\sigma')]_{PB} = \zeta^{\sigma}\sqrt{fg}X^{\mu\prime}(\sigma) \tag{4.15}$$

$$[\mathcal{P}^{\mu}(\sigma), \int d\sigma'\mathcal{H}^{bosonic}_{\sigma}(\sigma')\zeta^{\sigma}(\sigma')]_{PB} = \Big(\zeta^{\sigma}\sqrt{fg}\mathcal{P}^{\mu}(\sigma)\Big)' \tag{4.16}$$

and also

$$[\mathcal{H}^{bosonic}(\sigma), \int d\sigma'\mathcal{H}^{bosonic}_{\sigma}(\sigma')\zeta^{\sigma}(\sigma')]_{PB} = \Big(\zeta^{\sigma}\sqrt{fg}\mathcal{P}^{\mu}(\sigma)\Big)' \tag{4.17}$$

Notice that, these transformations are energy dependent.

4.1.1 Virasoro Generators

It is convenient to use the following linear combination of the constraints (4.2):

$$Q^{+}(\sigma) = 2\pi(\mathcal{H}^{bosonic} + \mathcal{H}^{bosonic}_{\sigma}) = P_{\mu}P^{\mu} \tag{4.18}$$

$$Q^{-}(\sigma) = 2\pi(\mathcal{H}^{bosonic} - \mathcal{H}^{bosonic}_{\sigma}) = S_{\mu}S^{\mu} \tag{4.19}$$

where it can be shown that

$$P_\mu(\sigma) = \sqrt{\frac{\pi f}{T}}\mathcal{P}_\mu + \sqrt{\pi T g}X'_\mu \tag{4.20}$$

$$S_\mu(\sigma) = \sqrt{\frac{\pi f}{T}}\mathcal{P}_\mu - \sqrt{\pi T g}X'_\mu \tag{4.21}$$

one can verify that:

$$\begin{aligned}\big[P_\mu(\sigma), P_\nu(\sigma')\big]_{PB} &= \big[\sqrt{\frac{\pi f}{T}}\mathcal{P}_\mu(\sigma) + \sqrt{\pi T g}X'_\mu(\sigma), \sqrt{\frac{\pi f}{T}}\mathcal{P}_\nu(\sigma') + \sqrt{\pi T g}X'_\nu(\sigma')\big]_{PB} \\ &= \pi\sqrt{fg}\big[\mathcal{P}_\mu(\sigma), X'_\nu(\sigma')\big]_{PB} + \pi\sqrt{fg}\big[X'_\mu(\sigma), \mathcal{P}_\nu(\sigma')\big]_{PB}\end{aligned}$$

So

$$\big[P_\mu(\sigma), P_\nu(\sigma')\big]_{PB} = 2\pi\sqrt{fg}\delta'(\sigma-\sigma')\eta_{\mu\nu} \tag{4.22}$$

$$\big[S_\mu(\sigma), S_\nu(\sigma')\big]_{PB} = -2\pi\sqrt{fg}\delta'(\sigma-\sigma')\eta_{\mu\nu} \tag{4.23}$$

$$\big[P_\mu(\sigma), S_\nu(\sigma')\big]_{PB} = 0 \tag{4.24}$$

The translation currents j^α_μ in terms of the light-like components take the form $j^\pm = j^0_\alpha \pm j^1_\alpha$, which are defined as follows

$$j^\pm_\mu(\sigma) = -\frac{\sqrt{2\alpha'}}{2}\Big(\mathcal{P}_\mu(\sigma) \pm \frac{1}{2\pi\alpha'}X'_\mu(\sigma)\Big) \tag{4.25}$$

so

$$[j^+_\mu(\sigma), j^+_\nu(\sigma')] = \sqrt{fg}\frac{i}{2\pi}\eta_{\mu\nu}\delta'(\sigma-\sigma') \tag{4.26}$$

$$[j^-_\mu(\sigma), j^-_\nu(\sigma')] = -\sqrt{fg}\frac{i}{2\pi}\eta_{\mu\nu}\delta'(\sigma-\sigma') \tag{4.27}$$

$$[j^+_\mu(\sigma), j^-_\nu(\sigma')] = 0 \tag{4.28}$$

In the other hand, the Virasoro algebra in terms of Q^+ and Q^- is:

$$[Q^+(\sigma), Q^+(\sigma')] = 4\pi\sqrt{fg}\Big(Q^+(\sigma) + Q^+(\sigma')\Big)\delta'(\sigma-\sigma') \tag{4.29}$$

$$[Q^+(\sigma), Q^-(\sigma')] = 0 \tag{4.30}$$

and

$$[Q^-(\sigma), Q^-(\sigma')] = -4\pi\sqrt{fg}\Big((Q^-(\sigma) + Q^-(\sigma')\Big)\delta'(\sigma-\sigma') \tag{4.31}$$

In the open string case we can use the following extension of the σ interval:

$$\mathcal{H}^{bosonic}(-\sigma) = \mathcal{H}^{bosonic}(\sigma), \quad \mathcal{H}^{bosonic}_\sigma(-\sigma) = -\mathcal{H}^{bosonic}_\sigma(\sigma), \quad Q^+(-\sigma) = Q^-(\sigma) \tag{4.32}$$

so, we can associate the two conditions $Q^\pm$ in a single one, in the interval $[-\pi,\ \pi]$ of the σ parameter;

$$Q^+(\sigma) = 0, \quad -\pi \le \sigma \le \pi \tag{4.33}$$

The Virasoro generators of the open string are defined by

$$L_n = \frac{1}{4\pi}\int_{-\pi}^{\pi} e^{in\sigma}Q^+(\sigma)d\sigma \tag{4.34}$$

and satisfy

$$L_{-n} = L_n^*. \tag{4.35}$$

The classical Virasoro algebra is

$$[L_m, L_n]_{PB} = i\sqrt{fg}(n-m)L_{n+m} \tag{4.36}$$

While in the closed string case the constraints $Q^+(\sigma) = 0$ and $Q^-(\sigma) = 0$ are still independent. So we must define two kinds on the Virasoro generators, the left handed;

$$L_n = \frac{1}{4\pi}\int_{0}^{2\pi} e^{in\sigma}Q^+(\sigma)d\sigma \tag{4.37}$$

and the right handed ones

$$\bar{L}_n = \frac{1}{4\pi}\int_{0}^{2\pi} e^{-in\sigma}Q^-(\sigma)d\sigma \tag{4.38}$$

The Virasoro algebra becomes

$$[L_m, L_n]_{PB} = i(n-m)\sqrt{fg}L_{n+m} \tag{4.39}$$

$$[\bar{L}_m, \bar{L}_n]_{PB} = i(n-m)\sqrt{fg}\bar{L}_{n+m} \tag{4.40}$$

$$[L_m, \bar{L}_n]_{PB} = 0 \tag{4.41}$$

where the Virasoro generators satisfy

$$L_m = 0, \qquad \bar{L}_m = 0 \tag{4.42}$$

and define the physical part of the phase space.

4.2 Equations of motion

The dynamics is generated by the Hamiltonian constraints:

$$H^{bosonic} = \int d\sigma \mathcal{H}^{bosonic} \tag{4.43}$$

so for the canonical variables we have:

$$\begin{aligned}
\dot{X}^0(\sigma) &= \int d\sigma' [X^0(\sigma), \mathcal{H}^{bosonic}(\sigma')]_{PB} \\
&= \int d\sigma' \Bigg(\frac{f(E)}{2T} [X^0(\sigma), \mathcal{P}^2(\sigma')]_{PB} + \frac{\mathcal{P}^2(\sigma')}{2T} [X^0(\sigma), f(E)]_{PB} \\
&+ \frac{T}{2} X'^2(\sigma') [X^0(\sigma), g(E)]_{PB} \Bigg) \\
&= \int d\sigma' \left(\frac{f(E)}{T} \delta(\sigma - \sigma') \mathcal{P}^0(\sigma') + \frac{\mathcal{P}^2(\sigma')}{2T} \frac{df(E)}{dE} + \frac{T}{2} X'^2(\sigma') \frac{dg(E)}{dE} \right) \\
&= \frac{f(E)}{T} \mathcal{P}^0(\sigma) + \int d\sigma' \left(\frac{\mathcal{P}^2(\sigma')}{2T} \frac{df(E)}{dE} + \frac{T}{2} X'^2(\sigma') \frac{dg(E)}{dE} \right)
\end{aligned} \tag{4.44}$$

and

$$\begin{aligned}
\dot{\mathcal{P}}^0(\sigma) &= \int d\sigma' [\mathcal{P}^0(\sigma), \mathcal{H}^{bosonic}(\sigma')]_{PB} \\
&= \int d\sigma' \frac{Tg(E)}{2} [\mathcal{P}^0(\sigma), X'^2(\sigma')]_{PB} \\
&= Tg(E) \partial_\sigma^2 X^0
\end{aligned} \tag{4.45}$$

the same method gives,

$$\dot{X}^a(\sigma) = \frac{f(E)}{T} \mathcal{P}^a(\sigma) \tag{4.46}$$

$$\dot{\mathcal{P}}^a(\sigma) = Tg(E) \partial_\sigma^2 X^a \tag{4.47}$$

where $a = 1, \ldots, D-1$ denotes the spatial index.
The relations (4.44) and (4.46) read in abbreviated notation,

$$\dot{X}^\mu(\sigma) = \frac{f(E)}{T} \mathcal{P}^\mu(\sigma) + \delta_0^\mu \int d\sigma' \left(\frac{\mathcal{P}^2(\sigma')}{2T} \frac{df(E)}{dE} + \frac{T}{2} X'^2(\sigma') \frac{dg(E)}{dE} \right) \tag{4.48}$$

The nonlinear term in (4.48) is eliminated if $f = g$, but there is another case:

$$\begin{cases} \frac{df}{dE} = \frac{E}{\lambda} \\ \frac{dg}{dE} = \frac{E}{\lambda} \end{cases} \tag{4.49}$$

with the solutions:

$$\begin{cases} f = K_1 e^{\frac{E}{\lambda}} \\ g = K_2 e^{\frac{E}{\lambda}} \end{cases} \tag{4.50}$$

where K_1 and K_2 are the constants of integration. The equation of motion,

$$\ddot{X}^\mu - fg \partial_\sigma^2 X^\mu = 0 \tag{4.51}$$

is a massless Klein-Gorden wave equation where the velocity of waves is $\sqrt{fg}$. In the other hand, we can consider the nonlinear term in the equation of motion as an interaction term, then the equation (4.51) became the free part [17] (Appendix B).

4.3 Boundary conditions

The generators must be conserved under variation of fields, this property supposes boundary conditions on the canonical variables and Lagrange multipliers. The variation of $\int N\mathcal{H}^{bosonic}d\sigma$ is

$$\begin{aligned}\delta\int_0^\pi N\mathcal{H}^{bosonic}d\sigma &= \int_0^\pi d\sigma N\Big(\frac{f(E)}{T}\mathcal{P}_\mu\delta\mathcal{P}^\mu + Tg(E)X'_\mu\delta X'^\mu\Big)\\ &= \Big[Tg(E)NX'\delta X\Big]_0^\pi + \int_0^\pi d\sigma\Big(N\frac{f(E)}{T}\mathcal{P}_\mu\delta\mathcal{P}^\mu - Tg(E)N'X''_\mu\delta X^\mu\Big)\end{aligned}$$

In the conformal gauge where $N=1$ at the boundaries, we can distinguish between two cases, first the Neumann boundary conditions

$$\Big[\partial_\sigma X^\mu\Big]_0^\pi = 0 \tag{4.52}$$

and second; the Dirichlet boundary conditions

$$\big[\delta X^\mu\big]_0^\pi = 0 \tag{4.53}$$

The variation of $\int_0^\pi N^\sigma\mathcal{H}_\sigma^{bosonic}d\sigma$ gives

$$\begin{aligned}\delta\int_0^\pi N^\sigma\mathcal{H}_\sigma^{bosonic}d\sigma &= \int_0^\pi d\sigma N^\sigma\sqrt{fg}\Big(\mathcal{P}^\mu\delta X'_\mu + X'_\mu\delta\mathcal{P}^\mu\Big)\\ &= \Big[\sqrt{fg}N^\sigma\mathcal{P}^\mu\delta X_\mu\Big]_0^\pi + \int_0^\pi d\sigma N^\sigma\sqrt{fg}\Big(-\mathcal{P}^{\mu\prime}\delta X_\mu + X'_\mu\delta\mathcal{P}^\mu\Big)\end{aligned}$$

The surface term can be eliminated in the conformal gauge, where $N^\sigma = 0$.

4.4 Solutions

4.4.1 Open string

Let us now try to give the solution of the wave equation (4.51), where the most general solution is

$$X^\mu(\tau,\sigma) = \frac{1}{2}\big(X_L^\mu(\sqrt{fg}\tau+\sigma) + X_R^\mu(\sqrt{fg}\tau-\sigma)\big) \tag{4.54}$$

the Neumann boundary conditions (4.52) allow to write, (at $\sigma=0$)

$$\frac{\partial X^\mu}{\partial\sigma}(\tau,0) = \frac{1}{2}\big(X_L^{\mu\prime}(\sqrt{fg}\tau) - X_R^{\mu\prime}(\sqrt{fg}\tau)\big) = 0 \tag{4.55}$$

this leads to

$$X_R^\mu = X_L^\mu + c^\mu \tag{4.56}$$

where c^μ is a constant of integration. So the result became

$$X^\mu(\tau,\sigma) = \frac{1}{2}\big(X_L^\mu(\sqrt{fg}\tau+\sigma) + X_L^\mu(\sqrt{fg}\tau-\sigma)\big) \tag{4.57}$$

In the other hand, the Neumann boundary condition at $\sigma=\pi$ gives

$$\frac{\partial X}{\partial\sigma}(\tau,\pi) = \frac{1}{2}\big(X_L^{\mu\prime}(\sqrt{fg}\tau+\pi) - X_L^{\mu\prime}(\sqrt{fg}\tau-\pi)\big) \tag{4.58}$$

then

$$X_L^{\mu\prime} = x_1^\mu + \sum_{n=1}^{\infty}(a_n^\mu \cos nu - b_n^\mu \sin nu). \tag{4.59}$$

The integration with respect to σ gives:

$$X_L^\mu(u) = x_0^\mu + x_1^\mu u + \sum_{n=1}^{\infty}(A_n^\mu \cos nu + B_n^\mu \sin nu) \tag{4.60}$$

so

$$X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{fg}x_1^\mu\tau + \sum_{n=1}^{\infty}\left(A_n^\mu\cos(\sqrt{fg}n\tau) + B_n^\mu\sin(\sqrt{fg}n\tau)\right)\cos(n\sigma) \tag{4.61}$$

Let us now use the following convention:

$$A_n^\mu\cos(\sqrt{fg}n\tau) + B_n^\mu\sin(\sqrt{fg}n\tau) \equiv -i\sqrt{\frac{2\alpha'}{n}}(a_n^{\mu*}e^{in\sqrt{fg}\tau} - a_n^\mu e^{-in\sqrt{fg}\tau}) \tag{4.62}$$

Taking $x_1^\mu = \frac{1}{\pi T}\sqrt{fg}p^\mu$. From [17], the Neumann boundary conditions give the following solutions of the canonical variables

$$X^\mu(\tau,\sigma) = x^\mu + \frac{p^\mu}{\pi T}\tau + \frac{i}{\sqrt{\pi T}}\sum_{n\neq0}\frac{\alpha_n^\mu}{n}e^{-in\sqrt{fg}\tau}cos(n\sigma) \tag{4.63}$$

$$\mathcal{P}^\mu(\tau,\sigma) = \frac{T}{f}\dot{X}^\mu = \frac{p^\mu}{\pi f} + \sqrt{\frac{gT}{\pi f}}\sum_{n\neq0}\alpha_n^\mu e^{-in\sqrt{fg}\tau}cos(n\sigma) \tag{4.64}$$

While with Dirichlet-Dirichlet boundary conditions, we have the solutions

$$X^\mu(\tau,\sigma) = x_0^\mu + \frac{1}{\pi}(x_1^\mu - x_0^\mu)\sigma + \sqrt{2\alpha'}\sum_{n\neq0}\frac{1}{n}\alpha_n^\mu e^{-in\sqrt{fg}\tau}sin(n\sigma) \tag{4.65}$$

4.4.2 Closed string

In the closed string case, the coordinates satisfy the periodicity conditions:

$$X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+2\pi) \tag{4.66}$$

Hence, the left and right handed wave solutions are

$$X^\mu_L(\tau,\sigma) = \frac{x^\mu}{2} + \frac{l_s^2}{2}p^\mu(\tau+\sigma) + i\frac{l_s}{\sqrt{2}}\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}e^{-in(\sqrt{fg}\tau+\sigma)} \tag{4.67}$$

$$X^\mu_R(\tau,\sigma) = \frac{x^\mu}{2} + \frac{l_s^2}{2}\bar{p}^\mu(\tau-\sigma) + i\frac{l_s}{\sqrt{2}}\sum_{n\neq 0}\frac{\bar{\alpha}^\mu_n}{n}e^{-in(\sqrt{fg}\tau-\sigma)} \tag{4.68}$$

where

$$X^\mu(\tau,\sigma) = X^\mu_L(\tau,\sigma) + X^\mu_R(\tau,\sigma) \tag{4.69}$$

which takes the form

$$X^\mu(\tau,\sigma) = x^\mu + \alpha' p^\mu\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-in\sqrt{fg}\tau}}{n}(\alpha^\mu_n e^{in\sigma} + \bar{\alpha}^\mu_n e^{-in\sigma}). \tag{4.70}$$

