REPUBLIQUE ALGERIENNE DEMOCRATIQUE ET POPULAIRE

MINISTERE DE L'ENSEIGNEMENT SUPERIEUR ET DE LA RECHERCHE SCIENTIFIQUE

UNIVERSITE MENTOURI CONSTANTINE & DEPARTEMENT DE MATHEMATIQUES

FACULTE DE SCIENCES EXACTES

N° d'Ordre :…………..

N° de Série :…………..

Thèse

Présentée pour l'obtention du diplôme de Doctorat

En : Mathématiques

Spécialité : Mathématiques appliquées

PAR

Karima Kimouche

Sujet

ANALYSE SPECTRALE DANS LES CHAMPS ALEATOIRES NON LINEAIRES

COMPOSITION DU JURY :

Mr.	F.L. RAHMANI	Président	Professeur	U. Mentouri
Mr.	A. BIBI	Rapporteur	Professeur	U. Mentouri
Mme.	S. DABO-NIANG	Examinatrice	Professeur	U.C.D. Lille 3
Mr.	A. BENNIA	Examinateur	Professeur	U. Mentouri
Mlle.	H. GUERBYENNE	Examinatrice	Professeur	U.S.T.H.B. Alger
Mr.	F. HAMDI	Examinateur	M. C	U.S.T.H.B. Alger

Soutenu le : 30- 06- 2013

To my parents

Acknowledgements

I am heartily thankful to my Professor A. Bibi, whose encouragement, guidance and support from the initial to the final level enabled me to develop an understanding of the subject.

I would like to express the deepest appreciation to my committee chair, Mr. F.L. Rahmani, Professor at the University Mentouri Constantine.

I would like to thank Professor S.D. Niang in university U.C.D. Lille 3, for honoring me with her presence on my thesis committee.

I am greatly honored that Mr. A. Bennia, Professor at the University Mentouri Constantine accepted to be the members of my thesis committee,

I wish also to thank Ms. H. Guerbyenne, Professor at the University USTHB Alger for agreeing to judge this work by being part of my thesis committee.

In addition, a thank you to Mr. F. Hamdi, M.C at the University USTHB Alger, who has accepted to judge my research work.

This thesis is dedicated to my parents who have given me the opportunity of an education from the best institutions and support throughout my life.

Lastly, I offer my regards and blessings to all of those who supported me in any respect during the completion of the thesis.

Contents

Introduction

1. Motivation

In many areas of the sciences as oceanography, electrical, engineering, geophysics, astronomy and hydrology, spectral analysis finds frequent and extensive use; it is a well established standard for use in many areas. In fact it facilitates the exchange of ideas across a broad array of scientific projects. In the 1960's spectral analysis was designed to be applied primarily to processes with relatively simple spectra. Thus, interest in spectral analysis to bidimensional processes has attracted considerable interest among mathematicians, probabilists, and statisticians, as Priestly (1964), Whittle (1954), Pierson and Tick (1957).

Spectral analysis for random fields which is a natural extension for a times series, has grown substantially over the last few years. Markov and Gibbs fields are the most studied models in the literature, with the main applications (see Guyon (1995)). For a random fields, in many others disciplines are non linear and may be non Gaussian and the second order statistics does not contain any information about the nonlinearity. Recently, considerable attention has been paid to nonlinear models in sever storms, earthquakes, spread of cancerous cells, regional economics, ecology.

The aim of our work is to contribute to the study of spectral analysis for random fields through a Fourier and Wavelet analysis based on probabilistic structures and statistical inference. This thesis allows to review the current state of research for random fields on some points not yet treated and indispensable for understanding these fields.

Fourier analysis is an established subject in the core of pure and applied mathematical analysis. From this analysis and thorough the study probabilistic, we are interested in a class of nonlinear fields called spatial bilinear processes which the extension of popular BL models. We derive necessary and sufficient conditions for the stability, stationary, regularly and ergodic solutions for some SBL models based on their associated transfer functions and we discover a group of Yule-Walker -type difference equations for third-order cumulant. therefore, several types of spatial linear models have been studied by several authors including Moor (1988), Gaetan and Guyon (2010), Tjostheim (1978, 1983), Yao and Brockwell (2006), Guo and Billard (1998), Dimitriou (2009) and the references therein.

Then, we obtain some asymptotic properties of spectral density estimation which is important in many fields including astronomy, meteorology, seismology, communication, economics, speech analysis, medical imaging, radar, and underwater acoustics. One of the most pioneering work in this field is due to Rosenblatt (1985). He proved that under strong mixing condition and the summability condition of cumulants up to the eighth order the estimate of density spectral is asymptotically normal, this work was generalized in various aspects. Bradley (1992), proved the asymptotic normality of weakly dependent random fields, Robinson (2006), shows that under some circumstance the bias of the choice of kernel and bandwidth can be dominated by the edge effect. The spectral density estimation for random fields was developed by many researches, including Alekseev (1973, 1990), Crujeiras and Fernandez-Casal (2009), Yuan and Subba-Rao (1993). Rachdi and Sabre (2008), are also interested in estimating the spectral density of the absolutely continuous measure by using the double kernel method. According to our modest knowledge there is some theoretical result on the estimation of the bispectral density and parameter for random fields which allows us to do a statistical inference study in particular, parameter estimation, despite the studies that have been achieved in the time series (see Van ness (1966), Rosenblatt and Vann ness (1965), Lii and Rosenblatt (1990), Glindemann et al. (1992), Berg and Politis (2009) and Terdik (1991)).

It is a fact that classical Fourier analysis assumes that signals are infinite in time or periodic, while many signals in practice are of short duration, and change substantially over their duration. Also low frequency pieces tend to last for a long interval, whereas high frequencies occur in general for a short moment only. For example, human speech signals are typical in this respect. Clearly Fourier analysis is highly unstable with respect to perturbation, because of its global character. Facing these problems, signal analysts turn to more sophisticated techniques which are a very popular topic of conversations in many scientific and engineering gatherings these days, Wavelet analysis. It is a particular time- or space-scale representation of signals that has found a wide range of applications in physics, signal processing and applied mathematics in the last few years.