4.5 Canonical quantization

4.5.1 Open string

In the canonical quantization, we replace Poisson brackets by commutators:

$$[,]_{PB} \to -i[,] \tag{4.71}$$

so

$$[X^\mu(\sigma,\tau), \mathcal{P}^\nu(\sigma',\tau)]_- = i\delta(\sigma-\sigma')\eta^{\mu\nu} \tag{4.72}$$

Let us take:

$$\alpha^\mu_n = \frac{1}{\pi l_s}\int_0^\pi d\sigma\cos n\sigma\left(\sqrt{\frac{f}{g}}\frac{\mathcal{P}^\mu}{T} - inX^\mu(\sigma)\right) \tag{4.73}$$

$$x^\mu = \frac{1}{\pi}\int_0^\pi d\sigma X^\mu(\sigma) \tag{4.74}$$

$$p^\mu = \frac{1}{Tl_s^2\pi}f\int_0^\pi d\sigma\mathcal{P}^\mu(\sigma) \tag{4.75}$$

also we need the relation,

$$\int_0^\pi \cos m\sigma\cos n\sigma d\sigma = \frac{\pi}{2}\delta_{mn} \tag{4.76}$$

then we find that

$$[x^\mu, p_\nu]_- = i f \delta^\mu_\nu \tag{4.77}$$

$$[\alpha^\mu_n, \alpha^\nu_{-n}]_- = \sqrt{\frac{f}{g}} n \eta^{\mu\nu} \tag{4.78}$$

Note that the relation (4.77) gives energy dependent Planck constant [34], while the choices $f = g$ and (4.50) can eliminate the energy deformation in (4.78). Let us use the definitions

$$\beta^\mu_n = \left(\frac{g}{f}\right)^{\frac{1}{4}} \alpha^\mu_n, \quad for \quad n \in \mathbb{Z} \tag{4.79}$$

$$\beta^\mu_0 = \frac{1}{\sqrt{\pi T}} \tilde{p}^\mu \tag{4.80}$$

and

$$\alpha^\mu_0 = \frac{p^\mu}{\sqrt{\pi T f g}}, \tag{4.81}$$

which lead to

$$\tilde{p}^\mu = f^{-\frac{3}{4}} g^{-\frac{1}{4}} p^\mu \tag{4.82}$$

and give

$$[x^\mu, \tilde{p}_\nu]_- = i \left(\frac{f}{g}\right)^{\frac{1}{4}} \delta^\mu_\nu \tag{4.83}$$

$$[\beta^\mu_n, \beta^\nu_m]_- = n \eta^{\mu\nu} \delta_{n+m} \tag{4.84}$$

Note that, in the conformal gauge we can take the momentum (4.64). Therefore, the relation (4.20) reads

$$P_\mu(\sigma) = \sqrt{\pi T} \Big(\frac{1}{\sqrt{f}} \dot{X}_\mu + \sqrt{g} X'_\mu \Big). \tag{4.85}$$

We use the solution (4.63), hence we write

$$\dot{X}^\mu(\tau, \sigma) = \frac{p^\mu}{\pi T} + \sqrt{\frac{fg}{\pi T}} \sum_{\neq 0} \alpha^\mu_n e^{-in\sqrt{fg}\tau} \cos(n\sigma) \tag{4.86}$$

$$X^{\mu\prime}(\tau, \sigma) = -\frac{i}{\sqrt{\pi T}} \sum_{n \neq 0} \alpha^\mu_n e^{-in\sqrt{fg}\tau} \sin(n\sigma). \tag{4.87}$$

then we obtain

$$P^\mu(\sigma) = \sqrt{g} \sum_{n=-\infty}^{+\infty} \alpha^\mu_n e^{-in\sigma} \tag{4.88}$$

Form (4.18) one gets, furthermore, that

$$Q^+(\sigma) = g \sum_{m,n} \alpha_m \cdot \alpha_n e^{-i(m+n)\sigma} \tag{4.89}$$

Since the Virasoro generators of the open string is given by (4.34), we obtain

$$\begin{aligned}L_n &= \frac{g}{\pi}\int_{-\pi}^{\pi} d\sigma e^{+in\sigma}\Big(\sum_{l,m}\alpha_l\cdot\alpha_m e^{-i(r+s)\sigma}\Big)\\ &= \frac{g}{2}\Big(\sum_{l,m}\alpha_l\cdot\alpha_m\delta(l+m-n)\Big)\\ &= \frac{g}{2}\sum_{-\infty}^{+\infty}\alpha_{n-m}\cdot\alpha_m\\ &= \frac{\sqrt{fg}}{2}\sum_{-\infty}^{+\infty}\beta_{n-m}\cdot\beta_m \end{aligned}\tag{4.90}$$

in the second line of the above expression we have used the fact that

$$\int d\sigma e^{i(n+m)\sigma} = 2\pi\delta(n+m) \tag{4.91}$$

where $\delta(n+m)$ is the Kronecker delta function.
By the use of the generators (4.90) (after quantization) and the redefinitions

$$L_n = \sqrt{fg}\tilde{L}_n, \tag{4.92}$$

The operators $\tilde{L}_n$ satisfy the ordinary Virasoro algebra, so this latter takes the form

$$[L_n, L_m]_- = \sqrt{fg}(n-m)L_{n+m} + fg\frac{D}{12}(n^3-n)\delta_{n,-m} \tag{4.93}$$

Notice that the above equation has an energy dependent central charge.

4.5.2 Closed string

From (4.70) and (4.72) we can write the following commutation relations

$$[x^\mu, p_\nu]_- = if\delta^\mu_\nu \tag{4.94}$$

$$[\alpha^\mu_n, \alpha^\nu_{-n}]_- = \sqrt{\frac{f}{g}}n\eta^{\mu\nu} \tag{4.95}$$

$$[\bar{\alpha}^\mu_n, \bar{\alpha}^\nu_{-n}]_- = \sqrt{\frac{f}{g}}n\eta^{\mu\nu} \tag{4.96}$$

We have also

$$P^\mu(\sigma) = \sqrt{g}\sum_{n=-\infty}^{+\infty}\alpha^\mu_n e^{-in\sigma} \tag{4.97}$$

$$S^\mu(\sigma) = \sqrt{g} \sum_{n=-\infty}^{+\infty} \bar{\alpha}_n^\mu e^{in\sigma} \tag{4.98}$$

which lead to

$$Q^+(\sigma) = g \sum_{m,n} \alpha_m \cdot \alpha_n e^{-i(m+n)\sigma} \tag{4.99}$$

$$Q^-(\sigma) = g \sum_{m,n} \bar{\alpha}_m \cdot \bar{\alpha}_n e^{i(m+n)\sigma} \tag{4.100}$$

then one can write

$$L_n = \frac{\sqrt{fg}}{2} \sum_{-\infty}^{+\infty} \beta_{n-m} \cdot \beta_m \tag{4.101}$$

and

$$\bar{L}_n = \frac{\sqrt{fg}}{2} \sum_{-\infty}^{+\infty} \bar{\beta}_{n-m} \cdot \bar{\beta}_m \tag{4.102}$$

where

$$\bar{\beta}_0^\mu = \beta_0^\mu = \frac{1}{\sqrt{\pi T}} \tilde{p}^\mu \tag{4.103}$$

and

$$\bar{\alpha}_0^\mu = \alpha_0^\mu = \frac{p^\mu}{\sqrt{\pi T f g}}, \tag{4.104}$$

which lead to

$$\tilde{p}^\mu = f^{-\frac{3}{4}} g^{-\frac{1}{4}} p^\mu \tag{4.105}$$

and give

$$[x^\mu, \tilde{p}_\nu]_- = i \left(\frac{f}{g}\right)^{\frac{1}{4}} \delta_\nu^\mu \tag{4.106}$$

$$[\beta_n^\mu, \beta_m^\nu]_- = n \eta^{\mu\nu} \delta_{n+m} \tag{4.107}$$

$$[\bar{\beta}_n^\mu, \bar{\beta}_m^\nu]_- = n \eta^{\mu\nu} \delta_{n+m} \tag{4.108}$$

The Virasoro algebra is

$$[L_n, L_m]_- = \sqrt{fg}(n-m) L_{n+m} + fg \frac{D}{12}(n^3 - n)\delta_{n,-m} \tag{4.109}$$

$$[\bar{L}_n, \bar{L}_m]_- = \sqrt{fg}(n-m) \bar{L}_{n+m} + fg \frac{D}{12}(n^3 - n)\delta_{n,-m} \tag{4.110}$$

$$[L_m, \bar{L}_n]_- = 0. \tag{4.111}$$

Again we find energy dependent central charges.

4.6 Dispersion relation

The Hamiltonian is the L_0 Virasoro generator, so from the equation (4.90):

$$H^{bosonic} = \int d\sigma \mathcal{H}^{bosonic} = L_0 = \frac{\sqrt{fg}}{2} \sum_{m=-\infty}^{+\infty} : \beta_{-m} \cdot \beta_m : \tag{4.112}$$

In order to construct the relativistic energy-momentum dispersion relation, we take

$$H^{bosonic}|\psi\rangle = 0 \tag{4.113}$$

then we find that

$$\sum_{m=-\infty}^{+\infty} : \beta_{-m} \cdot \beta_m := 0 \tag{4.114}$$

It directly follows that

$$\tilde{p}^2 = -M^2 = (2\pi T)(N + a) \tag{4.115}$$

or

$$p^2 = -f^{\frac{3}{2}} g^{\frac{1}{2}} M^2 = f^{\frac{3}{2}} g^{\frac{1}{2}} (2\pi T)(N + a) \tag{4.116}$$

where

$$N = \sum_{n>0} \beta_{-n} \cdot \beta_n \tag{4.117}$$

is the number operator.

4.7 Possibility of the noncommutative geometry

The question is what happens if we use the transformations

$$\breve{\mathcal{P}}^\mu = \sqrt{f} \mathcal{P}^\mu \tag{4.118}$$

$$\breve{X}^\mu = \sqrt{g} X^\mu \tag{4.119}$$

which give

$$\begin{cases} \mathcal{H}^{bosonic} = \frac{1}{2T} \breve{\mathcal{P}}_\mu \breve{\mathcal{P}}^\mu + \frac{T}{2} \breve{X}'_\mu \breve{X}'^\mu \\ \mathcal{H}_\sigma^{bosonic} = \breve{\mathcal{P}}^\mu \breve{X}'_\mu \end{cases} \tag{4.120}$$

The above equation has the ordinary form of the string constraints. When we use the transformations in Appendix C, we can find the deformed Poisson brackets in the new phase space as follows

$$\begin{aligned} [\breve{X}^\mu(\sigma), \breve{X}^\nu(\sigma')]_{PB} &= [\sqrt{g} X^\mu(\sigma), \sqrt{g} X^\nu(\sigma')]_{PB} \\ &= \sqrt{g} X^\nu(\sigma') [X^\mu(\sigma), \sqrt{g}]_{PB} + \sqrt{g} X^\mu(\sigma) [\sqrt{g}, X^\nu(\sigma')]_{PB} \\ &= \sqrt{g} X^\nu(\sigma') \frac{d\sqrt{g(E)}}{dE} \eta^{\mu 0} - \sqrt{g} X^\mu(\sigma) \frac{d\sqrt{g(E)}}{dE} \eta^{\nu 0} \\ &= \breve{X}^\nu(\sigma') \frac{d\sqrt{g(E)}}{dE} \eta^{\mu 0} - \breve{X}^\mu(\sigma) \frac{d\sqrt{g(E)}}{dE} \eta^{\nu 0} \end{aligned}$$

the next Poisson bracket is

$$\begin{aligned}[\breve{X}^\mu(\sigma), \breve{\mathcal{P}}^\nu(\sigma')]_{PB} &= [\sqrt{g}X^\mu(\sigma), \sqrt{f}\mathcal{P}^\nu(\sigma')]_{PB} \\ &= \sqrt{fg}[X^\mu(\sigma), \mathcal{P}^\nu(\sigma')]_{PB} + \sqrt{g}\mathcal{P}^\nu(\sigma')[X^\mu(\sigma), \sqrt{f}]_{PB} \\ &= \sqrt{fg}\delta(\sigma-\sigma')\eta^{\mu\nu} + \sqrt{\frac{g}{f}}\frac{d\sqrt{f}}{dE}\breve{\mathcal{P}}^\nu(\sigma')\eta^{\mu 0}\end{aligned}$$

while the last one is

$$[\breve{\mathcal{P}}^\mu(\sigma), \breve{\mathcal{P}}^\nu(\sigma')]_{PB} = 0 \tag{4.121}$$

in other words

$$[\breve{X}^\mu(\sigma), \breve{X}^\nu(\sigma')]_{PB} = \Big(\breve{X}^\nu(\sigma')\eta^{\mu 0} - \breve{X}^\mu(\sigma)\eta^{\nu 0}\Big)\frac{d\sqrt{\breve{g}(\breve{E})}}{d\breve{E}} \tag{4.122}$$

$$[\breve{X}^\mu(\sigma), \breve{\mathcal{P}}^\nu(\sigma')]_{PB} = \sqrt{\breve{f}\breve{g}}\delta(\sigma-\sigma')\eta^{\mu\nu} + \sqrt{\frac{\breve{g}}{\breve{f}}}\frac{d\sqrt{\breve{f}}}{d\breve{E}}\breve{\mathcal{P}}^\nu(\sigma')\eta^{\mu 0} \tag{4.123}$$

$$[\breve{\mathcal{P}}^\mu(\sigma), \breve{\mathcal{P}}^\nu(\sigma')]_{PB} = 0 \tag{4.124}$$

where $\breve{E} = \sqrt{f(E)}E$ with the function $\sqrt{f(E)}E$ must be reversible and unique, and $\breve{f}$ and $\breve{g}$ are functions of the energy $\breve{E}$.

In this stage, the equation of motion is a very important step, then the use of the deformed Poisson brackets and the Hamiltonian constraint leads to

$$\begin{aligned}\dot{\breve{X}}^0(\sigma) &= \int_0^\pi d\sigma'[\breve{X}^0(\sigma), \mathcal{H}(\sigma')]_{PB} \\ &= \int_0^\pi d\sigma'[\breve{X}^0(\sigma), \frac{1}{2T}\breve{\mathcal{P}}_\mu\breve{\mathcal{P}}^\mu + \frac{T}{2}\breve{X}'_\mu\breve{X}'^\mu]_{PB} \\ &= \int_0^\pi d\sigma'\Big(\frac{1}{T}\breve{\mathcal{P}}_\mu(\sigma')[\breve{X}^0(\sigma), \breve{\mathcal{P}}^\mu(\sigma')]_{PB} + T\breve{X}'_\mu[\breve{X}^0(\sigma), \breve{X}'^\mu(\sigma')]_{PB}\Big) \\ &= \int_0^\pi d\sigma'\Big(\frac{1}{T}\breve{\mathcal{P}}_\mu(\sigma')\big(\sqrt{fg}\delta(\sigma-\sigma')\eta^{0\mu} - \sqrt{\frac{g}{f}}\frac{d\sqrt{f}}{dE}\breve{\mathcal{P}}^\mu(\sigma')\big) - T\breve{X}'^\mu(\sigma')\breve{X}'_\mu(\sigma')\frac{d\sqrt{g}}{dE}\Big) \\ &= \frac{1}{T}\sqrt{fg}\breve{\mathcal{P}}^0(\sigma) - \int_0^\pi d\sigma'\Big(\frac{1}{T}\sqrt{\frac{g}{f}}\frac{d\sqrt{f}}{dE}\breve{\mathcal{P}}^2(\sigma') - T\breve{X}'^2(\sigma')\frac{d\sqrt{g}}{dE}\Big)\end{aligned}$$

and

$$\begin{aligned}
\dot{\breve{\mathcal{P}}}^0(\sigma) &= \int_0^\pi d\sigma' [\breve{\mathcal{P}}^0(\sigma), \mathcal{H}(\sigma')]_{PB} \\
&= \int_0^\pi d\sigma' \frac{T}{2} [\breve{\mathcal{P}}^0(\sigma), \breve{X}'^2(\sigma')]_{PB} \\
&= \int_0^\pi d\sigma' T \breve{X}'_\mu [\breve{\mathcal{P}}^0(\sigma), \breve{X}(\sigma')'^\mu(\sigma')]_{PB} \\
&= \int_0^\pi d\sigma' T \breve{X}'_\mu \sqrt{fg}\delta'(\sigma - \sigma')\eta^{0\mu} \\
&= T\sqrt{fg}\partial_\sigma^2 \breve{X}^0
\end{aligned}$$

while the spacial part gives

$$\ddot{\breve{X}}^a - fg\partial_\sigma^2 \breve{X}^a = 0 \tag{4.125}$$

where $a = 1 \ldots D$, and if we take $f = g$ the equation of motion in the space-time is

$$\ddot{\breve{X}}^\mu - fg\partial_\sigma^2 \breve{X}^\mu + \delta_0^\mu \frac{\partial}{\partial \tau} \int_0^\pi d\sigma' \Big(\frac{1}{T} \sqrt{\frac{g}{f}} \frac{d\sqrt{f}}{dE} \breve{\mathcal{P}}^2(\sigma') - T \frac{d\sqrt{g}}{dE} \breve{X}'^2(\sigma') \Big) = 0. \tag{4.126}$$

The above equation has the form of (4.51) which is given by the deformed constraints and the ordinary Poisson brackets between the phase space variables X^μ and $\mathcal{P}^\mu$, and if one splits the system into free and interacting parts, then we can conclude the equivalence between the two models at least on the free part:

$$\ddot{\breve{X}}^\mu - fg\partial_\sigma^2 \breve{X}^\mu = 0. \tag{4.127}$$

In such a situation the solution of (4.127) is

$$\breve{X}^\mu = \breve{x}^\mu + \frac{\breve{p}^\mu}{\pi T}\tau + \frac{i}{\sqrt{\pi T}} \sum_{n \neq 0} \frac{\breve{\alpha}_n^\mu}{n} e^{-in\sqrt{fg}\tau} \cos(n\sigma) \tag{4.128}$$

$$\breve{\mathcal{P}}^\mu = \frac{\breve{p}^\mu}{\pi T \sqrt{fg}} + \sqrt{\frac{T}{\pi}} \sum_{n \neq 0} \breve{\alpha}_n^\mu e^{-in\sqrt{fg}\tau} \cos(n\sigma) \tag{4.129}$$

with

$$\breve{\alpha}_n^\mu = \frac{1}{\pi l_s} \int_0^\pi d\sigma \cos(n\sigma) \Big(\frac{\breve{\mathcal{P}}^\mu}{T} - in\breve{X}^\mu \Big) \tag{4.130}$$

$$\breve{x}^\mu = \frac{1}{\pi} \int_0^\pi d\sigma \breve{X}^\mu(\sigma) \tag{4.131}$$

$$\breve{p}^\mu = \frac{\sqrt{fg}}{T l_s^2 \pi} \int_0^\pi d\sigma \breve{\mathcal{P}}^\mu(\sigma) \tag{4.132}$$

So

$$[\breve{x}^\mu, \tilde{x}^\nu]_{PB} = (\breve{x}^\nu \eta^{\mu 0} - \breve{x}^\mu \eta^{\nu 0}) \frac{d\sqrt{g}}{dE} \tag{4.133}$$

$$[\breve{x}^\mu, \tilde{p}^\nu]_{PB} = fg\eta^{\mu\nu} + g\frac{d\sqrt{f}}{dE}\breve{p}^\nu\eta^{\mu 0} \tag{4.134}$$

Notice that the Poisson brackets of the center of mass variables are deformed; for example the relation (4.132) describes non-commutative space time.