Similarly to the study of the first part based on probabilistic properties, we are interested in Wavelet transform and random field in $\mathbb{Z}^d$ which is the extending study to wavelet transform and times series that is studied by Subba Rao and Indukumar (1996), Chiann (1998). In the literature, Wavelet transform and random field in $\mathbb{R}^d$ is widely study by several authors (for example see Antoine et al. (2004)), for instance Masry gives the second-order properties of the wavelet transform of second order random fields in $\mathbb{R}^d$ (see Masry (1998)). However, spectral density which is the main purpose of chapter 5 attracted Neumann who considers nonlinear wavelet estimators of the spectral density of times series and has shown that optimality thresholded wavelet attains the minimax rate of convergence (see Neumann (1996)), several authors are concerned by this research as Clouet et al. (1995), Huang and Chen (2009), Failla et al. (2011).

2. Historical perspective

In 1807, Fourier's efforts with frequency analysis lead to what we now know as Fourier Analysis. His work is based on the fact that functions can be represented as the sum of sines and cosines. Another contribution of Joseph Fourier's was the Fourier Transform. It transforms a function that depends on time into a new function, which depends on frequency.

The first mention of wavelets appeared in an appendix to the thesis of A. Haar (1909). One property of the Haar wavelet is that it has compact support, which means that it vanishes outside of a finite interval. Unfortunately, Haar wavelets are not continuously differentiable which is something that limits their applications.

In the 1930s, several groups working independently researched the representation of functions using scale -varying basis functions. By using a scale -varying basis function called the Haar basis function Paul Levy, a physicist, investigated Brownian motion, a type of random signal. He discovered that the scale-varying basis functions created by Haar (i.e. Haar wavelets) were a better basis than the Fourier basis functions. Unlike the Haar basis function, which can be chopped up into different intervals.

Between 1960 and 1980, mathematicians Guido Weiss and Ronald R. Coifman studied the simplest elements of a function space, called atoms, with the goal of finding the atoms for a common function and finding the "assembly rules" that allow the reconstruction of all the elements of the function space using these atoms.

J. Morlet, a geophysical engineer, was faced with the problem of analyzing signals which have very high frequency components with short time spans, and low frequency component with long time spans. Short time Fourier transform (STFT) was able to analyze either high frequency components using narrow windows, or low frequency components using wide windows, but not both. He therefore came up with the ingenious idea of using a different window function for analyzing different frequency bands. Furthermore, these window functions had compact support both in time and in frequency. Due to the "small and oscillatory" nature of these window functions, Morlet named his basis functions as *Wavelet of constant shape.* Just like Fourier, Morlet faced much criticism from his Colleagues. In 1980, looking for help to find a mathematically rigorous basis to his approach, Morlet met A. Grossman, a theoretical physicist of quantum mechanics who helped him to formalize the transformation and devise the inverse transformation (see Grossmann and Morlet (1985)).

The next two important contributors to the field of wavelets are Yves Meyer and Stephane Mallat; they realized that the multiresolution with wavelets was a different version of an approach that has long been applied by electrical engineers and image processors. At the end of their research, Multiresolution Analysis for wavelets was born. This idea of multiresolution analysis was a big step in the research of wavelets.

While Mallat first worked on truncated versions of infinite wavelets, Daubechies used the idea of multiresolution analysis to create her own family of wavelets (see Mallat (2009)). These wavelets were of course named the Daubechies Wavelets which satisfies a number of wavelet properties. They have compact support, orthogonality, regularity, and continuity. Daubechies wavelets provide the smallest support for the given number of vanishing moments (see Daubechies (1990). In 1989, Coifman suggested to Daubechies that it might be worthwhile to construct orthogonal wavelet bases with vanishing moments not only for the wavelet, but also for the scaling function. Daubechies constructed the resulting wavelets in 1993 and named them coiflets (see Daubechies (1990).

Around this time, wavelet analysis evolved from a mathematical curiosity to a major source of new signal processing algorithms. The subject branched out to construct wavelet bases with very specific properties, including orthogonal and biorthogonal wavelets, compactly supported, periodic or interpolating wavelets, separable and non separable wavelets for multiple dimensions, multiwavelets, and wavelet packets, which are preferred by many researchers.

3. Thesis outline

In this thesis, we present the study of spectral analysis for random fields based on two analyses: Fourier analysis and wavelet analysis. This study contains the probabilistic structure and inference statistical. The thesis is divided into five chapters:

Chapter 1: In this chapter, we present on $\mathbb{L}_2$ structure of bilinear models on $\mathbb{Z}^d$ and the probabilistic properties based on its associated transfer functions. In particular we describe the spatial subdiagonal bilinear process with respect to its transfer functions, and we use this representation to give sufficient and necessary conditions ensuring the existence of regular second order stationary and ergodic solutions for several subclass especially for $SGARCH$ models. We also discuss the third order probabilistic structure for the model and we discover a group of Yule-Walker-type difference equation for third-order cumulants.

Chapter 2: We consider the spectral density estimate based on class of strictly stationary nonlinear spatial process and on class of nonlinear random fields that satisfy the geometric-moment contraction condition and we establish the asymptotic normality. Then we obtain the asymptotic distribution of certain estimates of the bispectral density this estimate would have distribution which tend to complex normal distributions under a uniform summability condition on the first six cumulants and the strong mixing condition. We also propose an estimator of the fourth-order cumulant spectral density and we demonstrate under the above conditions the asymptotic normality, this latter study is generalized in p-order case.

Chapter 3: Treats the methods of parameter estimation based on a functional of the spectrum and bispectrum for a random field depending on an unknown parameter θ. The estimation of the parameter of non Gaussian fields constructed by the minimization of the functional and the explicit expression for the asymptotic variance of the estimator calculate for both the cases when the spectra are estimated by the peridogram and by the smoothed periodogram. The consistency and asymptotic normality are proved.

Chapter 4: In this chapter we introduce notation and briefly review for the multiresolution analysis in $\mathbb{R}^d$ and we develop an alternative procedure in which a continuous random fields is first generated by interpolation of the discrete random fields. We obtain explicit expressions for the second and third order covariances between wavelet and scaling coefficient of discrete random fields and the dependence structure between wavelet coefficients is closely related to the dependence of scaling coefficients. Hence, the second order properties of the discrete wavelet transform are determined.

Chapter 5: In this chapter we consider nonlinear wavelet estimators of the spectral density random fields and we obtain empirical wavelet coefficients of the spectral density which are then treated with the same shrinkage methods as Neumann, then we state the asymptotic normality. We have shown also that optimality thresholded wavelet attains the minimax rate of convergence in a large scale of Besov smoothness classes. In addition we propose a wavelet-thresholding estimator of the bispectra and we show that this estimator reaches minimax rate on Sobolev spaces, which is not attained by linear (kernel or spiline) estimators.

Part I

Fourier Analysis

Chapter 1

On $\mathbb{L}_2$ structure of bilinear models on $\mathbb{Z}^d$

1.1 Introduction

A process $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ is called multidimensionally indexed (or spatial), when the variable $\mathbf{t}$ has several components $t_1, t_2, ..., t_d$ say. Multidimensionally indexed processes arise naturally in the study of random fields as well as in modeling some spatial data. Spatial data can be viewed either as a set of time series collected simultaneously at a number of sites (locations) or as sets of spatial data collected at several and different number of time points. For the statistical analysis, it is often assumed that the spatial data under consideration as in environmental monitoring studies, meteorology, oceanography, geology, biology, among others, are linear and may be Gaussian. Recent studies have shown that some crucial spatial data we come across as in digital image processing are neither linear nor Gaussian as for instance, spatial data collected from satellites, sever storms, earthquakes, spread of cancerous cells, regional economics, ecology and from multichannel EEG digital signal processing. Hence, extending one-dimensional nonlinear time series models to multidimensional one, yields novel clutter models which are capable of taking into account the non Gaussianity and spatiality dependence. However, the modeling of this type of data by a spatial non-linear models has become an appealing and popular tool for investigating both spatiality and non-Gaussianity patterns in time series analysis. Indeed, Amirmazlaghani and Amindavar (2007) have used two dimensional $GARCH$ model for wavelet coefficients modeling to perform the image denoising. In image anomaly detection, Noiboar and Cohen (2007) have proposed an approach based on $GARCH$ random field to distilling a small number of clustered pixels. Dai and Billard (1998, 2003) have introduced a class of spatio-temporal bilinear models to model the spatial spread of monthly surveillance data for mumps over 1971-1988 in twelve states of the U.S.A.

The literature for linear spatial models is very widespread and includes for instance Moor (1988), Gaetan and Guyon (2010), Tjostheim (1978, 1983), Yao and Brockwell (2006), Guo and Billard (1998), Dimitriou (2009) and the references therein which concern some studies on probabilistic structures and statistical inference. Unfortunately, only a few studies on the probabilistic structures or in statistical inference of spatial nonlinear models were investigated.

Some notations and concepts are used throughout: for any positive integer d, set $\mathbf{0} = (0, ..., 0)$ be the zeros vector of $\mathbb{Z}^d$, for any $\mathbf{k} = (k_1, ..., k_d)$ and $\mathbf{l} = (l_1, ..., l_d)$ belonging to $\mathbb{Z}^d$, we write $\mathbf{k} \preceq \mathbf{l}$ (resp. $\mathbf{k} \prec \mathbf{l}$) if and only if $k_m \leq l_m$ (resp. $k_m < l_m$) for $m = 1, ..., d$. However, for $\mathbf{p} \in \mathbb{N}^d$, the following indexing subsets in $\mathbb{N}^d$ will be considered $\Gamma[\mathbf{p}] = \left\{\mathbf{x} \in \mathbb{N}^d \ / \ \mathbf{0} \preceq \mathbf{x} \preceq \mathbf{p}\right\}$, $\Gamma]\mathbf{p}] = \Gamma[\mathbf{p}] \setminus \{\mathbf{0}\}$(see Dimitriou (2009) for an extensive discussion on the interest choice of the order in the lattice $\mathbb{Z}^d$) and for any $\mathbf{i} = (i_1, ..., i_d) \in \mathbb{Z}^d$ and $\mathbf{z} = (z_1, ..., z_d) \in \mathbb{C}^d$, write $\mathbf{z}^{\mathbf{i}} = \prod_{j=1}^{d} z_j^{i_j}$.

In this chapter we present a powerful frame for the study of spatial nonlinear processes based on its associated transfer functions. This approach allows us to distinguish between linear and nonlinear and between regular and singular processes. We describe the spatial subdiagonal bilinear process with respect to its transfer functions, we then use this representation to give sufficient and necessary conditions ensuring the existence of regular second order stationary and ergodic solutions for several subclass especially for $SGARCH$ models. Our approach is based on the observation that a number of $SGARCH$ models can be written as a diagonal SBL models. This relationship has already been observed by a number of authors (e.g., see Terdik (2000)). Then, we obtain the autocovariance function and the spectral density function , and we derive the Yule-Walker-type difference equations for autocovariance by means of the spectral density function. Concerning the second order probabilistic structure, the model is similar to an spatial $ARMA$ model. Hence, we discuss for the third order probabilistic structure and we discover a group of Yule-Walker-type difference equations for third-order cumulants.

1.2 The multidimensional Wiener-Itô representation

For any Gaussian white noise $(e(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ with mean 0 and variance σ^2, we associate its spectral representation (see [87]), i.e., $e(\mathbf{t}) = \int_{\boldsymbol{\pi}} e^{i\mathbf{t}.\boldsymbol{\lambda}} dZ(\boldsymbol{\lambda})$ in which $\mathbf{t}.\boldsymbol{\lambda} = \sum_{i=1}^{d} \lambda_i t_i$ for any $\mathbf{t} = (t_1, ..., t_d) \in \mathbb{Z}^d$, $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_d) \in \boldsymbol{\pi} = [-\pi, \pi[\times ... \times [-\pi, \pi[$, $d-$times and $Z(.)$ is a Gaussian orthogonal stochastic measure with $E\{dZ(\boldsymbol{\lambda})\} = 0$ and spectral measure $E\left\{|dZ(\boldsymbol{\lambda})|^2\right\} = dF(\boldsymbol{\lambda}) = \frac{\sigma^2}{(2\pi)^d} d\boldsymbol{\lambda}$ where $d\boldsymbol{\lambda}$ means the Lebesgue measure on $\mathbb{R}^d$. Consider the real Hilbert space $\mathcal{H} = L_2(\boldsymbol{\pi}, \mathcal{B}_{\boldsymbol{\pi}}, F)$