Exemples

In the case where:

$$f = g = 1 - \frac{E}{E_p} \tag{4.135}$$

with $E = \int_0^\pi d\sigma \mathcal{P}^0(\sigma)$, we obtain

$$[\breve{X}^\mu(\sigma), \breve{X}^\nu(\sigma')]_{PB} = -\frac{1}{2E_p}\Big(1 - \frac{E}{E_p}\Big)^{-\frac{1}{2}}\Big(\breve{X}^\nu(\sigma')\eta^{\mu 0} - \breve{X}^\mu(\sigma)\eta^{\nu 0}\Big) \tag{4.136}$$

$$[\breve{X}^\mu(\sigma), \breve{\mathcal{P}}^\nu(\sigma')]_{PB} = \Big(1 - \frac{E}{E_p}\Big)\delta(\sigma - \sigma')\eta^{\mu\nu} - \frac{1}{2E_p}\Big(1 - \frac{E}{E_p}\Big)^{-\frac{1}{2}}\breve{\mathcal{P}}^\nu(\sigma')\eta^{\mu 0} \tag{4.137}$$

$$[\breve{\mathcal{P}}^\mu(\sigma), \breve{\mathcal{P}}^\nu(\sigma')]_{PB} = 0 \tag{4.138}$$

Let us take

$$f = g = (1 - L_p E)^{-2} \tag{4.139}$$

so one can write

$$\breve{E} = (1 - L_p E)^{-1} E \tag{4.140}$$

which allows to obtain

$$E = \frac{\breve{E}}{1 + L_p\breve{E}} \tag{4.141}$$

then the phase space Poisson brackets are

$$[\breve{X}^\mu(\sigma), \breve{X}^\nu(\sigma')]_{PB} = L_P(1 + L_P\breve{E})^2\Big(\breve{X}^\nu(\sigma')\eta^{\mu 0} - \breve{X}^\mu(\sigma)\eta^{\nu 0}\Big) \tag{4.142}$$

$$[\breve{X}^\mu(\sigma), \breve{\mathcal{P}}^\nu(\sigma')]_{PB} = (1 + L_P\breve{E})^2\Big(\delta(\sigma - \sigma')\eta^{\mu\nu} + L_P\breve{\mathcal{P}}^\nu(\sigma')\eta^{\mu 0}\Big) \tag{4.143}$$

$$[\breve{\mathcal{P}}^\mu(\sigma), \breve{\mathcal{P}}^\nu(\sigma')]_{PB} = 0 \tag{4.144}$$

Energy dependent string tension:

The string tension T is the only unknown parameter in string theory, but what happens if it is energy dependent. First let us take

$$f(E) = \frac{1}{g(E)} \tag{4.145}$$

so one can find that

$$\sqrt{f} = \frac{1}{\sqrt{g}} \tag{4.146}$$

the Poisson brackets are

$$[\breve{X}^\mu(\sigma), \breve{X}^\nu(\sigma')]_{PB} = \breve{X}^\nu(\sigma')\frac{d\sqrt{g(E)}}{dE}\eta^{\mu 0} - \breve{X}^\mu(\sigma)\frac{d\sqrt{g(E)}}{dE}\eta^{\nu 0} \tag{4.147}$$

$$[\breve{X}^\mu(\sigma), \breve{\mathcal{P}}^\nu(\sigma')]_{PB} = \delta(\sigma - \sigma')\eta^{\mu\nu} - \frac{d\sqrt{g}}{dE}\breve{\mathcal{P}}^\nu(\sigma')\eta^{\mu 0} \tag{4.148}$$

$$[\breve{\mathcal{P}}^\mu(\sigma), \breve{\mathcal{P}}^\nu(\sigma')]_{PB} = 0 \tag{4.149}$$

and the equation of motion is

$$\ddot{\breve{X}}^\mu - \partial_\sigma^2 \breve{X}^\mu - \delta_0^\mu \frac{\partial}{\partial\tau}\int_0^\pi d\sigma' \frac{d\sqrt{g}}{dE}\Big(\frac{1}{T}\breve{\mathcal{P}}^2(\sigma') + T\breve{X}'^2(\sigma')\Big) = 0 \tag{4.150}$$

the non-linear term in the above equation is clearly proportional to the Hamiltonian constraint, so one can find the usual bosonic string equation of motion.

4.8 Paraquantization

The paraquantization of a theory is given by the introduction of trilinear commutation relations between the dynamical variables $X^\mu(\tau, \sigma)$ and $\mathcal{P}^\mu(\tau, \sigma)$ [7]

$$\big[X^\mu(\tau,\sigma), [\mathcal{P}^\nu(\tau,\sigma'), \mathcal{P}^\rho(\tau,\sigma'')]_+\big] = 2ig^{\mu\nu}\mathcal{P}^\rho\delta(\sigma-\sigma') + 2ig^{\mu\rho}\mathcal{P}^\nu\delta(\sigma-\sigma'') \tag{4.151}$$

$$\big[\mathcal{P}^\mu(\tau,\sigma), [X^\nu(\tau,\sigma'), X^\rho(\tau,\sigma'')]_+\big] = -2ig^{\mu\nu}X^\rho\delta(\sigma-\sigma') - 2ig^{\mu\rho}X^\nu\delta(\sigma-\sigma'') \tag{4.152}$$

$$\big[X^\mu(\tau,\sigma), [X^\nu(\tau,\sigma'), \mathcal{P}^\rho(\tau,\sigma'')]_+\big] = 2ig^{\mu\rho}X^\nu\delta(\sigma-\sigma'') \tag{4.153}$$

$$\big[\mathcal{P}^\mu(\tau,\sigma), [X^\nu(\tau,\sigma'), \mathcal{P}^\rho(\tau,\sigma'')]_+\big] = 2ig^{\mu\nu}\mathcal{P}^\rho\delta(\sigma-\sigma') \tag{4.154}$$

In terms of modes and center of mass variables, by the use of the relations (4.73)-(4.75), the trilinear relations (4.151)-(4.154); are equivalent to (Appendix D):

$$\Big[x^\mu, [x^\nu, p^\rho]_+\Big] = 2ifg^{\mu\rho}x^\nu \tag{4.155}$$

$$\Big[x^\mu, [p^\nu, p^\rho]_+\Big] = 2if(g^{\mu\nu}p^\rho + g^{\mu\rho}p^\nu) \tag{4.156}$$

$$\Big[x^\mu, [p^\nu, \alpha_n^\rho]_+\Big] = 2ifg^{\mu\nu}\alpha_n^\rho \tag{4.157}$$

$$\Big[p^\mu, [x^\nu, p^\rho]_+\Big] = -2ifg^{\mu\nu}p^\rho \tag{4.158}$$

$$\Big[p^\mu, [x^\nu, x^\rho]_+\Big] = -2if(g^{\mu\nu}x^\rho + g^{\mu\rho}x^\nu) \tag{4.159}$$

$$\big[\alpha_n^\mu, [\alpha_m^\nu, \alpha_l^\rho]_+\big] = 2\sqrt{\frac{f}{g}}lg^{\mu\nu}\alpha_n^\rho\delta_{m+l,0} + 2\sqrt{\frac{f}{g}}lg^{\mu\rho}\alpha_n^\nu\delta_{m+l,0} \tag{4.160}$$

$$\big[\alpha_n^\mu, [p^\nu, \alpha_m^\rho]_+\big] = 2n\sqrt{\frac{f}{g}}g^{\mu\rho}\delta_{n+m,0}p^\nu \tag{4.161}$$

$$\big[\alpha_n^\mu, [x^\nu, \alpha_m^\rho]_+\big] = 2n\sqrt{\frac{f}{g}}g^{\mu\rho}\delta_{n+m,0}x^\nu \tag{4.162}$$

With the use of Green decomposition:

$$x^\mu = \sum_{\alpha=1}^{Q} x^{\mu(\alpha)} \tag{4.163}$$

$$p^\mu = \sum_{\alpha=1}^{Q} p^{\mu(\alpha)} \tag{4.164}$$

$$\alpha_n^\mu = \sum_{\alpha=1}^{Q} \alpha_n^{\mu(\alpha)} \tag{4.165}$$

where Q is the order of paraquantization, the trilinear relations (4.155)-(4.162) are equivalent to the following bilinear commutations and anticommutation:

$$[x^{\mu(\alpha)}, p^{\nu(\alpha)}] = ifg^{\mu\nu} \tag{4.166}$$

$$[x^{\mu(\alpha)}, p^{\nu(\beta)}]_+ = 0, \qquad \alpha \neq \beta \tag{4.167}$$

$$[x^{\mu(\alpha)}, x^{\nu(\alpha)}] = [p^{\mu(\alpha)}, p^{\nu(\alpha)}] = 0 \tag{4.168}$$

$$[x^{\mu(\alpha)}, x^{\nu(\beta)}]_+ = [p^{\mu(\alpha)}, p^{\nu(\beta)}]_+ = 0 \qquad \alpha \neq \beta \tag{4.169}$$

$$[\alpha_n^{\mu(\alpha)}, \alpha_m^{\nu(\alpha)}] = i\sqrt{\frac{f}{g}}g^{\mu\nu} \tag{4.170}$$

$$[\alpha_n^{\mu(\alpha)}, \alpha_m^{\nu(\beta)}]_+ = 0 \qquad \alpha \neq \beta \tag{4.171}$$

$$[x^{\mu(\alpha)}, \alpha_n^{\nu(\beta)}]_+ = [p^{\mu(\alpha)}, \alpha_n^{\nu(\alpha)}] = 0 \tag{4.172}$$

$$[x^{\mu(\alpha)}, \alpha_n^{\nu(\beta)}]_+ = [p^{\mu(\alpha)}, \alpha_n^{\nu(\beta)}]_+ = 0 \qquad \alpha \neq \beta \tag{4.173}$$

4.8.1 Mass operator for the para-bosonic string

The Virasoro generators became

$$L_n = \frac{\sqrt{fg}}{2} \sum_{m=-\infty}^{+\infty} [\beta^{\mu}_{n-m}, \beta^{\mu}_{m}]_+ \tag{4.174}$$

then

$$L_0 = \frac{\sqrt{fg}}{2} \sum_{m-\infty}^{+\infty} [\beta^{\mu}_{-m}, \beta^{\mu}_{m}]_+ \tag{4.175}$$

and with Green decompositions, one can write

$$L_0 = \frac{\sqrt{fg}}{2} \sum_{\alpha=1}^{Q} \sum_{\beta=1}^{Q} \sum_{m-\infty}^{+\infty} [\beta^{\mu(\alpha)}_{-m}, \beta^{\mu(\beta)}_{m}]_+ \tag{4.176}$$

where

$$\beta^{\mu(\alpha)}_{n} = \left(\frac{g}{f}\right)^{\frac{1}{4}} \alpha^{\mu(\alpha)}_{n}, \quad for \quad n \in \mathbb{Z} \tag{4.177}$$

and satisfies

$$[\beta^{\mu(\alpha)}_{n}, \beta^{\nu(\alpha)}_{m}]_- = n\eta^{\mu\nu}\delta_{n+m} \tag{4.178}$$

$$[\beta^{\mu(\alpha)}_{n}, \beta^{\nu(\alpha)}_{m}]_+ = n\eta^{\mu\nu}\delta_{n+m} \qquad \alpha \neq \beta \tag{4.179}$$

then

$$L_0 = \frac{\sqrt{fg}}{2} \sum_{\alpha=1}^{Q} \sum_{m-\infty}^{+\infty} [\beta^{\mu(\alpha)}_{-m}, \beta^{\mu(\alpha)}_{m}]_+ \tag{4.180}$$

While the Virasoro algebra is:

$$[L_n, L_m]_- = \sqrt{fg}(n-m)L_{n+m} + fg\frac{QD}{12}(n^3 - n)\delta_{n,-m} \tag{4.181}$$

The deformed dispersion relation for the para-bosonic string theories is:

$$p^2 = -\frac{f^{\frac{3}{2}} g^{\frac{1}{2}}}{2\alpha'} M^2 = \frac{f^{\frac{3}{2}} g^{\frac{1}{2}}}{2\alpha'} \sum_{n=1}^{+\infty} [\beta^{\mu}_{-n}, \beta_{\mu,n}]_+ \tag{4.182}$$

This relation characterizes the spectrum of the parabosonic string.

4.9 Conclusion

In this chapter, we introduced the bosonic string theories with deformed relativistic relations between energy and momentum (dispersion relations); We have followed the development in [17]. The modifications encoded in deformed usual bosonic string constraints, which generate reparametrization transformations on the world-sheet depends on energy, and hold a closed classical algebra. The canonical quantization on these theories leads to energy dependent Virasoro central charge, and deformed mass-shell condition. We studied also the possibilities, in such theories, to describe equivalent ones with ordinary bosonic string constraints but in a non-commutative target space-time. Finally, we paraquantised these theories. In the next chapter, we will extend these theories to fermionic string ones.

Chapter 5

Fermionic string theories with deformed energy momentum relations

The bosonic string theories with deformed relativistic dispersion relations was formulated in the previous chapter. In this chapter; which is based on [24], we study the fermionic string extension of the previous bosonic models.

5.1 Square roots of the deformed Virasoro constraints and deformed fermionic string constraints

The Hamiltonian formalism has several options, from one viewpoint, it helps to study the constrained physical systems; notice that string theory is a totally constrained system as the relativistic particle and general relativity, on the other hand, the canonical quantization depends on this formalism, and it is a very elegant method for going from the classical theory to the quantum one [38, 39, 44].

One can study the fermionic string theory as two dimensions supergravity. To do this, and in addition to the bosonic degrees of freedom, one also needs fermionic ones, so let us introduce the real anti-commuting variables $\psi^\mu_a(\sigma)$ (where $a = 1, 2$), which represent Majorana spinors in the world-sheet. After appropriate simplifications [9], one can write the fermionic string action:

$$S_F = \int_0^\pi d\sigma d\tau (\dot{X}^\mu \mathcal{P}_\mu + \bar{\pi}^\mu \dot{\psi}_\mu - N\mathcal{H} - N^\sigma \mathcal{H}_\sigma - \bar{M}\mathcal{S}) \tag{5.1}$$

where $\pi(\psi^\mu_a)$ is the canonical conjugate momentum of ψ^μ, the constraints $\mathcal{H}$ and $\mathcal{H}_\sigma$ generate the reparametrizations, $\mathcal{S}$ is the fermionic constraint and generates the local supersymmetry in the world sheet, and $\bar{M}$ is the fermionic Lagrange multiplier. Let us start our discussion of the extension of the bosonic string; which is given in chapter 4, by introducing the following definition of anticommuting Poisson brackets:

$$\left[\psi^\mu_a(\sigma), \psi^\nu_b(\sigma')\right]_{PB} = -4\pi i \eta^{\mu\nu} \delta_{ab} \delta(\sigma - \sigma') \tag{5.2}$$

Then let us introduce the energy deformed fermionic constraints as the definitions:

$$\mathcal{S}_1(\sigma) = \psi_1^\mu(\sigma) P_\mu(\sigma) \tag{5.3}$$

$$\mathcal{S}_2(\sigma) = \psi_2^\mu(\sigma) S_\mu(\sigma) \tag{5.4}$$

Where P_μ and S_μ are given by (4.20) and (4.21). Notice that the Eqs. (5.3) and (5.4) generate the local supersymmetry transformations in the two dimensions world sheet. Indeed

$$\begin{aligned} \delta\psi_1^\mu(\sigma) &= \Big[\psi_1^\mu(\sigma), \int_0^\pi \epsilon^1(\sigma')\psi_1^\mu(\sigma')P_\mu(\sigma')d\sigma'\Big]_{PB} \\ &= 4\pi i \epsilon^1(\sigma) P^\mu(\sigma) \end{aligned}$$

the same thing holds for ψ_2^μ,

$$\delta\psi_2^\mu(\sigma) = 4\pi i \epsilon^2(\sigma) S^\mu(\sigma) \tag{5.5}$$

while

$$\delta X^\mu(\sigma) = \pi\sqrt{2f}\alpha'\epsilon^a(\sigma)\psi_a^\mu(\sigma) \tag{5.6}$$

With the help of the Poisson brackets (5.2) and (4.22), we can write

$$\begin{aligned} \big[\mathcal{S}_1(\sigma), \mathcal{S}_1(\sigma')\big]_{PB} &= \big[\psi_1^\mu(\sigma)P_\mu(\sigma), \psi_1^\nu(\sigma')P_\nu(\sigma')\big]_{PB} \\ &= \psi_1^\mu(\sigma)\psi_1^\nu(\sigma')\big[P_\mu(\sigma), P_\nu(\sigma')\big]_{PB} + P_\mu(\sigma)P_\nu(\sigma')\big[\psi_1^\mu(\sigma), \psi_1^\nu(\sigma')\big]_{PB} \\ &= \psi_1^\mu(\sigma)\psi_1^\nu(\sigma')\Big(- 2\pi\sqrt{fg}\frac{d\delta(\sigma-\sigma')}{d\sigma'}\eta_{\mu\nu}\Big) \\ &\quad + P_\mu(\sigma)P_\nu(\sigma')\Big(- 4\pi i\eta^{\mu\nu}\delta(\sigma-\sigma')\Big) \\ &= \psi_1^\mu(\sigma)\frac{d\psi_1^\nu(\sigma)}{d\sigma}\Big(2\pi\sqrt{fg}\eta_{\mu\nu}\delta(\sigma-\sigma')\Big) \\ &\quad + P_\mu(\sigma)P_\nu(\sigma')\Big(- 4\pi i\eta^{\mu\nu}\delta(\sigma-\sigma')\Big) \end{aligned}$$

so

$$\big[\mathcal{S}_1(\sigma), \mathcal{S}_1(\sigma')\big]_{PB} = -4\pi i\Big(P_\mu P^\mu + \frac{i}{2}\sqrt{fg}\psi_1^\mu\frac{d\psi_{1\mu}}{d\sigma}\Big)\delta(\sigma-\sigma'). \tag{5.7}$$

The energy deformed Hamiltonian and spatial diffeomorphism constraints on the fermionic string are:

$$\mathcal{H} = \mathcal{H}^{bosonic} + \frac{i\sqrt{fg}}{8\pi}(\psi_1^\mu\frac{d\psi_{1\mu}}{d\sigma} - \psi_2^\mu\frac{d\psi_{2\mu}}{d\sigma}) \tag{5.8}$$

$$\mathcal{H}_\sigma = \mathcal{H}_\sigma^{bosonic} + \frac{i\sqrt{fg}}{8\pi}(\psi_1^\mu\frac{d\psi_{1\mu}}{d\sigma} + \psi_2^\mu\frac{d\psi_{2\mu}}{d\sigma}) \tag{5.9}$$

It is also convenient to define the linear combinations,

$$Q^+(\sigma) = 2\pi(\mathcal{H} + \mathcal{H}_\sigma) = P_\mu P^\mu + \frac{i}{2}\sqrt{fg}\psi_1^\mu \frac{d\psi_{1\mu}}{d\sigma} \tag{5.10}$$

$$Q^-(\sigma) = 2\pi(\mathcal{H} - \mathcal{H}_\sigma) = S_\mu S^\mu - \frac{i}{2}\sqrt{fg}\psi_2^\mu \frac{d\psi_{2\mu}}{d\sigma} \tag{5.11}$$

Then the right-hand side of (5.7) is clearly proportional to Q^+. For example one can also verify that

$$\begin{aligned}
\left[Q^+(\sigma), \mathcal{S}_1(\sigma')\right]_{PB} &= \left[P_\mu(\sigma)P^\mu(\sigma) + \frac{i}{2}\sqrt{fg}\psi_1^\mu(\sigma)\frac{d\psi_{1\mu}(\sigma)}{d\sigma}, \psi_1^\nu(\sigma')P_\nu(\sigma')\right]_{PB} \\
&= 2\psi_1^\nu(\sigma')P^\mu(\sigma)\Big[P_\mu(\sigma), P_\nu(\sigma')\Big]_{PB} + \frac{i}{2}\sqrt{fg}P_\nu(\sigma')\Big[\psi_1^\mu(\sigma)\frac{d\psi_{1\mu}(\sigma)}{d\sigma}, \psi_1^\nu(\sigma')\Big]_{PB} \\
&= 2\psi_1^\mu(\sigma')P_\mu(\sigma)\big(2\pi\sqrt{fg}\frac{d\delta(\sigma-\sigma')}{d\sigma}\big) \\
&\quad + \frac{i}{2}\sqrt{fg}P_\mu(\sigma')\Bigg(\psi_1^\mu(\sigma)\Big(-4\pi i\frac{d\delta(\sigma-\sigma')}{d\sigma}\Big) - \frac{d\psi_1^\mu(\sigma)}{d\sigma}\Big(-4\pi i\delta(\sigma-\sigma')\Big)\Bigg) \\
&= 2\pi\sqrt{fg}\big(2\psi_1(\sigma)\cdot P(\sigma) + \psi_1(\sigma')\cdot P(\sigma')\big)\frac{d\delta(\sigma-\sigma')}{d\sigma}
\end{aligned}$$

One then arrives at:

$$\Big[Q^+(\sigma), \mathcal{S}_1(\sigma')\Big]_{PB} = 2\pi\sqrt{fg}\Big(2\mathcal{S}_1(\sigma) + \mathcal{S}_1(\sigma')\Big)\delta'(\sigma-\sigma') \tag{5.12}$$

and

$$\Big[Q^+(\sigma), Q^+(\sigma')\Big]_{PB} = 4\pi\sqrt{fg}\Big(Q^+(\sigma) + Q^+(\sigma')\Big)\delta'(\sigma-\sigma'), \tag{5.13}$$

and likewise for the constraints $\mathcal{S}_2(\sigma)$ and $Q^-(\sigma)$, where the super-algebra is

$$\big[\mathcal{S}_2(\sigma), \mathcal{S}_2(\sigma')\big]_{PB} = -4\pi i Q^- \delta(\sigma-\sigma') \tag{5.14}$$

$$\Big[Q^-(\sigma), \mathcal{S}_2(\sigma')\Big]_{PB} = -2\pi\sqrt{fg}\Big(2\mathcal{S}_2(\sigma) + \mathcal{S}_2(\sigma')\Big)\delta'(\sigma-\sigma') \tag{5.15}$$

$$\Big[Q^-(\sigma), Q^-(\sigma')\Big]_{PB} = -4\pi\sqrt{fg}\Big(Q^-(\sigma) + Q^-(\sigma')\Big)\delta'(\sigma-\sigma') \tag{5.16}$$

The expressions (5.7), (5.12), (5.13), (5.14), (5.15), and (5.16) construct a consistent system of constraints; which is akin to the first class constraints system with the energy dependent factors which are represented by the functions f and g.

5.2 The equations of motion and solutions

5.2.1 Equations of motion

The hamiltonian constraint (5.8) generates the dynamics of string , so for the bosonic field X^μ, the equations of motion can be given by the Poisson brackets,

$$\begin{aligned}\dot{X}^0 &= \int_0^\pi d\sigma' [X^0(\sigma), \mathcal{H}(\sigma')]_{PB} \\ &= \frac{f}{T}\mathcal{P}^0(\sigma) + \int_0^\pi d\sigma' \bigg(\frac{\mathcal{P}^2}{2T}\frac{df}{dE} + \frac{T}{2}X'^2\frac{dg}{dE} \\ &\qquad + \frac{i}{8\pi}\frac{d\sqrt{fg}}{dE}(\psi_1^\mu \frac{d\psi_{1\mu}}{d\sigma'} - \psi_2^\mu \frac{d\psi_{2\mu}}{d\sigma'}) \bigg)\end{aligned}$$

and

$$\begin{aligned}\dot{\mathcal{P}}^0(\sigma) &= \int_0^\pi d\sigma' [\mathcal{P}^0(\sigma), \mathcal{H}(\sigma')]_{PB} \\ &= Tg(E)\partial_\sigma^2 X^0\end{aligned}$$

the same method reads,

$$\dot{X}^i(\sigma) = \frac{f(E)}{T}\mathcal{P}^i(\sigma) \tag{5.17}$$

$$\dot{\mathcal{P}}^i(\sigma) = Tg(E)\partial_\sigma^2 X^i \tag{5.18}$$

where $i = 1, \ldots, D-1$ denotes the spatial index. So one obtains

$$\begin{aligned}\dot{X}^\mu = \frac{f}{T}\mathcal{P}^\mu + \delta_0^\mu \int d\sigma' \bigg(\frac{\mathcal{P}^2}{2T}\frac{df}{dE} + \frac{T}{2}X'^2\frac{dg}{dE} \\ + \frac{i}{8\pi}\frac{d\sqrt{fg}}{dE}(\psi_1^\mu \frac{d\psi_{1\mu}}{d\sigma'} - \psi_2^\mu \frac{d\psi_{2\mu}}{d\sigma'}) \bigg)\end{aligned} \tag{5.19}$$

$$\dot{\mathcal{P}}^\mu(\sigma) = Tg(E)\partial_\sigma^2 X^\mu \tag{5.20}$$

The nonlinear term in (5.19) can be eliminated if $f = g$. There are, however, other possibilities if

$$\begin{cases} \frac{df}{dE} = hf(E) \\ \frac{dg}{dE} = hg(E) \end{cases} \tag{5.21}$$

with the solutions:

$$\begin{cases} f = k_f e^{hE} \\ g = k_g e^{hE} \end{cases} \tag{5.22}$$

where h, k_f and k_g are constants. The equations of motion became as in (4.51). And with the help of (5.2), one can read

$$\begin{aligned}\frac{\partial\psi_1^\mu(\sigma,\tau)}{\partial\tau} &= \int_0^\pi d\sigma'\Big[\psi_1^\mu(\sigma,\tau),\mathcal{H}(\sigma')\Big]_{PB}\\ &= \sqrt{fg}\frac{\partial\psi_1^\mu(\sigma,\tau)}{\partial\sigma}\end{aligned}$$

So

$$(\partial_\tau - \sqrt{fg}\partial_\sigma)\psi_1^\mu = 0 \tag{5.23}$$

and the same method leads to

$$(\partial_\tau + \sqrt{fg}\partial_\sigma)\psi_2^\mu = 0 \tag{5.24}$$

Here, $\sqrt{fg}$ represents an energy dependent speed of propagation of the fermionic and bosonic waves in the world-sheet; as mentioned in [17] for the bosonic string case.

Therefore, the definition:

$$\sigma^\pm = \sqrt{fg}\tau \pm \sigma \tag{5.25}$$

leads to

$$\frac{\partial}{\partial\sigma^\pm} = \frac{1}{2}\left(\frac{1}{\sqrt{fg}}\frac{\partial}{\partial\tau} \pm \frac{\partial}{\partial\sigma}\right) \tag{5.26}$$

thus

$$\begin{cases}\partial_-\psi_1^\mu(\tau,\sigma) = 0\\ \partial_+\psi_2^\mu(\tau,\sigma) = 0\end{cases} \tag{5.27}$$

which means that, ψ_1^μ depends only on σ^+, while ψ_2^μ depends on σ^-.

5.2.2 Boundary conditions

Open string

The boundary conditions are important for writing the solutions . The functionals $\int_0^\pi N\mathcal{H}d\sigma$, $\int_0^\pi N^\sigma\mathcal{H}_\sigma d\sigma$, and $\int_0^\pi \bar{M}\mathcal{S}d\sigma$ must be well defined as generators, which must be weakly zero (weak equality in Dirac terminology) [38, 39, 44], and it must be also a constant of motion. Let us mention that the variation of the action equals zero, and this property leads to the equations of motion, the constraints, and boundary conditions. The first variation is

$$\begin{aligned}\delta\int_0^\pi d\sigma N\mathcal{H} = \int_0^\pi d\sigma N\Bigg(&\delta\mathcal{H}^{bosonic} + \frac{i}{8\pi}\sqrt{fg}\delta\psi_1^\mu\frac{d\psi_{1\mu}}{d\sigma}\\ &- \frac{i}{8\pi}\sqrt{fg}\delta\psi_2^\mu\frac{d\psi_{2\mu}}{d\sigma} + \frac{i}{8\pi}\sqrt{fg}\psi_1^\mu\frac{d\delta\psi_{1\mu}}{d\sigma}\\ &- \frac{i}{8\pi}\sqrt{fg}\psi_2^\mu\frac{d\delta\psi_{2\mu}}{d\sigma}\Bigg)\end{aligned} \tag{5.28}$$

In general, the N and N^σ functions relate to the world-sheet metric; and the physics is independent of this metric. So in the conformal gauge $N = 1$ and $N^\sigma = 0$, at the boundaries.

Thus the surface term which is given by $\mathcal{H}^{bosonic}$ can be eliminated by the same boundary conditions of the bosonic string in chapter 4.

Let us now focus on the surface terms which is related to ψ^μ and M. Here we follow the same method; concerning the usual fermionic open string, shown in [9]. The surface term of ψ^μ is obtained from the second part of the right hand side of the equation (5.28),

$$\left[\sqrt{fg}N\Big(\psi_{1\mu}\delta\psi_1^\mu - \psi_{2\mu}\delta\psi_2^\mu\Big)\right]_0^\pi \tag{5.29}$$

the root $\sqrt{fg}$ is a multiplicative factor, we obtain then the usual Ramond ψ^μ boundary conditions

$$\psi_1^\mu(0) = \psi_2^\mu(0) \tag{5.30}$$

$$\psi_1^\mu(\pi) = \psi_2^\mu(\pi) \tag{5.31}$$

and the Neveu-Schwarz ones

$$\psi_1^\mu(0) = \psi_2^\mu(0) \tag{5.32}$$

$$\psi_1^\mu(\pi) = -\psi_2^\mu(\pi) \tag{5.33}$$

With the same method, the variation $\delta(N^\sigma\mathcal{H}_\sigma)$ gives the boundary term

$$\Big[N^\sigma \mathcal{P}'_\mu \delta X^\mu\Big]_0^\pi = 0 \tag{5.34}$$

This surface term vanishes because $N^\sigma = 0$ at the boundaries, one can remember that N and N^σ verify the same boundary conditions, as in the ordinary bosonic string [9], what remains is the contribution of $\delta\int_0^\pi \bar{M}\mathcal{S}d\sigma = \delta\int_0^\pi (M^1\mathcal{S}_1 + M^2\mathcal{S}_2)d\sigma$, which gives:

$$\int_0^\pi d\sigma\Big[M^1\Big(\delta\psi_1\cdot P + \sqrt{\frac{\pi f}{T}}\psi_1\cdot\delta\mathcal{P} + \sqrt{\pi T g}\psi_1^\mu\cdot\delta X'\Big) \\ + M^2\Big(\delta\psi_2\cdot S + \sqrt{\frac{\pi f}{T}}\psi_2\cdot\delta\mathcal{P} - \sqrt{\pi T g}\psi_2\cdot\delta X'\Big)\Big]$$

So the surface term can be written after partial integration of the third and the sixth terms in the last expression as follows,

$$\sqrt{\pi T g}\Big[(M^1\psi_1^\mu - M^2\psi_2^\mu)\delta X_\mu\Big]_0^\pi \tag{5.35}$$

which can be vanished if one takes $\delta X_\mu = 0$ in the boundaries or

$$M^1 = M^2 \quad at \quad \sigma = 0 \tag{5.36}$$

$$M^1 = M^2(R) \quad or \quad M^1 = -M^2(NS) \quad at \quad \sigma = \pi \tag{5.37}$$

which are consistent with the R and NS boundary conditions on ψ_1^μ and ψ_2^μ.