of the complex squared integrable functions f satisfying $f(-\boldsymbol{\lambda}) = \overline{f(\boldsymbol{\lambda})}$ for any $\boldsymbol{\lambda} \in \boldsymbol{\pi}$. For any $n \geq 1$, we associated three real Hilbert spaces based on $\mathcal{H}$, the first is $\mathcal{H}_n = \mathcal{H}^{\otimes n}$ the $n-$fold tensor product of $\mathcal{H}$ endowed by the inner product $\langle f_n, g_n \rangle_{\otimes} = \int_{\boldsymbol{\pi}^n} f_n\left(\boldsymbol{\lambda}_{(n)}\right) \overline{g_n\left(\boldsymbol{\lambda}_{(n)}\right)} dF\left(\boldsymbol{\lambda}_{(n)}\right)$ where $\boldsymbol{\lambda}_{(n)} = (\boldsymbol{\lambda}_1, ..., \boldsymbol{\lambda}_n) \in \boldsymbol{\pi}^n$, $f_n\left(-\boldsymbol{\lambda}_{(n)}\right) = \overline{f_n\left(\boldsymbol{\lambda}_{(n)}\right)}$, $\|f_n\|^2 < \infty$ and $dF\left(\boldsymbol{\lambda}_{(n)}\right) = \prod_{i=1}^{n} dF\left(\boldsymbol{\lambda}_i\right)$. The second one is $\widehat{\mathcal{H}}_n = \mathcal{H}^{\oplus n} \subset \mathcal{H}_n$ the $n-$fold symmetrized tensor product of $\mathcal{H}$ defined by $f_n \in \widehat{\mathcal{H}}_n$ if and only if f_n is invariant under permutation of their arguments i.e., $f_n\left(\boldsymbol{\lambda}_{(n)}\right) = sym\left\{f_n\left(\boldsymbol{\lambda}_{(n)}\right)\right\}$ $= \frac{1}{n!}\sum_{p\in\mathcal{P}(n)} f_n\left(\boldsymbol{\lambda}_{(p(n))}\right)$ where $\mathcal{P}(n)$ denotes the group of all permutation of the set $\{1, ..., n\}$ with an inner product $\langle f_n, g_n \rangle_{\oplus} = n! \langle f_n, g_n \rangle_{\otimes}$ for $f_n, g_n \in \widehat{\mathcal{H}}_n$. The third space is called Fock space over $\mathcal{H}$ denoted by $\Im(\mathcal{H})$ and defined by $\Im(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \widehat{\mathcal{H}}_n$ in which $\bigoplus$ denotes the direct orthogonal sum, whose elements are $f := (f_0, f_1, f_2, ...)$ with $f_n \in \widehat{\mathcal{H}}_n$, $\widehat{\mathcal{H}}_0 = \mathcal{H}_0 = \mathbb{R}$ and satisfying $\|f\|^2 = \sum_{n\geq 0} \|f_n\|^2 < +\infty$. The corresponding orthogonal decomposition is called Wiener's chaos decomposition.

Let $\Im = \Im(e) := \sigma\left(e(\mathbf{t}), \mathbf{t} \in \mathbb{Z}^d\right)$ the $\sigma-$algebra generated by all $e(\mathbf{t}), \mathbf{t} \in \mathbb{Z}^d$, $\Im_{\mathbf{t}} := \sigma\left(e(\mathbf{s}), \mathbf{s} \preceq \mathbf{t}\right)$ and $\mathbb{L}_2(\Im)$ be the real Hilbert space of $\mathbb{L}_2-$functional of $e(\mathbf{t}), \mathbf{t} \in \mathbb{Z}^d$. It is well known (see Bibi (2006) for further details) that $\mathbb{L}_2(\Im)$ is isometrically isomorphic to $\Im(\mathcal{H})$, i.e., for any random field $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ of $\mathbb{L}_2(\Im)$ admits the so-called Wiener-Itô orthogonal representation

$$X(\mathbf{t}) = f_0 + \sum_{r\geq 1} \int_{\boldsymbol{\pi}^r} f_r\left(\boldsymbol{\lambda}_{(r)}\right) e^{i\sum_{j=1}^{r} \mathbf{t}.\boldsymbol{\lambda}_j} dZ(\boldsymbol{\lambda}_{(r)}), \tag{1.2.1}$$

where $f_0 = E\{X(\mathbf{t})\}$ and $dZ(\boldsymbol{\lambda}_{(r)}) = \prod_{i=1}^{r} dZ\left(\boldsymbol{\lambda}_i\right)$, $f_r \in \widehat{\mathcal{H}}_r$ are uniquely determined and the integrals are the so-called multiple Wiener-Itô stochastic integrals with respect to the Gaussian stochastic measure Z. The following theorem gives some important properties related to Wiener-Itô stochastic integrals which we shall apply throughout. For the proof we refer to Major (1981).

Theorem 1.1 *1. [Itô's formula] The Itô's formula state that*

$$\prod_{i=1}^{k} h_{n_i}\left(\int_{\boldsymbol{\pi}} \varphi_i\left(\boldsymbol{\lambda}\right) dZ(\boldsymbol{\lambda})\right) = \int_{\boldsymbol{\pi}^n} \prod_{i=1}^{k} \prod_{j=1}^{n_i} \varphi_i\left(\boldsymbol{\lambda}_{n_{i-1}+j}\right) dZ(\boldsymbol{\lambda}_{(n)}) = \int_{\boldsymbol{\pi}^n} sym\left\{\prod_{j=1}^{n} \varphi_j\left(\boldsymbol{\lambda}_j\right)\right\} dZ(\boldsymbol{\lambda}_{(n)}),$$

where $(\varphi_i)_{1\leq i\leq k}$ is an orthonormal system in $\mathcal{H}$, $n_1, ..., n_k$ are positive integers $(n_0 = 0)$ with $n = n_1 + ... + n_k$ and h_j denotes the $j-th$ Hermite polynomial with leading coefficient 1, i.e., $h_j(x) = (-1)^j e^{\frac{x^2}{2}} \frac{d^j}{dx^j} e^{-\frac{x^2}{2}}$, $x \in \mathbb{R}$.