Closed string

For eliminating the surface term concerning the closed string, one can analyze both periodicity and anti-periodicity of ψ_a^μ [9], So

$$\begin{cases} \psi_1^\mu(\sigma = 0) = \psi_1^\mu(\sigma = 2\pi) \\ \psi_2^\mu(\sigma = 0) = \psi_2^\mu(\sigma = 2\pi) \end{cases} \tag{5.38}$$

$$\begin{cases} \psi_2^\mu(\sigma = 0) = -\psi_1^\mu(\sigma = 2\pi) \\ \psi_2^\mu(\sigma = 0) = \psi_2^\mu(\sigma = 2\pi) \end{cases} \tag{5.39}$$

$$\begin{cases} \psi_1^\mu(\sigma = 0) = -\psi_1^\mu(\sigma = 2\pi) \\ \psi_2^\mu(\sigma = 0) = -\psi_2^\mu(\sigma = 2\pi) \end{cases} \tag{5.40}$$

The building of the closed string theory is akin to what is already given in the open string. The main differences are achieved by the boundary conditions, and the extended range of σ, which leads to double the modes and constraints compared with the open string. The closed string coordinates satisfy the periodicity condition $X^\mu(\sigma) = X^\mu(\sigma + 2\pi)$.

5.2.3 Solutions

Open string

In the case of the Neumann boundary conditions X^μ and its momentum conjugate $\mathcal{P}^\mu$ are given by (4.63) and (4.64) respectively, While the solutions in Dirichlet-Dirichlet boundary conditions are given by (4.65).

From now on, we will consider the only Neumann-Neumann boundary conditions in the open string case. The Fourier mode expansions of ψ_a^μ, in the Neveu-Schwarz sector are:

$$\psi_1^\mu(\tau, \sigma) = \sqrt{2} \sum_s b_s^\mu e^{-is(\sqrt{fg}\tau + \sigma)} \tag{5.41}$$

$$\psi_2^\mu(\tau, \sigma) = \sqrt{2} \sum_s b_s^\mu e^{-is(\sqrt{fg}\tau - \sigma)} \tag{5.42}$$

with $s = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$
From (5.2), one can find the following Poisson brackets between modes;

$$[b_s^\mu, b_r^\nu]_{PB} = -i\eta^{\mu\nu}\delta_{s,-r} \tag{5.43}$$

In the Ramond sector, we can write:

$$\psi_1^\mu(\tau, \sigma) = \sqrt{2} \sum_n d_n^\mu e^{-in(\sqrt{fg}\tau + \sigma)} \tag{5.44}$$

$$\psi_2^\mu(\tau, \sigma) = \sqrt{2} \sum_n d_n^\mu e^{-in(\sqrt{fg}\tau - \sigma)} \tag{5.45}$$

with $n = 0, \pm 1, \pm 2, \ldots$, and

$$[d_n^\mu, d_m^\nu]_{PB} = -i\eta^{\mu\nu}\delta_{n,-m} \tag{5.46}$$

Closed string

Using the equation (5.27), and the conditions (5.38), (5.39), and (5.40), one can give the modes expansion of the closed string fermionic variables ψ_a^μ as follows,

$$\psi_1^\mu(\sigma,\tau) = \sqrt{2}\sum_r d_r^\mu e^{-ir(\sqrt{fg}\tau+\sigma)} \tag{5.47}$$

$$\psi_2^\mu(\sigma,\tau) = \sqrt{2}\sum_r \tilde{d}_r^\mu e^{-ir(\sqrt{fg}\tau-\sigma)} \tag{5.48}$$

So the Ramond sector; as in the open string case, gives,: $r = 0, \pm 1, \pm 2, \ldots$, while with the Neveu-Schwarz sector, we have: $r = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$.
In the closed string the right and left movers are independent, so if one of them takes the modes of one sector the other one can take the modes of any sector, therefore, we can distinguish four cases,

- NS in the left and NS in the right is space-time bosons.
- R in the left and R in the right is space-time bosons.
- NS in the left and R in the right is space-time fermions.
- R in the left and NS in the right is space-time bosons.

5.3 Classical super-Virasoro generators

The super-Virasoro generators are the Fourier modes of the original deformed constraints of the fermionic string, for the open string we have:

$$L_n = \frac{1}{4\pi}\int_{-\pi}^{+\pi} d\sigma e^{in\sigma} Q^+(\sigma) \tag{5.49}$$

$$G_s = \frac{1}{2\pi\sqrt{2}}\int_{-\pi}^{+\pi} d\sigma e^{is\sigma} \mathcal{S}_1(\sigma) \quad (Neveu - Schwarz) \tag{5.50}$$

$$F_n = \frac{1}{2\pi\sqrt{2}}\int_{-\pi}^{+\pi} d\sigma e^{in\sigma} \mathcal{S}_1(\sigma) \quad (Ramond) \tag{5.51}$$

Therefore, these generators in terms of modes can be obtained by using the solutions (4.63), (4.64), (5.44), (5.45), (5.41), and (5.42), and the definition (4.81) as follows

$$L_n = \frac{g}{2}\sum_{m=-\infty}^{+\infty} \alpha_{n-m}\cdot\alpha_m + \frac{\sqrt{fg}}{2}\sum_{s\in\mathbb{Z}+\frac{1}{2}} (s+\frac{n}{2}) b_{-s}\cdot b_{n+s} \tag{5.52}$$

$$G_s = \sqrt{g}\sum_{m=-\infty}^{+\infty} b_{m+s}\cdot\alpha_m \tag{5.53}$$

for the Neveu-Schwars sector, and

$$L_n = \frac{g}{2} \sum_{m=-\infty}^{+\infty} \alpha_{n-m} \cdot \alpha_m + \frac{\sqrt{fg}}{2} \sum_{m \in \mathbb{Z}} (m + \frac{n}{2}) d_{-m} \cdot d_{n+m} \tag{5.54}$$

$$F_n = \sqrt{g} \sum_{m=-\infty}^{+\infty} d_{m+n} \cdot \alpha_m \tag{5.55}$$

for Ramond sector.
The algebra of these classical constraints became:

$$[G_r, G_s] = -2iL_{r+s} \tag{5.56}$$

$$[L_m, G_r] = i\sqrt{fg}(r - \frac{m}{2})G_{m+r} \tag{5.57}$$

$$[F_m, F_n] = -2iL_{m+n} \tag{5.58}$$

$$[L_m, F_n] = i\sqrt{fg}(n - \frac{m}{2})F_{m+n} \tag{5.59}$$

$$[L_m, L_n] = i\sqrt{fg}(n - m)L_{m+n} \tag{5.60}$$

Again the above super-Virasoro algebra is closed and energy deformed.

5.4 Canonical quantization and super-Virasoro algebras

In terms of modes, the commutator (4.72) gives again (4.77) and (4.78) and the anti-commutators;

$$\left[\psi_a^\mu(\sigma,\tau), \psi_b^\nu(\sigma',\tau)\right]_+ = 4\pi\eta^{\mu\nu}\delta_{ab}\delta(\sigma - \sigma') \tag{5.61}$$

gives

$$[b_s^\mu, b_r^\nu]_+ = \eta^{\mu\nu}\delta_{s,-r} \tag{5.62}$$

for the NS sector, and

$$[d_n^\mu, d_m^\nu]_+ = \eta^{\mu\nu}\delta_{n,-m} \tag{5.63}$$

for the R sector.
Notice that the anti-commutators (5.62) and (5.63) follow the usual form of fermionic string theory.

Again the redefinitions [17], (4.79) and (4.80) and the help of (4.81), one readily shows that

$$[x^\mu, \tilde{p}_\nu]_- = i\left(\frac{f}{g}\right)^{\frac{1}{4}}\delta_\nu^\mu \tag{5.64}$$

$$[\beta_n^\mu, \beta_m^\nu]_- = n\eta^{\mu\nu}\delta_{n+m} \tag{5.65}$$

Notice again that the choices $f = g$ and (5.22) lead to eliminate the energy deformations from the equation (5.64).
By the use of the quantum version of the generators (5.52-5.55) and the redefinitions

$$L_n = \sqrt{fg}\tilde{L}_n, \tag{5.66}$$

$$G_s = f^{\frac{1}{4}} g^{\frac{1}{4}} \tilde{G}_s \tag{5.67}$$

$$F_n = f^{\frac{1}{4}} g^{\frac{1}{4}} \tilde{F}_n \tag{5.68}$$

The generators $\tilde{L}_n$, $\tilde{G}_s$, and $\tilde{F}_n$ satisfy the ordinary super-Virasoro algebra, so that this latter takes the form

$$[L_n, L_m]_- = \sqrt{fg}(n-m)L_{n+m} + fg\frac{D}{8}(n^3 - n)\delta_{n,-m} \tag{5.69}$$

$$[G_r, G_s]_+ = 2L_{r+s} + \sqrt{fg}\frac{D}{2}(r^2 - \frac{1}{4})\delta_{r,-s} \tag{5.70}$$

$$[L_m, G_r]_- = -\sqrt{fg}(r - \frac{m}{2})G_{m+r} \tag{5.71}$$

for the NS sector.

One can also obtain the following modified super-Virasoro algebra

$$[L_n, L_m]_- = \sqrt{fg}(n-m)L_{n+m} + fg\frac{D}{8}n^3\delta_{n,-m} \tag{5.72}$$

$$[F_n, F_m]_+ = 2L_{n+m} + \sqrt{fg}\frac{D}{2}n^2\delta_{n,-m} \tag{5.73}$$

$$[L_m, F_n]_- = -\sqrt{fg}(n - \frac{m}{2})F_{m+n} \tag{5.74}$$

for the Ramond sector. The super-algebras are deformed by the presence of the functions f and g. For positive and non-vanishing fg, the conventional super-Virasoro algebra with energy independent anomaly can be used to describe the physical states, which are as the usual ones, except that the center of mass energy and momentum are modified. Whereas, for finite p^0 where $fg = 0$, we can see (as in [17]) that the central charges are energy dependent.

5.5 Spectrum and GSO projection

From the Hamiltonian relation (5.8) and with the solutions (4.63), (4.64), (5.44), (5.45), (5.41), and (5.42), the Hamiltonian constraints, which are identical to L_0 in both sectors (5.52) and (5.54), can be obtained as follows

$$H = H_{bosonic} + \sqrt{fg}\frac{1}{2}\sum_{s\in\mathbb{Z}+\frac{1}{2}} s : b^{\mu}_{-s}b_{s,\mu} : \tag{5.75}$$

for the NS sector, and

$$H = H_{bosonic} + \sqrt{fg}\frac{1}{2}\sum_{n\neq 0} n : d^{\mu}_{-n}d_{n,\mu} : \tag{5.76}$$

for the R sector, where

$$H_{bosonic} = \frac{p^2}{2\pi fT} + \frac{g}{2}\sum_{n\neq 0} : \alpha_n^\mu \alpha_{n,\mu} : \tag{5.77}$$

Let us use the same definition of the mass squared as suggested in [17]

$$M^2 = -\frac{p^2}{fg}, \tag{5.78}$$

In the fermionic string case, this latter allows a same factor in the bosonic (in terms of β_n^μ) and the fermionic parts of the mass squared expression so that we can use the usual results for the spectrum. Indeed

$$M_{NS}^2 = \pi T \big[\sum_{n\neq 0} : \alpha_n^\mu \alpha_{\mu,n} : + \sqrt{\frac{f}{g}} \sum_{s\in\mathbb{Z}+\frac{1}{2}} s : b_{-s}^\mu b_{s,\mu} : \big] \tag{5.79}$$

$$M_R^2 = (\pi T) \sum_{n\neq 0} \big[: \alpha_n^\mu \alpha_{n,\mu} : + \sqrt{\frac{f}{g}} n : d_{-n}^\mu d_{n,\mu} : \big] \tag{5.80}$$

for NS and R sectors respectively, or with the redefinition (4.79) one can write

$$M_{NS}^2 = \pi T \sqrt{\frac{f}{g}} \big[\sum_{n\neq 0} : \beta_n^\mu \beta_{n,\mu} : + \sum_{s\in\mathbb{Z}+\frac{1}{2}} s : b_{-s}^\mu b_{s,\mu} : \big] \tag{5.81}$$

$$M_R^2 = \pi T \sqrt{\frac{f}{g}} \sum_{n\neq 0} \big[: \beta_n^\mu \beta_{n,\mu} : + n : d_{-n}^\mu d_{n,\mu} : \big] \tag{5.82}$$

We can conclude then that, in terms of the β_n^μ modes, the above relations have the same form of ordinary fermionic string mass squared in both sectors but with the energy dependent factor $\sqrt{\frac{f}{g}}$.

5.5.1 NS sector

In addition to the Hamiltonian condition, the physical states must obey the following equations

$$G_r|\phi\rangle = 0,\ r > 0; \quad L_n|\phi\rangle = 0,\ n \geq 1 \tag{5.83}$$

and the quantum vacuum satisfies

$$\beta_n^\mu |0,p\rangle = \left(\frac{g}{f}\right)^{\frac{1}{4}} \alpha_n^\mu |0,p\rangle = 0 \quad for \quad n \geq 1 \tag{5.84}$$

$$b_s^\mu |0,p\rangle = 0 \quad for \quad n \geq \frac{1}{2} \tag{5.85}$$

The spectrum of the NS sector is like a generalization of the bosonic string one. So the first few levels are written in the following general functional,

$$|\phi\rangle = (\Phi + i b_{-\frac{1}{2}}^\mu A_\mu + i\beta_{-1}^\mu B_\mu + \frac{1}{2} b_{-\frac{1}{2}}^\mu b_{-\frac{1}{2}}^\nu S_{\mu\nu} + \dots)|0,p\rangle \tag{5.86}$$

from the L_0 constraints, the dispersion relation is writing as

$$-\frac{p^2}{fg} = M_{NS}^2 = (2\pi T)\sqrt{\frac{f}{g}}[N_{NS} + a_{NS}] \tag{5.87}$$

where

$$N_{NS} = \sum_{n>0} \beta_{-n}^{\mu}\beta_{n,\mu} + \sum_{s\in\mathbb{Z}+\frac{1}{2}>0} sb_{-s}^{\mu}b_{s,\mu} \tag{5.88}$$

let us use the relation (4.82) which satisfies the expression

$$\tilde{p}^2 + \tilde{M}_{NS}^2 = 0 \tag{5.89}$$

the above equation has the form of the ordinary mass shell condition where

$$\tilde{M}_{NS}^2 = (2\pi T)[N_{NS} + a_{NS}] \tag{5.90}$$

Notice here that the relations (5.87) and (5.88) suggest the usual value $a_{NS} = -\frac{1}{2}$, so that the spectrum obtained is inspired by the ordinary fermionic string one.

The application of the Hamiltonian condition gives

$$(l^2\partial^2 + f\sqrt{fg}\frac{1}{2})\phi = 0 \tag{5.91}$$

$$\partial^2 A_\mu = 0 \tag{5.92}$$

$$(l^2\partial^2 - f\sqrt{fg}\frac{1}{2})S_{\mu\nu} = (l^2\partial^2 - f\sqrt{fg}\frac{1}{2})B_\mu = 0 \tag{5.93}$$

where $l^2 = \frac{1}{(2\pi T)}$.
The application of the constraint condition $G_{\frac{1}{2}}|\phi\rangle = 0$ leads to,

$$\partial^\mu A_\mu = 0 \tag{5.94}$$

$$\frac{\sqrt{2}l}{f^{\frac{3}{4}}g^{\frac{1}{4}}}\partial^\mu S_{\mu\nu} - B_\nu = 0 \tag{5.95}$$

From the relations (5.92) and (5.94), one can note that the massless vector state seems to be similar to the conventional fermionic string one with $D-2$ independent components; while the other ground and massive states are affected by the energy deformation (see the relations (5.91), (5.93), and (5.95)). Let us try to calculate the physical degrees of freedom of the vector state $N_{NS} = \frac{1}{2}$ (we follow [41]), with (5.93) and (5.95), which are equivalent to $p^2 = 0$ and $p_\mu A^\mu = 0$ respectively. The choice of the Lorentz frame where $p_\mu = (p_0, p_0, 0, \ldots, 0)$ is possible in the case of massless vector state, so one can takes [1],

$$A^0 + A^1 = 0 \tag{5.96}$$

The above expression leaves $D-1$ independent components in polarization vector; but we can also define the state

$$\frac{1}{\sqrt{2\alpha'}}i\epsilon G_{-\frac{1}{2}}|0,p\rangle = (ip_\mu\epsilon)b_{-\frac{1}{2}}^{\mu}|0,p\rangle \tag{5.97}$$

[1] because $p_0A^0 + p_0A^1 = 0$

this equation has the form of the vector state with a polarization $ip_\mu\epsilon$ and if $p^2 = 0$ the following equivalence can be written:

$$A_\mu b^\mu_{-\frac{1}{2}}|0,p\rangle \equiv (A_\mu + ip_\mu\epsilon)b^\mu_{-\frac{1}{2}}|0,p\rangle \tag{5.98}$$

let us use $p_\mu = (p_0, p_0, 0, \ldots, 0)$, which gives,

$$A^0 \equiv A^0 - ip_0\epsilon \tag{5.99}$$

$$A^1 \equiv A^1 + ip_0\epsilon \tag{5.100}$$

so we can find

$$A^0 - A^1 \equiv A^0 - A^1 - 2ip_0\epsilon \tag{5.101}$$

and with appropriate choice we can write

$$A^0 - A^1 = 0 \tag{5.102}$$

then we say that the vector state has $D-2$ independent polarization.