2. [Diagram formula] For any $f\in\mathcal{H}$ *and* $f_n\in\mathcal{H}_n$ *we have*

$$\begin{aligned}&\int_{\boldsymbol{\pi}} f(\boldsymbol{\lambda})\ dZ(\boldsymbol{\lambda}) \int_{\boldsymbol{\pi}^n} f_n\left(\boldsymbol{\lambda}_{(n)}\right)\ dZ(\boldsymbol{\lambda}_{(n)})\\ =\ &\int_{\boldsymbol{\pi}^{n+1}} f_n\left(\boldsymbol{\lambda}_{(n)}\right) f\left(\boldsymbol{\lambda}_{n+1}\right)\ dZ(\boldsymbol{\lambda}_{(n+1)}) + \sum_{k=1}^{n}\int_{\boldsymbol{\pi}^{n-1}}\int_{\boldsymbol{\pi}} f_n\left(\boldsymbol{\lambda}_{(n)}\right)\overline{f\left(\boldsymbol{\lambda}_k\right)} dF\left(\boldsymbol{\lambda}_k\right) dZ(\boldsymbol{\lambda}_{(n\backslash k)}),\end{aligned}$$

where $dZ(\boldsymbol{\lambda}_{(n\backslash k)}) := \prod_{i=1,i\neq k}^{n} dZ\left(\boldsymbol{\lambda}_i\right)$.

3. $\left[\textit{Orthogonality of } \widehat{\mathcal{H}}_n \textit{ spaces}\right]$ *For any* $f_n\in\mathcal{H}_n$ *and* $g_m\in\mathcal{H}_m$*, we have*

$$\begin{aligned}&E\left\{\int_{\boldsymbol{\pi}^n} f_n\left(\boldsymbol{\lambda}_{(n)}\right)\ dZ(\boldsymbol{\lambda}_{(n)}) \int_{\boldsymbol{\pi}^m} g_m\left(\boldsymbol{\lambda}_{(m)}\right)\ dZ(\boldsymbol{\lambda}_{(m)})\right\}\\ =\ &\delta_n^m n! \int_{\boldsymbol{\pi}^n} sym\left\{f_n\left(\boldsymbol{\lambda}_{(n)}\right)\right\}\overline{sym\left\{g_n\left(\boldsymbol{\lambda}_{(n)}\right)\right\}} dF(\boldsymbol{\lambda}_{(n)}),\end{aligned}$$

where δ_n^m *is the Kronecker symbol. This means that the spaces* $\widehat{\mathcal{H}}_n$ *are orthogonal.*

Remark 1.1 *Applying Itô's formula, it is easily seen that any random field* $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ *of* $\mathbb{L}_2(\Im)$ *is* $\Im_{\mathbf{t}}(e)$*-measurable (or causal) iff the Fourier coefficients with nonnegative indices of its transfer functions are only nonzero, i.e.,* $f_r\left(\boldsymbol{\lambda}_{(r)}\right) = \sum_{\mathbf{k}_1,\mathbf{k}_2,\ldots,\mathbf{k}_r\preceq\mathbf{0}} \widetilde{f}_r\left(\mathbf{k}_{(r)}\right) e^{i\sum_{i=1}^r \mathbf{k}_i.\boldsymbol{\lambda}_i}$ *and* $\widetilde{f}_r\left(\mathbf{k}_{(r)}\right) = \int_{\boldsymbol{\pi}^r} f_r\left(\boldsymbol{\lambda}_{(r)}\right) e^{-i\sum_{i=1}^r \mathbf{k}_i.\boldsymbol{\lambda}_i} dF(\boldsymbol{\lambda}_{(r)})$ *where* $\mathbf{k}_{(r)} = (\mathbf{k}_1,\ldots,\mathbf{k}_r) \in \left(\mathbb{Z}^d\right)^r$ *with* $\mathbf{k}_i\in\mathbb{Z}^d$, $i=1,\ldots,r$*. Hence the corresponding representation* (1.2.1) *will be referred to later as regular.*

Remark 1.2 *A necessary and sufficient condition that the random field* $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ *of* $\mathbb{L}_2(\Im)$ *admits a regular solution given by* (1.2.1)*, is that the transfer functions* f_r *satisfies Szegö's condition* $\int_{\boldsymbol{\pi}^r} \log\left|f_r\left(\boldsymbol{\lambda}_{(r)}\right)\right| dZ(\boldsymbol{\lambda}_{(r)}) > -\infty$.

Example 1.1 *A general class of nonlinear random fields* $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ *which admits a regular solution are the Wiener fields i.e.,*

$$X(\mathbf{t}) = g_{\mathbf{0}} + \sum_{r=1}^{\infty}\sum_{\mathbf{k}_{(r)}\in\left(\mathbb{N}^d\backslash\{\mathbf{0}\}\right)^r}\sum_{\mathbf{s}_{(r)}\in(\mathbb{N}^d_{\prec})^r} g_{\mathbf{k}_{(r)}}\left(\mathbf{s}_{(r)}\right)\prod_{j=1}^{r} h_{k_j}\left(e(\mathbf{t}-\mathbf{s}_j)\right), \tag{1.2.2}$$

for some stationary Gaussian random field $(e(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ *where* $\left(\mathbb{N}^d_{\prec}\right)^r := \left\{\mathbf{s}_{(r)}\in\left(\mathbb{N}^d\right)^r : \mathbf{0}\preceq\mathbf{s}_1\prec\mathbf{s}_2\prec\ldots\prec\mathbf{s}_r\right\}$ *and where the Volterra's kernels* $g_{\mathbf{k}_{(r)}}\left(\mathbf{s}_{(r)}\right)$ *are uniquely determined if there are assumed to be symmetric functions in their arguments. Hence, by applying Itô's formula, it is easily seen that* $X(\mathbf{t})$ *admits a Wiener-Itô orthogonal representation* (1.2.1).