In the other hand, the zero norm state

$$G_{-\frac{1}{2}}|\Omega\rangle \tag{5.103}$$

is a physical state with the conditions.

$$G_r|\Omega\rangle = 0, \ r > 0 \tag{5.104}$$

and

$$L_n|\Omega\rangle = 0, \ n \geq 0 \tag{5.105}$$

One can also construct another zero norm physical state:

$$|\psi\rangle = (G_{-\frac{3}{2}} + aL_{-1}G_{-\frac{1}{2}})|\Omega'\rangle \tag{5.106}$$

where the state $|\Omega'\rangle$ satisfies the following conditions

$$(L_0 + \sqrt{fg})|\Omega'\rangle = 0 \tag{5.107}$$

$$L_n|\Omega'\rangle = 0, \ n \geq 1 \tag{5.108}$$

$$G_r|\Omega'\rangle = 0, \ r \geq \frac{1}{2} \tag{5.109}$$

The constraint $G_{\frac{1}{3}}$ acts on $|\psi\rangle$ and gives:

$$G_{\frac{1}{2}}|\psi\rangle = (2 - \sqrt{fg}a)L_1|\Omega'\rangle, \tag{5.110}$$

so we can find that:

$$a = \frac{2}{\sqrt{fg}}. \tag{5.111}$$

We can use the constraint $G_{\frac{3}{2}}$ on the above state:

$$\begin{aligned} G_{\frac{3}{2}}(G_{-\frac{3}{2}} + aL_{-1}G_{-\frac{1}{2}})|\Omega'\rangle &= (G_{\frac{3}{2}}G_{-\frac{3}{2}} + G_{\frac{3}{2}}L_{-1}G_{-\frac{1}{2}})|\Omega'\rangle \\ &= (2L_0 + \sqrt{fg}D + a4L_0)|\Omega'\rangle \end{aligned} \tag{5.112}$$

$$|\psi\rangle = \sqrt{fg}(-2 + D - 8)|\Omega'\rangle \tag{5.113}$$

which implies that $D = 10$.

5.5.2 Ramond sector

The physical states obey the constraints conditions

$$L_n|\psi\rangle = 0 \tag{5.114}$$

and

$$F_n|\psi\rangle = 0 \tag{5.115}$$

for $n \geq 0$, while the ground state is defined by

$$\beta_n^\mu|0,p\rangle_R = \left(\frac{g}{f}\right)^{\frac{1}{4}}\alpha_n^\mu|0,p\rangle_R = 0 \tag{5.116}$$

$$d_n^\mu|0,p\rangle_R = 0 \tag{5.117}$$

also for $n \geq 0$.

The spinor wave functional is given by

$$|\Phi\rangle_\epsilon = (\lambda_\epsilon(x) + i\beta_{-1}^\mu\Psi_{\mu\epsilon}^1(x) + d_{-1}^\mu\Psi_{\mu\epsilon}^2(x) + \dots)|0;p\rangle \tag{5.118}$$

where $\epsilon = 1, 2, \dots, 2^{\frac{D}{2}}$.

The energy dispersion relation is

$$-\frac{p^2}{fg} = M_R^2 = (2\pi T)\sqrt{\frac{f}{g}}N_R \tag{5.119}$$

where

$$N_R = \sum_{n>0}\beta_{-n}^\mu\beta_{n,\mu} + \sum_{m\in\mathbb{Z}>0} m d_{-m}^\mu d_{m,\mu} \tag{5.120}$$

Let us apply the F_0 and F_1 conditions

$$F_0|\Phi\rangle_\epsilon = \sqrt{g}(\alpha_0^\mu\frac{\Gamma_\mu}{\sqrt{2}} + \alpha_{-1}^\mu d_{1\mu} + \alpha_1^\mu d_{-1\mu} + \dots)|\Phi\rangle_\epsilon = 0 \tag{5.121}$$

$$F_1|\Phi\rangle_\epsilon = \sqrt{g}(\alpha_0^\mu d_{1\mu} + \alpha_1^\mu\frac{\Gamma_\mu}{\sqrt{2}} + \dots)|\Phi\rangle_\epsilon = 0 \tag{5.122}$$

where the zero mode $d_0^\mu = \frac{1}{\sqrt{2}}\Gamma^\mu$. We obtain the following equations

$$\partial\!\!\!/\lambda = 0, \tag{5.123}$$

$$\frac{l}{f^{\frac{3}{4}}g^{\frac{1}{4}}}\partial\!\!\!/\Psi_\mu^1 = \Psi_\mu^2,\ -\frac{l}{f^{\frac{3}{4}}g^{\frac{1}{4}}}\partial\!\!\!/\Psi_\mu^2 = \Psi_\mu^1, \dots \tag{5.124}$$

and

$$-\frac{l}{f^{\frac{3}{4}}g^{\frac{1}{4}}}\partial^\mu\Psi_\mu^2 - \frac{\Gamma^\mu}{\sqrt{2}}\Psi_\mu^1 = 0 \tag{5.125}$$

where $\partial\!\!\!/ = \Gamma^\mu\partial_\mu$.

The massless state seems to be like the ordinary fermionic string one, but the behavior of the other states depends on the energy functions.

5.5.3 GSO projection

Let us recall that the mass squared operators in (5.87) and (5.119) keep the usual form in terms of β_n and the fermionic modes unchanged, but with the multiplicative factor $\sqrt{\frac{f}{g}}$, so in $D = 10$ the GSO projection appears to be not affected by the deformation, and the two projectors, which act on the NS and R sectors, have also the same form as the ordinary GSO ones. We find the same known results: the possibility of the tachyon elimination, mass levels with the factor $\sqrt{\frac{f}{g}}$ are well defined, and the space-time supersymmetry is preserved with such deformation. Indeed, in our case, the degrees of freedom appears to be similar to the ordinary fermionic string ones. One can see this precisely if we take the non-linear map (4.82), we obtain the conventional form of the constraints equations.
After the GSO projection, the ground state of the open fermionic string contains a massless spin-1 state

$$b^{\mu}_{-\frac{1}{2}}|0,p\rangle \tag{5.126}$$

and a massless spin-$\frac{1}{2}$ state

$$|\epsilon,p\rangle_{s,c} \tag{5.127}$$

as a Majorana spinor with a well defined chirality, while the next state is massive and described by

$$\{\beta^{\mu}_{-1}, b^{\mu}_{-\frac{1}{2}}b^{\nu}_{-\frac{1}{2}}\}|0;p\rangle \tag{5.128}$$

in the NS sector, and

$$\{\beta^{\mu}_{-1}, d^{\mu}_{-1}\}|\epsilon,p\rangle_{s,c} \tag{5.129}$$

in the R sector.

Of course, as in the bosonic case [17], the precedent reasoning supposes that $f\sqrt{fg} > 0$, is non-vanishing and non-singular, but in otherwise the ghosts must not be reintroduced into the theory by a choice of functions.

5.6 Examples

Let us first restrict ourself with the condition $f = g$, and take the example studied in the bosonic string case [17]

$$f^2(p_0) = 1 - (L_P p_0)^2 \tag{5.130}$$

The dispersion relation can be written as follows

$$p^2 + f^2 M^2 = 0 \tag{5.131}$$

in the rest reference frame, we obtain

$$E^2_{NS} = \frac{N_{NS} - \frac{1}{2}}{l^2 + L_p^2(N_{NS} - \frac{1}{2})} \tag{5.132}$$

for the NS sector and

$$E^2_{R} = \frac{N_R}{l^2 + L_p^2 N_R} \tag{5.133}$$

for the R sector.

Then lets see the first few levels, first in the NS sector

- $N_{NS} = 0$
$$E_{NS}^2 = \frac{1}{L_p^2 - 2l^2} > \frac{1}{L_P^2} \tag{5.134}$$
this state is non-tachyonic if $L_P > \sqrt{2}l$.
- $N_{NS} = \frac{1}{2}$
$$E_{NS}^2 = 0 \tag{5.135}$$
- $N_{NS} = 1$
$$E_{NS}^2 = \frac{1}{L_p^2 + 2l^2} < \frac{1}{L_P^2} \tag{5.136}$$
- $N_{NS} \geq \frac{3}{2}$
$$E_{NS}^2 < \frac{1}{L_P^2} \tag{5.137}$$

second in the R sector

- $N_R = 0$
$$E_R^2 = 0 \tag{5.138}$$
- $N_R = 1$
$$E_R^2 = \frac{1}{l^2 + L_P^2} < \frac{1}{L_P^2} \tag{5.139}$$

We note that the ground state in the NS sector can be non-tachyonic if $L_P > \sqrt{2}l$, in the same way as in the result found in the bosonic string case [17], but in the same time, it is the only state in the two sectors which can have a rest energy greater than E_P, while the other stats accumulate below L_P^{-1}, (see (5.132) and (5.133) where $\lim_{N\to+\infty} E_{NS} = \lim_{N\to+\infty} E_R = \frac{1}{L_P}$). In the other hand, the NS ground state has no equivalent state in the R sector, so that, after the GSO projection, all states accumulate below the Planck energy, and we can perform the first steps toward a space-time supersymmetric string theory with the deformed dispersion relation (5.131) and the function (5.130).

Other examples concern the cases (5.22), so in the rest frame, we can write

$$-E^2 + \frac{\lambda}{l^2}(N_{NS} - \frac{1}{2})\exp(2hE) = 0 \tag{5.140}$$

for the NS sector, and

$$-E^2 + \frac{\lambda}{l^2}N_R\exp(2hE) = 0 \tag{5.141}$$

for the R sector, where λ is a real positive number. Then, we can write the general form of the string energy with the Lambert W function, as follows

$$E_{nNS} = -\frac{1}{h}W\Big(\pm\frac{h}{l}\sqrt{\lambda(N_{NS} - \frac{1}{2})}\Big) \tag{5.142}$$

$$E_{nR} = -\frac{1}{h}W(\pm\frac{h}{l}\sqrt{\lambda N_R}). \tag{5.143}$$

5.7 Poincaré algebra

Let us consider the Lorentz generators

$$M^{\mu\nu} = \int_0^\pi (X^\mu \mathcal{P}^\nu - X^\nu \mathcal{P}^\mu) d\sigma + \frac{1}{4\pi i}\int_0^\pi \sum_a \psi_a^\mu \psi_a^\nu d\sigma \tag{5.144}$$

in terms of modes, one can write

$$M^{\mu\nu} = J^{\mu\nu} + I^{\mu\nu} \tag{5.145}$$

where

$$J^{\mu\nu} = \frac{1}{f}(x^\nu p^\mu - x^\mu p^\nu) - i\sqrt{\frac{g}{f}}\sum_{n>1}\frac{1}{n}(\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\mu \alpha_n^\nu) \tag{5.146}$$

$$I^{\mu\nu} = -i\sum_{s>0}(b_{-s}^\mu b_s^\nu - b_{-s}^\nu b_s^\mu) \tag{5.147}$$

for the Neveu-Schwarz sector and

$$I^{\mu\nu} = -i\sum_{n>0}(d_{-n}^\mu d_n^\nu - d_{-n}^\nu d_n^\mu) - i d_0^\mu d_0^\nu \tag{5.148}$$

for the Ramond sector.
By the use of (4.72) and (5.61), one can find the usual Poincaré algebra,

$$[p^\mu, p^\nu] = 0 \tag{5.149}$$

$$[p^\mu, M^{\nu\rho}] = i\eta^{\mu\rho}p^\nu - i\eta^{\mu\nu}p^\rho \tag{5.150}$$

$$[M^{\mu\nu}, M^{\rho\zeta}] = i\eta^{\mu\rho}M^{\nu\zeta} + i\eta^{\nu\zeta}M^{\mu\rho} - i\eta^{\nu\rho}M^{\mu\zeta} - i\eta^{\mu\zeta}M^{\nu\rho} \tag{5.151}$$

where

$$p^\mu = \int_0^\pi d\sigma \mathcal{P}^\mu(\sigma) \tag{5.152}$$

is the total momentum of the string.
Notice here that, while both the bosonic modes of the string and the center of mass propagate with deformed commutation relation (4.77) and (4.78), and the presence of the deformation functions f and g in (5.146), the Poincaré algebra is still unchanged.

5.8 Paraquantization

In addition to the parabosonic string trilenear commutation relations (4.151)-(4.154) which are given above, we need the following ones [45, 46]:

$$\left[\psi_a^\mu(\sigma), [\psi_b^\nu(\sigma'), \psi_c^\rho(\sigma'')]_-\right] = 8\pi\Big(\eta^{\mu\nu}\delta_{ab}\psi_c^\rho(\sigma'')\delta(\sigma-\sigma') - \eta^{\mu\rho}\delta_{ac}\psi_b^\nu(\sigma')\delta(\sigma-\sigma'')\Big) \tag{5.153}$$

$$\left[\psi_a^\mu(\sigma), [\psi_b^\nu(\sigma'), X^\rho(\sigma'')]_+\right]_+ = 8\pi\eta^{\mu\nu}\delta_{ab}X^\rho(\sigma'')\delta(\sigma-\sigma') \tag{5.154}$$

$$\left[\psi_a^\mu(\sigma), [\psi_b^\nu(\sigma'), \mathcal{P}^\rho(\sigma'')]_+\right]_+ = 8\pi\delta_{ab}\eta^{\mu\nu}\mathcal{P}^\rho(\sigma'')\delta(\sigma-\sigma') \tag{5.155}$$

$$\left[X^\mu(\sigma), [\psi_a^\nu(\sigma'), \mathcal{P}^\rho(\sigma'')]_+\right] = 2i\eta^{\mu\rho}\psi_a^\nu(\sigma')\delta(\sigma-\sigma'') \tag{5.156}$$

$$\left[\mathcal{P}^\mu(\sigma), [\psi_a^\nu(\sigma'), X^\rho(\sigma'')]_+\right] = -2i\eta^{\mu\rho}\psi_a^\nu(\sigma')\delta(\sigma-\sigma'') \tag{5.157}$$

Now, in order to compute the trilinear commutation relations of the modes we need to use,

$$b_r^\mu = \frac{1}{2\sqrt{2}\pi}\int_0^\pi d\sigma\big(\psi_1^\mu(\tau,\sigma)e^{ir(\sqrt{fg}\tau+\sigma)} + \psi_2^\mu(\tau,\sigma)e^{ir(\sqrt{fg}\tau-\sigma)}\big) \tag{5.158}$$

for Neveu-Schwarz boundary conditions, and

$$d_n^\mu = \frac{1}{2\sqrt{2}\pi}\int_0^\pi d\sigma\big(\psi_1^\mu(\tau,\sigma)e^{in(\sqrt{fg}\tau+\sigma)} + \psi_2^\mu(\tau,\sigma)e^{in(\sqrt{fg}\tau-\sigma)}\big) \tag{5.159}$$

for Ramond boundary conditions. Then

$$\left[\alpha_n^\mu, [\alpha_m^\nu, \alpha_l^\rho]_+\right] = 2\sqrt{\frac{f}{g}}(\eta^{\mu\nu}n\delta_{n+m,0}\alpha_l^\rho + \eta^{\mu\rho}n\delta_{n+l,0}\alpha_m^\nu) \tag{5.160}$$

$$\left[\alpha_n^\mu, [\alpha_m^\nu, A_i^\rho]_+\right] = 2n\sqrt{\frac{f}{g}}\eta^{\mu\nu}\delta_{n+m,0}A_i^\rho \tag{5.161}$$

$$\left[x^\mu, [p^\nu, p^\rho]_+\right] = 2if(\eta^{\mu\nu}p^\rho + \eta^{\mu\rho}p^\nu) \tag{5.162}$$

$$\left[x^\mu, [p^\nu, B_i^\rho]_+\right] = 2i\eta^{\mu\nu}fB_i^\rho \tag{5.163}$$

$$\left[p^\mu, [x^\nu, x^\rho]_+\right] = -2if(\eta^{\mu\nu}x^\rho + \eta^{\mu\rho}x^\nu) \tag{5.164}$$

$$\left[p^\mu, [C_i^\nu, x^\rho]_+\right] = -2if\eta^{\mu\nu}C_i^\nu \tag{5.165}$$

$$\left[b_r^\mu, [b_s^\nu, b_q^\rho]_-\right] = 2(\eta^{\mu\nu}\delta_{r+s,0}b_q^\rho - \eta^{\mu\rho}\delta_{r+q,0}b_s^\nu) \tag{5.166}$$

$$\left[b_n^\mu, [D_i^\mu, d_m^\rho]_+\right]_+ = 2\eta^{\mu\nu}\delta_{n+m,0}D_i^\nu \tag{5.167}$$

$$\left[b_n^\mu, [D_i^\mu, D_j^\rho]_+\right] = 0 \tag{5.168}$$

$$\left[D_i^\mu, [d_m^\mu, d_m^\rho]_-\right] = 0 \tag{5.169}$$

where the operators A_i^μ, B_i^μ, C_i^μ and D_i^μ represent the elements of the sets:

$$A^\mu = \{x^\mu, p^\mu, b_n^\mu\} \tag{5.170}$$

$$B^\mu = \{x^\mu, \alpha_n^\mu, b_n^\mu\} \tag{5.171}$$

$$C^\mu = \{p^\mu, \alpha_n^\mu, b_n^\mu\} \tag{5.172}$$

and

$$D^\mu = \{x^\mu, p_n^\mu, \alpha_n^\mu\} \tag{5.173}$$

respectively. And the same relations are satisfied by the modes d_n^μ.

General conclusion

The main goal of this dissertation is to give a possible modifications of the usual string theory, and emphasizing the differences and the identifications. In chapter 3 we explored the open bosonic string with dust field; which completes the work on closed one which is developed in [3]. The quantization of this bosonic string one; and with the help of the no-ghost theorem, leads to the possibility of non-critical string theory. We found that the algebra of the diffeomorphism generators is consistent and anomaly free. And unlike the closed string with dust where the equality of the anomaly terms of the Virasoro algebras must be used for consistent diffeomorphism algebra, the open string case can eliminate automatically the anomaly of the algebra. In this stage, we also study the bosonic string with dust field stretched between two parallel Dp-branes.

In chapter 4, we have studied the bosonic string theories with deformed dispersion relations, and we have shown that such models can be used to describe a non-commutative space-time. We paraquantize these models and have found the deformed dispersion relations.

In chapter 5, we have used the square root of the deformed bosonic string constraints [17] to get the modified mass-shell condition concerning fermionic string theories. These constraints are also redefined to fit the fermionic models by providing the consistency of the deformed constraints super-algebra. We have also found that the local supersymmetry transformations in the world-sheet depend on energy. We have obtained the equations of motions of the bosonic space-time coordinates, and found that they are linear for the two cases $f = g$ or $f' = hf$ and $g' = hg$ where h is a constant, while the fermionic coordinates ones are originally linear. We have also studied the open fermionic string surface terms of the constraints to get the Neumann and Dirichlet boundary conditions and define the R and NS sectors. We followed the canonical quantization method, the obtained super-Virasoro algebra has energy dependent central charges.

For $f = g$ where f is real, non-vanishing, and non-singular, we found that the usual results of fermionic string theory remain existing; as well as the tachyonic ground state of the NS sector, and the non-tachyonic spectrum in the R sector, and also the GSO projection is still possible for obtaining something like a space-time supersymmetry.

Notice here that there is a possibility of deformations; where $f^2 < 0$ near the string energy scale, leads to the non-tachyonic manifestation of the NS ground state, however the action became complex, and this poses a problem of a unitarity violation. In this context, the choice (5.130) seems to give a theory without ghosts, as usual, where the $N = 0$ state is no more a tachyon, but it has a rest energy greater than L_P^{-1}. The GSO projection gives a theory with space-time supersymmetry, and makes all the states levels below the energy scale L_P^{-1}, which implies that f^2 became always positive.

Appendix A

Vielbein

The general linear group $GL(D, \mathbb{R})$ is the group of diffeomorphism transformations in D dimensions (like in general relativity where $D = 4$). But no spinor representation is possible in such group, which makes the addition of fermionic degrees of freedom difficult. The subject became different in flat space-time, where the Lorentz group has spinor representations.

The general relativity for example lives in a manifold M, which is locally flat, so we can use this idea to find the spinor representation in the group $gl(D, \mathbb{R})$ with the expense of introducing a new gauge fields, which represent an orthonormal vector space $e^{\mu}_{m}(x)$ on every tangent space $T_x(M)$ on the point x in M. Which satisfies

$$g_{\mu\nu} e^{\mu}_{m} e^{\nu}_{n} = \eta_{mn} \tag{A.1}$$

where η_{mn} is the metric of Minkowski space, and e^{μ}_{m} is called a "Vielbein".
In the other hand, the cotangent space $T^{*}_{x}(M)$ has the orthonormal basis $e^{m}_{\mu}(x)$, which are the dual vectors space of $e^{\mu}_{m}(x)$, so

$$e^{\mu}_{m} e^{m}_{\nu} = g^{\mu}_{\nu} = \delta^{\mu}_{\nu} \tag{A.2}$$

$$e^{m}_{\mu} e^{\mu}_{n} = \eta^{m}_{n} = \delta^{m}_{n} \tag{A.3}$$

The vielbein verifies also the space-time coordinate transformations:

$$e^{\mu'}_{m} = \frac{\partial x^{\mu\prime}}{\partial x^{\mu}} e^{\mu}_{m} \tag{A.4}$$

Notice that the Lorentz transformation is satisfied for e^{μ}_{m}:

$$e^{\mu}_{m'} = \Lambda^{m}_{m'} e^{\mu}_{m} \tag{A.5}$$

where $\Lambda^{m'}_{m}$ is the Lorentz transformation in Minkowski space. And also the relation (A.1) is preserved under this Lorentz transformation:

$$g_{\mu\nu} e^{\mu}_{m'} e^{\nu}_{n'} = \eta_{m'n'}$$

In the other hand, the currant of the spinor ψ must satisfy the following vector transformation in space-time:

$$\bar{\psi}(x') \rho^{\mu'} \psi(x') = \frac{\partial x^{\mu\prime}}{\partial x^{\mu}} \bar{\psi}(x) \rho^{\mu} \psi(x) \tag{A.6}$$

where $\frac{\partial x^{\mu\prime}}{\partial x^{\mu}}$ can be represented by $D \times D$ invertible matrix which is an element of the group $GL(D, \mathbb{R})$. While in flat space-time the vector transformation is satisfied:

$$\bar{\psi}(x')\rho^{m'}\psi(x') = \Lambda^{m'}_{\ m}\bar{\psi}(x)\rho^{m}\psi(x) \tag{A.7}$$

where

$$[\rho^n, \rho^m]_+ = 2\eta^{nm} \tag{A.8}$$

Furthermore the spinor transformation are

$$\psi(x') = S\psi(x) \tag{A.9}$$

$$\bar{\psi}(x') = \bar{\psi}(x)S^{-1} \tag{A.10}$$

Notice that, the Eq. (A.7) leads to

$$S^{-1}\rho^{m'}S = \Lambda^{m'}_{\ m}\rho^{m} \tag{A.11}$$

where $S(\Lambda)$ is a spinorial representation of the Lorentz group. We can introduce the following definition:

$$\rho^{\mu}(x) = e^{\mu}_{m}(x)\rho^{m} \tag{A.12}$$

which satisfies (A.6) and

$$[\rho^{\mu}(x), \rho^{\nu}(x)]_+ = 2g^{\mu\nu}(x) \tag{A.13}$$

The above algebra is akin to the Clifford algebra which can characterize the spinor representation, and with the use of (A.12),the relation (A.6) is satisfied (for further details see [47]).

Appendix B

Nonlinearity in Φ^4 model

As an example, let us consider the action of the ϕ^4 model in two dimensions [48]:

$$S = \int dx^0 dx^1 \big[\frac{1}{2}(\partial_0 \Phi)^2 - \frac{1}{2}(\partial_1 \Phi)^2 - V(\Phi)\big] \tag{B.1}$$

where th potential term is

$$V(\Phi) = \frac{1}{4}\lambda(\Phi^2 + \frac{m^2}{\lambda})^2 \tag{B.2}$$

λ and m are constants. In this stage the Hamiltonian is,

$$H = \int dx^1 \big[\frac{1}{2}(\partial_0 \Phi)^2 + \frac{1}{2}(\partial_1 \Phi)^2 + V(\Phi)\big] \tag{B.3}$$

and its equation of motion,

$$\partial_\mu \partial^\mu \Phi - m^2 \Phi - \lambda \Phi^3 = 0. \tag{B.4}$$

If we neglect the interaction term $\lambda\Phi^3$ the above equation becomes linear as its solution and if λ is small we can use the perturbation method. The whole solution of the equation (B.4) with the interaction term can be for example a soliton (the Dp-brane is a soliton in string theory) which is a non-linear solution.

Appendix C

Poisson brackets transformations in the phase space

Let us start with a very brief overview on the geometric description of the phase space. This issue is similar to the Riemannian or pseudo-Riemannian manifold [49]. The metric in the Riemannian space usually reads

$$G_{\rho\sigma} = \frac{\partial X^{\mu}}{\partial x^{\rho}} \frac{\partial X^{\nu}}{\partial x^{\sigma}} \eta_{\mu\nu} \tag{C.1}$$

where $X^{\mu}(x^{\rho})$ are functions of the coordinates x^{ρ}. Let us use the following antisymmetric tensor

$$\omega^{\mu\nu} = \{x^{\mu}, x^{\nu}\} \tag{C.2}$$

which represents Poisson bracket in the phase space, then one can write the transformations

$$\bar{\omega}^{\mu\nu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}} \omega^{\alpha\beta}. \tag{C.3}$$

If we take the phase space coordinates (x^{μ}, p^{μ}) (which can be canonical or non-canonical), the Poisson bracket of two functions of these coordinates F and G [26, 44] satisfies

$$\{F(x,p), G(x,p)\} = \{F(x,p), G(x,p)\}_{SPB}\{x^{\mu}, p^{\nu}\} + \frac{\partial F}{\partial x^{\mu}} \frac{\partial G}{\partial x^{\nu}} \{x^{\mu}, x^{\nu}\} + \frac{\partial F}{\partial p^{\mu}} \frac{\partial G}{\partial p^{\nu}} \{p^{\mu}, p^{\nu}\}$$

where

$$\{F(x,p), G(x,p)\}_{SPB}\{x^{\mu}, p^{\nu}\} = \left(\frac{\partial F}{\partial x^{\mu}} \frac{\partial G}{\partial p^{\nu}} - \frac{\partial F}{\partial p^{\nu}} \frac{\partial G}{\partial x^{\mu}} \right) \{x^{\mu}, p^{\nu}\} \tag{C.4}$$

is the standard Poisson bracket. So the extension to the continuous systems or classical field theory gives

$$\begin{aligned} \{F(x(\sigma), p(\sigma)), G(x(\sigma'), p(\sigma'))\} = & \{F(x,p), G(x,p)\}_{SPB}\{x^{\mu}(a), p^{\nu}(b)\} + \\ & \int_0^{\pi} \int_0^{\pi} da db \frac{\partial F}{\partial x^{\mu}(a)} \frac{\partial G}{\partial x^{\nu}(b)} \{x^{\mu}(a), x^{\nu}(b)\} \\ & + \frac{\partial F}{\partial p^{\mu}(a)} \frac{\partial G}{\partial p^{\nu}(b)} \{p^{\mu}(a), p^{\nu}(b)\} \end{aligned} \tag{C.5}$$

with

$$\{F(x,p),G(x,p)\}_{SPB}\{x^{\mu}(a),p^{\nu}(b)\} = \int_0^{\pi}\int_0^{\pi} dadb\left(\frac{\partial F}{\partial x^{\mu}(a)}\frac{\partial G}{\partial p^{\nu}(b)} - \frac{\partial F}{\partial p^{\nu}(b)}\frac{\partial G}{\partial x^{\mu}(a)}\right)\{x^{\mu}(a),p^{\nu}(b)\} \tag{C.6}$$

Appendix D

Trilinear relations of the modes

D.1 Calculation of $\left[\alpha_l^\mu, [\alpha_m^\nu, \alpha_n^\rho]_+\right]$:

$$\begin{aligned}\left[\alpha_l^\mu, [\alpha_m^\nu, \alpha_n^\rho]_+\right] &= \frac{1}{(\pi l_s)^3}\int_0^\pi\int_0^\pi\int_0^\pi d\sigma d\sigma' d\sigma'' \cos(l\sigma)\cos(m\sigma')\cos(n\sigma'') \\ &\left[\Big(\sqrt{\frac{f}{g}}\frac{\mathcal{P}^\mu}{T} - ilX^\mu(\sigma)\Big), \left[\Big(\sqrt{\frac{f}{g}}\frac{\mathcal{P}^\nu}{T} - imX^\nu(\sigma')\Big), \Big(\sqrt{\frac{f}{g}}\frac{\mathcal{P}^\rho}{T} - inX^\rho(\sigma'')\Big)\right]_+\right] \\ &= \frac{1}{(\pi l_s)^3}\int_0^\pi\int_0^\pi\int_0^\pi d\sigma d\sigma' d\sigma'' \cos(l\sigma)\cos(m\sigma')\cos(n\sigma'') K\end{aligned}$$

where

$$\begin{aligned}K =& \Bigg\{\sqrt{\frac{f}{g}}\frac{1}{T}\Big(\frac{f}{gT^2}K_1 - in\sqrt{\frac{f}{g}}\frac{1}{T}K_2 - im\sqrt{\frac{f}{g}}\frac{1}{T}K_3 - mnK_4\Big) \\ &- il\Big(\frac{f}{gT^2}K_5 - in\sqrt{\frac{f}{g}}\frac{1}{T}K_6 - im\sqrt{\frac{f}{g}}\frac{1}{T}K_7 - mnK_8\Big)\Bigg\}\end{aligned}$$

$$\begin{aligned}K =& \Big(\frac{f}{g}\Big)^{\frac{3}{2}}\frac{1}{T^3}K_1 - in\frac{f}{g}\frac{1}{T^2}K_2 - im\frac{f}{g}\frac{1}{T^2}K_3 - mn\sqrt{\frac{f}{g}}\frac{1}{T}K_4 \\ &- il\frac{f}{gT^2}K_5 - lm\sqrt{\frac{f}{g}}\frac{1}{T}K_6 - lm\sqrt{\frac{f}{g}}\frac{1}{T}K_7 + ilmnK_8\end{aligned}$$

where

$$K_1 = \left[\mathcal{P}^\mu(\sigma), [\mathcal{P}^\nu(\sigma'), \mathcal{P}^\rho(\sigma'')]_+\right] = 0 \tag{D.1}$$

$$K_2 = \left[\mathcal{P}^\mu(\sigma), [\mathcal{P}^\nu(\sigma'), X^\rho(\sigma'')]_+\right] = 2ig^{\mu\rho}\mathcal{P}^\nu\delta(\sigma-\sigma'') \tag{D.2}$$

$$K_3 = \left[\mathcal{P}^\mu(\sigma), [X^\nu(\sigma'), \mathcal{P}^\rho(\sigma'')]_+\right] = 2ig^{\mu\nu}\mathcal{P}^\rho\delta(\sigma-\sigma'') \tag{D.3}$$

$$K_4 = \left[\mathcal{P}^\mu(\sigma), [X^\nu(\sigma'), X^\rho(\sigma'')]_+\right] = -2ig^{\mu\nu}X^\rho\delta(\sigma-\sigma') - 2ig^{\mu\rho}X^\nu\delta(\sigma-\sigma'') \tag{D.4}$$

$$K_5 = \left[X^\mu(\sigma), [\mathcal{P}^\nu(\sigma'), \mathcal{P}^\rho(\sigma'')]_+\right] = 2ig^{\mu\nu}\mathcal{P}^\rho\delta(\sigma-\sigma') + 2ig^{\mu\rho}\mathcal{P}^\nu\delta(\sigma-\sigma'') \tag{D.5}$$

$$K_6 = \big[X^\mu(\sigma), [\mathcal{P}^\nu(\sigma'), X^\rho(\sigma'')]_+\big] = 2ig^{\mu\nu}X^\rho\delta(\sigma-\sigma') \tag{D.6}$$

$$K_7 = \big[X^\mu(\sigma), [X^\nu(\sigma'), \mathcal{P}^\rho(\sigma'')]_+\big] = 2ig^{\mu\rho}X^\nu\delta(\sigma-\sigma'') \tag{D.7}$$

$$K_8 = \big[X^\mu(\sigma), [X^\nu(\sigma'), X^\rho(\sigma'')]_+\big] = 0 \tag{D.8}$$

for example

$$\begin{aligned}
&\frac{1}{(\pi l_s)^3}\int_0^\pi\int_0^\pi\int_0^\pi d\sigma d\sigma' d\sigma'' \cos(l\sigma)\cos(m\sigma')\cos(n\sigma'') \\
&\qquad(-in\frac{f}{g}\frac{1}{T^2}K_2 - im\sqrt{\frac{f}{g}}\frac{1}{T}K_7) \\
&\qquad= \frac{1}{(\pi l_s)^3}\int_0^\pi\int_0^\pi\int_0^\pi d\sigma d\sigma' d\sigma'' \cos(l\sigma)\cos(m\sigma')\cos(n\sigma'') \\
&\qquad(-in\frac{f}{g}\frac{1}{T^2}2ig^{\mu\rho}\mathcal{P}^\nu\delta(\sigma-\sigma'') - lm\sqrt{\frac{f}{g}}\frac{1}{T}2ig^{\mu\rho}X^\nu\delta(\sigma-\sigma'')) \qquad (D.9) \\
&\qquad= \sqrt{\frac{f}{g}}g^{\mu\rho}\delta_{ln}n\frac{1}{\pi l_s}\int_0^\pi d\sigma'\cos(m\sigma')\left(\sqrt{\frac{f}{g}}\frac{\mathcal{P}^\nu}{T} - miX^\nu\right) \\
&\qquad= \sqrt{\frac{f}{g}}g^{\mu\rho}\delta_{ln}n\alpha_m^\nu
\end{aligned}$$

finally one can write,

$$\big[\alpha_l^\mu, [\alpha_m^\nu, \alpha_n^\rho]_+\big] = 2\sqrt{\frac{f}{g}}lg^{\mu\nu}\alpha_n^\rho\delta_{m+l,0} + 2\sqrt{\frac{f}{g}}lg^{\mu\rho}\alpha_n^\nu\delta_{m+l,0} \tag{D.10}$$