1.3 Wiener-Itô solution for spatial subdiagonal bilinear random fields

The class of spatial Wiener's models (1.2.2) can describe general non linear models with great accuracy and can be enlarged to include the random field $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ solving the following recursive equation

$$X(\mathbf{t}) = f(X(\mathbf{t}-\mathbf{s}), e(\mathbf{t}-\mathbf{r}), \mathbf{0} \prec \mathbf{s} \preceq \mathbf{P}, \mathbf{0} \prec \mathbf{r} \preceq \mathbf{Q}) + e(\mathbf{t}), \tag{1.3.1}$$

for some white noise field $(e(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ and polynomial function f. The main objective here is to derive the system of transfer functions associated with (1.3.1) and thus we establish the necessary and sufficient condition ensuring the existence of regular second order stationary solutions. For this purpose we shall restrict ourself to the so-called spatial subdiagonal bilinear model. A $\mathbb{R}-$valued random field $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ defined on a probability space $(\Omega, \Im, P)$ is called spatial subdiagonal bilinear process denoted by $SBL_d(\mathbf{p}, \mathbf{q}, \mathbf{P}, \mathbf{Q})$ if it is solution of the following stochastic difference equation

$$X(\mathbf{t}) = \sum_{\mathbf{i}\in\Gamma]\mathbf{p}]} a_{\mathbf{i}} X(\mathbf{t}-\mathbf{i}) + \sum_{\mathbf{j}\in\Gamma[\mathbf{q}]} b_{\mathbf{j}} e(\mathbf{t}-\mathbf{j}) + \sum_{\mathbf{i}\in\Gamma]\mathbf{P}],} \sum_{\mathbf{j}\in\Gamma[\mathbf{Q}]} c_{\mathbf{i},\mathbf{j}} X(\mathbf{t}-\mathbf{i}-\mathbf{j}) e(\mathbf{t}-\mathbf{i}). \tag{1.3.2}$$

In (1.3.2) $(e(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ is a Gaussian field white noise defined on the same probability $(\Omega, \Im, P)$ with zero mean and variance σ^2. The assumption of subdiagonality is technical because it is difficult to handle the product terms like $X(\mathbf{t})e(\mathbf{t}-\mathbf{i})$, $\mathbf{i} \succ \mathbf{0}$. Noting that different SBL_d representations appear to depend on the lexicographic order chosen on $\mathbb{Z}^d$. Define the functions

$$\begin{aligned}
\Theta(\boldsymbol{\lambda}) &= 1 - \sum_{\mathbf{i}\in\Gamma]\mathbf{p}]} a_{\mathbf{i}} e^{-i\mathbf{i}.\boldsymbol{\lambda}}, \Phi(\boldsymbol{\lambda}) = \sum_{\mathbf{j}\in\Gamma[\mathbf{q}]} b_{\mathbf{j}} e^{-i\mathbf{j}.\boldsymbol{\lambda}}, \\
\Psi_0(\boldsymbol{\lambda}) &= \sum_{\mathbf{i}\in\Gamma]\mathbf{P}]} c_{\mathbf{i}\mathbf{0}} e^{-i\mathbf{i}.\boldsymbol{\lambda}}, \Psi(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \sum_{\mathbf{i}\in\Gamma]\mathbf{P}]} \sum_{\mathbf{j}\in\Gamma[\mathbf{Q}]} c_{\mathbf{i},\mathbf{j}} e^{-i(\mathbf{i}+\mathbf{j}).\boldsymbol{\lambda}} e^{-i\mathbf{j}.\boldsymbol{\mu}}.
\end{aligned}$$

We seek necessary and sufficient conditions ensuring the existence of regular second order stationary solution of (1.3.2) in the Form (1.2.1). Throughout the paper, we shall assume the following condition

Condition 1.1 *all the characteristic roots of the polynomial* $\Theta(\mathbf{z}) = 1 - \sum_{\mathbf{i}\in\Gamma]\mathbf{p}]} a_{\mathbf{i}} \mathbf{z}^{\mathbf{i}}$ *are outside the unit circle, in the sense that* $\Theta(\mathbf{z}) \neq 0$ *for* $|z_i| \leq 1, i = 1, ..., d$.

Other conditions ensuring the existence of the roots of polynomial $\Theta(\mathbf{z})$ outside the circles $|z_i| \leq 1, i = 1, ..., d$, can be found in Tjostheim (1983) and in Yao and Brackwell (2006). For instance, a necessary and sufficient conditions for the equation $\Theta(z_1, z_2) = 1 - a_1 z_1 - a_2 z_2 - a_3 z_1 z_2$ to have its roots outside the circles $|z_i| \leq 1, i = 1, 2$ (see Basu and Reinsel (1992), Proposition 1) are (i) $|a_i| < 1$, $i = 1, 2, 3$ (ii) $(1 + a_1^2 - a_2^2 - a_3^2)^2 - 4(a_1 + a_2 a_3)^2 > 0$ and (iii) $1 - a_2^2 > |a_1 + a_2 a_3|$. In particular, the special case where $a_3 = -a_1 a_2$, the above conditions reduce to $|a_i| < 1$, $i = 1, 2$. Noting that a multivariable polynomial can be factored into factors which are themselves multivariable polynomials but which cannot be further factored, and these irreducible polynomials are unique to multiplicative constants.

Lemma 1.1 *Assume that the SBL_d Model* (1.3.2) *has regular second order stationary solution, then the transfer functions of this solution are given by the symmetrization of the following functions defined recursively by*

$$f_r\left(\boldsymbol{\lambda}_{(r)}\right) := \begin{cases} \sigma^2 \dfrac{\Psi_0(\mathbf{0})}{\Theta(\mathbf{0})}, \text{ if } r = 0, \\ \dfrac{\Phi^*(\boldsymbol{\lambda})}{\Theta(\boldsymbol{\lambda})}, \text{ if } r = 1, \\ \dfrac{\Psi\left(\sum_{j=1}^{r-1} \boldsymbol{\lambda}_j, \boldsymbol{\lambda}_r\right)}{\Theta\left(\sum_{j=1}^{r} \boldsymbol{\lambda}_j\right)} f_{r-1}\left(\boldsymbol{\lambda}_{(r-1)}\right), \text{ if } r \geq 2, \end{cases} \tag{1.3.3}$$

with $\Phi^*(\boldsymbol{\lambda}) = \Phi(\boldsymbol{\lambda}) + f_0 \Psi(\mathbf{0}, \boldsymbol{\lambda})$.