D.2 Calculation of $\big[x^\mu, [p^\nu, \alpha_n^\rho]_+\big]$:

$$\begin{aligned}
\big[x^\mu, [p^\nu, \alpha_n^\rho]_+\big] =& \frac{1}{\pi^3 T l_s^3}f\int_0^\pi\int_0^\pi\int_0^\pi d\sigma d\sigma' d\sigma'' \cos(n\sigma'') \\
&\big[X^\mu(\sigma), [\mathcal{P}^\nu(\sigma'), \sqrt{\frac{f}{g}}\frac{\mathcal{P}^\rho(\sigma'')}{T} - inX^\rho(\sigma'')]_+\big] \\
&= \frac{1}{\pi^3 T l_s^3}f\int_0^\pi\int_0^\pi\int_0^\pi d\sigma d\sigma' d\sigma'' \cos(n\sigma'') \\
&\frac{1}{T}\sqrt{\frac{f}{g}}\big[2ig^{\mu\nu}\mathcal{P}^\rho\delta(\sigma-\sigma') + 2ig^{\mu\rho}\mathcal{P}^\nu\delta(\sigma-\sigma'')\big] + 2g^{\mu\nu}X^\rho\delta(\sigma-\sigma') \\
&= \frac{1}{\pi^3 T l_s^3}f\int_0^\pi\int_0^\pi\int_0^\pi d\sigma d\sigma' d\sigma'' \cos(n\sigma'') \\
&\Big(\frac{1}{T}\sqrt{\frac{f}{g}}2ig^{\mu\nu}\mathcal{P}^\rho\delta(\sigma-\sigma') + 2ng^{\mu\nu}X^\rho\delta(\sigma-\sigma')\Big) \\
&= 2ifg^{\mu\nu}\alpha^\rho
\end{aligned}$$

D.3 Calculation of $\left[\alpha_n^\mu, [p^\nu, \alpha_m^\rho]_+\right]$:

$$\begin{aligned}
\left[\alpha_n^\mu, [p^\nu, \alpha_m^\rho]_+\right] &= \Big(\frac{1}{\pi l_s}\Big)^2 \frac{1}{T l_s^2 \pi} \int_0^\pi \int_0^\pi \int_0^\pi d\sigma d\sigma' d\sigma'' \cos(n\sigma)\cos(m\sigma'') \\
&\quad \Big[\sqrt{\frac{f}{g}}\frac{\mathcal{P}^\mu(\sigma)}{T} - inX^\mu(\sigma), [f\mathcal{P}^\mu(\sigma'), \sqrt{\frac{f}{g}}\frac{\mathcal{P}^\rho(\sigma'')}{T} - imX^\rho(\sigma'')]_+\Big] \\
&= \Big(\frac{1}{\pi l_s}\Big)^2 \frac{1}{T l_s^2 \pi} \int_0^\pi \int_0^\pi \int_0^\pi d\sigma d\sigma' d\sigma'' \cos(n\sigma)\cos(m\sigma'') \\
&\quad \Big\{ - imf\sqrt{\frac{f}{g}}\frac{1}{T}\Big(2ig^{\mu\rho}\mathcal{P}^\nu \delta(\sigma-\sigma'')\Big) - inf\sqrt{\frac{f}{g}}\frac{1}{T}\Big(2ig^{\mu\nu}\mathcal{P}^\rho\delta(\sigma-\sigma') \\
&\quad + 2ig^{\mu\rho}\mathcal{P}^\nu\delta(\sigma-\sigma'')\Big) - inmf\Big(2ig^{\mu\nu}X^\rho\delta(\sigma-\sigma')\Big)\Big\} \\
&= 2n\sqrt{\frac{f}{g}} g^{\mu\rho}\delta_{n+m,0}p^\nu
\end{aligned}$$

D.4 Calculation of $\left[\alpha_n^\mu, [x^\nu, \alpha_m^\rho]_+\right]$:

$$\begin{aligned}
\left[\alpha_n^\mu, [x^\nu, \alpha_m^\rho]_+\right] &= \Big(\frac{1}{\pi l_s}\Big)^2 \int_0^\pi \int_0^\pi \int_0^\pi d\sigma d\sigma' d\sigma'' \cos(n\sigma)\cos(m\sigma'') \\
&\quad \Big[\sqrt{\frac{f}{g}}\frac{\mathcal{P}^\mu(\sigma)}{T} - inX^\mu(\sigma), [X^\nu(\sigma'), \sqrt{\frac{f}{g}}\frac{\mathcal{P}^\rho(\sigma'')}{T} - imX^\rho(\sigma'')]_+\Big] \\
&= \Big(\frac{1}{\pi l_s}\Big)^2 \int_0^\pi \int_0^\pi \int_0^\pi d\sigma d\sigma' d\sigma'' \cos(n\sigma)\cos(m\sigma'') \\
&\quad \Big\{ \frac{f}{g}\frac{1}{T^2}\Big(2ig^{\mu\nu}\mathcal{P}^\rho\delta(\sigma-\sigma')\Big) - in\sqrt{\frac{f}{g}}\frac{1}{T}\Big(2ig^{\mu\rho}X^\nu\delta(\sigma-\sigma'')\Big) \\
&\quad - im\sqrt{\frac{f}{g}}\frac{1}{T}\Big(- 2ig^{\mu\nu}X^\rho\delta(\sigma-\sigma') - 2ig^{\mu\rho}X^\nu\delta(\sigma-\sigma'')\Big)\Big\} \\
&= 2n\sqrt{\frac{f}{g}} g^{\mu\rho}\delta_{n+m,0}x^\nu
\end{aligned}$$

Bibliography

[1] I. Nikitin, *Quantization of Non-Critical Bosonic Open String Theory*, [arXiv:hep-th/0401002].

[2] K. Becker, M. Becker and J H. Schwarz, *String Theory and M-Theory A Modern Introduction*, Cambridge University Press, New York, USA, (2007).

[3] J. Gegenberg, V. Husain, *Bosonic string theory with dust*, Class. Quantum Grav. **31**, 075008 (2014), [arXiv:1212.0271v2].

[4] V. Husain and T. Pawłowski,*Time and a physical Hamiltonian for quantum gravity*, Phys. Rev. Lett. **108**, 141301 (2012), [arXiv:1108.1145v2].

[5] Y. Ohnuki and S. Kamefuchi, *Quantum Field Theory and parastatistics*, Springer-Verlag, (1982).

[6] F. Ardalan and F. Mansouri, *Quantum theory of dual relativistic parastring models*, Phys. Rev. D **9**, 3341 (1974).

[7] N. Belaloui and H. Bennacer, *Poincaré Algebra and Space-Time Critical Dimensions for Parabosonic Strings*, Czech. J. Phys. **53**, 769 (2003), [arXiv:hep-th/0210191v2] .

[8] N. Belaloui and H. Bennacer, *Poincaré Algebra and Space-Time Critical Dimensions for Paraspinnings Strings*, Czech. J.**54**, 621 (2004),[arXiv:hep-th/0210255v2].

[9] L. Brink and M. Henneaux, *Principles of string theory*, New York: Plenum, (1988).

[10] G. Amelino-Camelia, J. Ellis, N.E. Mavromatos and D.V. Nanopoulos, *Distance Measurement and Wave Dispersion in a Liouville-String Approach to Quantum Gravity*, Int. J. Mod.Phys. A **12**, 607 (1997), [arXiv: hep-th/9605211v1].

[11] G. 't Hooft, *How quantization of gravity leads to a discrete space-time*, J. Phys. 701, 012014 (2016).

[12] J. Magueijo, *New varying speed of light theories*, Rep. Prog. Phys. **66**, 2025 (2003), [arXiv:astro-ph/0305457v3].

[13] G. Amelino-Camelia, *Relativity: Special treatment*, Nature (London) **418**, 34 (2002); G. Amelino-Camelia, *Doubly Special Relativity*, [arXiv:gr-qc/0207049v1].

[14] G. Amelino-Camelia, *Relativity in space-times with short-distance structure governed by an observer-independent (Planckian) length scale*, Int. J. Mod. Phys. D **11**, 35 (2002), [arXiv:gr-qc/0012051].

[15] G. Amelino-Camelia and T. Piran, *Cosmic rays and TeV photons as probes of quantum properties of space-time*, Phys. Lett. B **497**, 265 (2001), [arXiv:hep-ph/0006210v1].

[16] J. Maguejo and L. Smolin, *Lorentz invariance with an invariant energy scale*, Phys. Rev. Lett. **88**, 190403 (2002), [arXiv:hep-th/0112090v2].

[17] J. Magueijo and L. Smolin, *String theories with deformed energy momentum relations, and a possible non-tachyonic bosonic string*. Phys. Rev. D **71**, 026010 (2005), [arXiv:hep-th/0401087v2].

[18] A. F. Ali, *Minimal Length in Quantum Gravity Equivalence Principle and Holographic Entropy Bound*, Classical Quantum Gravity **28**, 065013 (2011), [arXiv:1101.4181v1].

[19] A. F. Ali, S. Das and E. C. Vagenas, *Proposal for testing Quantum Gravity in lab*, Phys. Rev. D **84**, 044013 (2011), [arXiv:1107.3164v2].

[20] L. Perivolaropoulos, *Cosmological Horizons, Uncertainty Principle and Maximum Length Quantum Mechanics*, Phys. Rev. D **95**, 103523 (2017), [arXiv:1704.05681v1].

[21] P. West, *Introduction to Strings and Branes*, Cambridge University Press, (2012).

[22] R. Blumenhagen, D. Lüst and S. Theisen, *Basic Concepts of String Theory*, Springer-Verlag Berlin Heidelberg, (2013).

[23] J. G. Kelly and S. S. Seahra, *Velocity-dependent deformations of the energy spectrum of quantum cavity from Lorentz symmetry violations*, Phys. Rev. D **100**, (2019), [arXiv:1812.06047v2].

[24] H. Randji and N. Belaloui, Fermionic string theories with deforme dispersion relations, Phys.Rev.D **100**, 126003 (2019).

[25] N. Takka, *Le formalisme non commutatif en théorie de la Relativite*, thèse doctorat soutenue à l'université A. Mira de Bejaia, Algérie, (2018).

[26] T. Foughali, *Sur les Difféeretes Extensions de la Relativity Générale*, thèse doctorat soutenue à l'université A. Mira de Bejaia, Algérie, (2014).

[27] Z. Belhadi, *Application de la Mécanique Quantique Non Commutative en Relativité et Quantification des Systèmes avec Constraintes*, thèse doctorat soutenue à l'université Mouloud Mammeri Tizi-Ouzou, Algérie, (2015).

[28] S. Ghosh, *Lagrangian for doubly special relativity particle and the role of noncommutativity*, Phys. Rev. D **74**, 084019 (2006), [arXiv: hep-th/0608206v4].

[29] A. Zee, *Einstein Gravity in a Nutshell*, Princeton University Press, (2013).

[30] A. Zee, *Group Theory in a Nutshell for Physicists*, Princeton University Press, (2016).

[31] P. Di Francesco, P. Mathieu and D. Sénéchal, *Conformal Field Theory* , Springer, (1997).

[32] W. Greiner, *Relativistic quantum Mechanics* , Springer-Verlag Berlin Heidelberg. (2000).

[33] N. Jafari and A. Shariati, *Doubly Special Relativity: A New Relativity or Not*, AIP Conference Proceedings **841**, 462 (2006), [arXiv: gr-qc/0602075v1].

[34] J. Magueijo and L. Smolin, *Generalized Lorentz invariance with an invariant energy scale*, Phys. Rev. D **67**, 044017 (2003), [arXiv: gr-qc/0207085v1].

[35] M. Baumgartl, I. Brunner and M. Haack Editors, *Strings and Fundamental Physics* Springer-Verlag Berlin Heidelberg, Germany, (2012).

[36] G. Arutyunova, *Lectures on String Theory*, Ultrecht University, (2009).

[37] C. Teitelboim, *Supergravity and Square Roots of Constraints*, Phys.Rev.Lett. **38**. 1106, (1977).

[38] K. Sundermeyer, *Constrained Dynamics, Lecture Notes in Physics, vol. 169*, Springer, Heidelberg, (1982).

[39] P.A.M. Dirac, *Lectures on quantum mechanics*, Belfer Graduate School of Science, Yeshiva University, New York, (1964).

[40] V. Schomerus, *Primer on string theory*, Cambridge University Press, (2017).

[41] B. Zwiebach, *A First Course in String Theory*, Second Edition, Cambridge University Press, (2009).

[42] T. Lancaster and S.J. Blundell, *Quantum Field Theory for the Gifted Amateur*, Oxford University Press, (2014).

[43] W. Greiner and J. Reinhardt, *Field Quantization*, Springer-Verlag Berlin Heidelberg. (1996).

[44] A. Hanson, T. Regge, and C. Teitelboim, *Constrained Hamiltonian Systems*, Accademia Nazionale dei Lincei, Roma, (1976).

[45] D. Hamam and N. Belaloui, *Ramond and Neveu Schwarz paraspinning strings in presence of D-branes*, Int.J.Mod.Phys. A **33** , No. 08, 1850048 (2018).

[46] L. Khodja, *Théories des Cordes Etendues (Membranes et Supermembranes) et Supercordes dans le Formalisme de la Paraquantification*, thèse doctorat soutenue à l'université Mentouri, Constantine, Algérie, (2009-2010).

[47] B. Hatfield, *Quantum Field Theory Of Point Particles And Strings*, Perseus Books, (1991).

[48] K. Hashimoto, *D-Brane Superstring and New Perspective of Our World*, Springer-Verlag Berlin Heidelberg, (2012).

[49] M. Henneaux and C. Teitelboim, *Quantization of gauge systems*, Princeton University Press, (1994).

العنوان : نظريات الأوتار في صيغ معدلة

ملخص :

تتناول هذه الأطروحة نظريات الأوتار في صيغ معدلة. ندرس أولا الوتر البوزوني المفتوح في حقل غبار هذا النموذج يعطي امكانية بناء وتر لا حرج. في نفس السياق نعطي الوتر الممتد بين ديبرانين (غشائين) متوازيين حيث نهتم بامكانية وجود وتر لا حرج بلا تكويونات و لا أشباح.

ثانيا، ندرس نظريات الوتر البوزوني ذوات علاقات طاقة زخم نسبوية مشوهة والتي طورت من قبل جو ماغويجو و لي سماولين. نبين قابلية هذه النظريات لوصف وتر بقيود غير مشوهة في زمكان لا تبديلي. كما ندرس ايضا التمديد الشبه الكمي.

ثالثا، ندرس نظريات وتر معدلة بعلاقات تشتت مشوهة نستعمل الجذور التربيعية لقيود الوتر البوزوني المشوهة لنجد كل قيود هذه النظريات و التي تحقق جبر مغلق يتعلق بالطاقة. نكمم هذه النظريات فنجد ان خصائص الطيف تتعلق بدوال الطاقة الكلية للوتر. في قسم من هذه النماذج تبقى نتائج الوتر البوزوني العادي ممكنة بما في ذلك عدم وجود أشباح و وجود التناظر الفائق في الزمكان و عدم وجود تاكيونات (بعد اسقاط GSO).

<u>كلمات مفتاحية</u> وتر، تناظر فائق، اسقاط GSO ، بعد حرج .

Titre : Théories des Cordes dans des Formulations modifiées

Résumé

Le propos de cette thèse est de développer des formulations modifiées de théories des cordes. D'abord, en étudiant la théorie de la corde bosonique ouverte en présence d'un champ scalaire (Dust field), ce modèle donne des possibilités de construction de théories non critiques. On étudie en particulier le cas où la corde est située entre 2 Dp-branes parallèles, sans dimension critiques, sans tachyons et sans ghosts.

En second lieu, On étudie des théories de cordes bosoniques modifiées avec des relations de dispersion déformées, modèle développé par Joào Magueijo et Lee Smolin où on a mis en évidence la correspondance avec la théorie avec des contraintes non déformées dans un espace-temps non commutatif. L'extension paraquantique a été également étudiée.

En troisième lieu, On étudie des théories de cordes fermioniques modifiées avec des relations de dispersion déformées. On utilise les racines carrées des contraintes de la corde bosonique déformée pour obtenir la totalité des contraintes de cette théorie, qui vérifient une algèbre fermée dépendante de l'énergie totale de la corde. On quantifie ces théories et on s'aperçoit que les caractéristiques du spectre dépendent des fonctions d'énergie totale de la corde. Pour une partie de ces modèles, les résultats ordinaires de la corde fermionique restent possibles, incluant, des théories sans le ghost, avec des espaces-temps supersymétriques, et sans tachyons. (après projection GSO).

<u>Mots clés</u>: corde, supersymétrie, Projection GSO, dimension critique.

Abstract :

This thesis deals with string theories in modified formulations. First, we study the open bosonic string with a dust field, this model gives the possibility of construction of a non-critical string. In the same context, we give the string stretching between two parallel Dp-branes where we care about the possibility of non-critical string with no tachyons and no-ghosts.

Second, we study bosonic string theories with deformed relativistic momentum energy relations which were developed by Joào Magueijo and Lee Smolin. We show the ability of such theories to describe a string with non deformed constraints in a non-commutative space-time . We study also the paraquantum extension.

Third, we study modified fermionic string theories with deformed dispersion relations. We use the square roots of the bosonic string deformed constraints to obtain the whole constraints of these theories, which verify energy dependent closed algebra. We quantize these theories and we find that the characteristics of the spectrum change with respect to the total energy functions. In a subset of these models, the ordinary fermionic string results remain possible, including theories with no ghost, with space-time supersymmetry, and without tachyons (after the GSO projection).

Keywords: String, supersymmetry, GSO projection, critical dimension.