Proof. Assume that the SBL_d Model (1.3.2) has a Wiener-Itô representation (1.2.1). Then by the diagram formula (2.3), we get

$$\begin{aligned} X(\mathbf{t}-\mathbf{i}-\mathbf{j})e(\mathbf{t}-\mathbf{i}) &= \left(f_0 + \sum_{r=1}^{\infty} \int_{\boldsymbol{\pi}^r} f_r\left(\boldsymbol{\lambda}_{(r)}\right) e^{i\sum_{j=1}^{r}(\mathbf{t}-\mathbf{i}-\mathbf{j}).\boldsymbol{\lambda}_j} dZ\left(\boldsymbol{\lambda}_{(r)}\right)\right) \int_{\boldsymbol{\pi}} e^{i(\mathbf{t}-\mathbf{i}).\boldsymbol{\lambda}} dZ(\boldsymbol{\lambda}) \\ &= f_0 \int_{\boldsymbol{\pi}} e^{i(\mathbf{t}-\mathbf{i}).\boldsymbol{\lambda}} dZ(\boldsymbol{\lambda}) + \sum_{r=1}^{\infty} \int_{\boldsymbol{\pi}^{r+1}} f_r\left(\boldsymbol{\lambda}_{(r)}\right) e^{i\sum_{j=1}^{r+1}(\mathbf{t}-\mathbf{i}).\boldsymbol{\lambda}_j - \sum_{j=1}^{r}\mathbf{j}.\boldsymbol{\lambda}_j} dZ\left(\boldsymbol{\lambda}_{(r+1)}\right) \\ &\quad + \frac{\sigma^2}{(2\pi)^d} \int_{\boldsymbol{\pi}^{r-1}} e^{i\sum_{j=1}^{r-1}(\mathbf{t}-\mathbf{i}-\mathbf{j}).\boldsymbol{\lambda}_j} \int_{\boldsymbol{\pi}} f_r(\boldsymbol{\lambda}_r) e^{-i\mathbf{j}.\boldsymbol{\lambda}_r} d\boldsymbol{\lambda}_r dZ\left(\boldsymbol{\lambda}_{(r-1)}\right). \end{aligned}$$

Since a regular solution is independent of random fields $e(\mathbf{s})$, $\mathbf{s} \succ \mathbf{t}$ and depends linearly on $e(\mathbf{t})$, then similar argument to Terdik (2000) show that $\frac{1}{(2\pi)^d} \int_{\boldsymbol{\pi}} f_r(\boldsymbol{\lambda}_r) e^{-i\mathbf{j}.\boldsymbol{\lambda}_r} d\boldsymbol{\lambda}_r = 0$ if $\mathbf{j} \succ \mathbf{0}$ or $\mathbf{j} = \mathbf{0}$ and $r > 1$. Using (1.3.2), Condition 1.1 and the uniqueness of the symmetrized transfer functions we get the recursion (1.3.3).

Remark 1.3 *It not difficult to see that the symmetrized transfer functions are given by*

$$
\begin{aligned}
& sym\left\{f_r\left(\boldsymbol{\lambda}_{(r)}\right)\right\} \\
: &= \begin{cases}
\sigma^2 \dfrac{\Psi_0(\mathbf{0})}{\Theta(\mathbf{0})}, \ \text{if } r=0, \\
\dfrac{\Phi^*(\boldsymbol{\lambda})}{\Theta(\boldsymbol{\lambda})}, \ \text{if } r=1, \\
\Theta^{-1}\left(\sum_{j=1}^{r}\boldsymbol{\lambda}_j\right) \sum\limits_{\mathbf{j}\in\Gamma[\mathbf{Q}]} e^{-i\sum_{j=1}^{r}\mathbf{j}.\boldsymbol{\lambda}_j} \sum\limits_{\mathbf{i}\in\Gamma]\mathbf{P}]} c_{\mathbf{i},\mathbf{j}} sym\left\{f_{r-1}\left(\boldsymbol{\lambda}_{(r-1)}\right)e^{-i\sum_{j=1}^{r-1}\mathbf{i}.\boldsymbol{\lambda}_j}\right\}, \ \text{if } r\geq 2.
\end{cases}
\end{aligned}
$$

Lemma 1.2 *Under the conditions of Lemma 1.1, we have* $\|f_r\|^2 \leq r!\,\|sym\{f_r\}\|^2 \leq 2\|f_r\|^2$ *for any* $r \geq 1$.

Proof. The proof is similar as that of Lemma 1 in Terdik and Subba Rao (1989).
We are now in a position to state our first result.

Theorem 1.2 *A necessary and sufficient condition for the existence of regular second order stationary solution for* $SBL_d(\mathbf{p},\mathbf{q},\mathbf{P},\mathbf{Q})$ *model* (1.3.2) *is that*

$$\sum_{r\geq 0}\|f_r\|^2 < +\infty \tag{1.3.4}$$

where the transfer functions $f_r\left(\boldsymbol{\lambda}_{(r)}\right)$ *are given by* (1.3.3).

Proof. To prove Theorem 1.2, we use the Lemmas 1.1, 1.2 and the fact that $Var\{X(\mathbf{t})\}$ is finite if and only if the Condition (1.3.4) holds true.

Corollary 1.1 *A simple sufficient condition for* (1.3.4) *is*

$$\frac{\sigma^2}{(2\pi)^d}\int\limits_{\boldsymbol{\pi}}\left|\frac{\Psi(\boldsymbol{\lambda},\boldsymbol{\mu})}{\Theta(\boldsymbol{\lambda}+\boldsymbol{\mu})}\right|^2 d\boldsymbol{\mu} = c < 1, \boldsymbol{\lambda}\in\boldsymbol{\pi}.$$

Proof. Consider the norm

$$
\begin{aligned}
& \int\limits_{\boldsymbol{\pi}^r}\left|f_r\left(\boldsymbol{\lambda}_{(r)}\right)\right|^2 dF\left(\boldsymbol{\lambda}_{(r)}\right) \\
= & \int_{\boldsymbol{\pi}^{r-1}}\left\{\int_{\boldsymbol{\pi}}\left|\frac{\Psi(\boldsymbol{\lambda},\boldsymbol{\mu})}{\Theta(\boldsymbol{\lambda}+\boldsymbol{\mu})}\right|^2 dF(\boldsymbol{\mu})\right\}\left|f_{r-1}\left(\boldsymbol{\lambda}_{(r-2)},\boldsymbol{\lambda}-\overline{\boldsymbol{\lambda}}_{(r-2)}\right)\right|^2 dF\left(\boldsymbol{\lambda}_{(r-2)}\right)dF(\boldsymbol{\lambda}),
\end{aligned}
$$

where $\overline{\boldsymbol{\lambda}}_{(r)} := \sum_{i=1}^{r} \boldsymbol{\lambda}_i$. Now if there exist c such that $\frac{\sigma^2}{(2\pi)^d} \int_{\boldsymbol{\pi}} \left| \frac{\Psi(\boldsymbol{\lambda}, \boldsymbol{\mu})}{\Theta(\boldsymbol{\lambda} + \boldsymbol{\mu})} \right|^2 d\boldsymbol{\mu} = c < 1$ for any $\boldsymbol{\lambda} \in \boldsymbol{\pi}$, then

$$\int\limits_{\boldsymbol{\pi}^r} \left| f_r\left(\boldsymbol{\lambda}_{(r)}\right) \right|^2 dF\left(\boldsymbol{\lambda}_{(r)}\right) \leq c \int\limits_{\boldsymbol{\pi}^r} \left| f_{r-1}\left(\boldsymbol{\lambda}_{(r-1)}\right) \right|^2 dF\left(\boldsymbol{\lambda}_{(r-1)}\right) \leq \frac{\sigma^2 c^{r-1}}{(2\pi)^d} \int_{\boldsymbol{\pi}} \left| \frac{\Phi^*(\boldsymbol{\lambda})}{\Theta(\boldsymbol{\lambda})} \right|^2 d\boldsymbol{\lambda},$$

so the Condition (1.3.4) holds true.

Corollary 1.2 *[Diagonal models] Consider the model*

$$X(\mathbf{t}) = \sum_{\mathbf{i} \in \Gamma]\mathbf{p}]} a_{\mathbf{i}} X(\mathbf{t} - \mathbf{i}) + \sum_{\mathbf{j} \in \Gamma[\mathbf{q}]} b_{\mathbf{j}} e(\mathbf{t} - \mathbf{j}) + \sum_{\mathbf{i} \in \Gamma]\mathbf{Q}]} c_{\mathbf{i}} X(\mathbf{t} - \mathbf{i}) e(\mathbf{t} - \mathbf{i}), \tag{1.3.5}$$

then a necessary and sufficient condition that the model (1.3.5) *has a regular second order stationary solution is that*

$$\frac{\sigma^2}{(2\pi)^d} \int\limits_{\boldsymbol{\pi}} \left| \frac{\Psi_0(\boldsymbol{\lambda})}{\Theta(\boldsymbol{\lambda})} \right|^2 d\boldsymbol{\lambda} < 1.$$

Proof. It is easy to see that the transfer functions associated with the Model (1.3.5) are given by

$$f_0 = \sigma^2 \frac{\Psi_0(\mathbf{0})}{\Theta(\mathbf{0})}, f_1(\boldsymbol{\lambda}) = \frac{\Phi^*(\boldsymbol{\lambda})}{\Theta(\boldsymbol{\lambda})}, f_r\left(\boldsymbol{\lambda}_{(r)}\right) = \frac{\Psi_0\left(\sum_{j=1}^{r} \boldsymbol{\lambda}_j\right)}{\Theta\left(\sum_{j=1}^{r} \boldsymbol{\lambda}_j\right)} f_{r-1}\left(\boldsymbol{\lambda}_{(r-1)}\right), r \geq 2.$$

Hence, we obtain for $r \geq 2$ after repeated substitution

$$\left(\frac{\sigma^2}{(2\pi)^d}\right)^r \int\limits_{\boldsymbol{\pi}^r} \left| f_r\left(\boldsymbol{\lambda}_{(r)}\right) \right|^2 d\boldsymbol{\lambda}_{(r)} = \frac{\sigma^2}{(2\pi)^d} \int\limits_{\boldsymbol{\pi}} \left| f_1(\boldsymbol{\lambda}) \right|^2 d\boldsymbol{\lambda} \left\{ \frac{\sigma^2}{(2\pi)^d} \int\limits_{\boldsymbol{\pi}} \left| \frac{\Psi_0(\boldsymbol{\lambda})}{\Theta(\boldsymbol{\lambda})} \right|^2 d\boldsymbol{\lambda} \right\}^{r-1},$$

and the necessary and sufficient conditions follows from the convergence of the geometrical series.

Example 1.2 *Consider the diagonal model*

$$X(\mathbf{t}) = a_1 X(\mathbf{t} - \mathbf{e}_1) + a_2 X(\mathbf{t} - \mathbf{e}_2) - a_1 a_2 X(\mathbf{t} - \mathbf{1}) + c X(\mathbf{t} - \mathbf{l}) e(\mathbf{t} - \mathbf{l}) + e(\mathbf{t}),$$

where $\mathbf{e}_1 = (1,0), \mathbf{e}_2 = (0,1), \mathbf{1} = (1,1), \mathbf{l} = (l_1, l_2)$ *and* $l_1 \geq 1, l_2 \geq 2$. *For this model, we assume that* $\max\{|a_1|, |a_2|\} < 1$ *that ensure that the roots of the polynomial* $\Theta(\mathbf{z})$ *are outside the unit* *circles* $|z_i| \leq 1, i = 1,2$. *From the Theorem 1.2, it follows that the transfer functions of a regular stationary solution are*

$$\begin{aligned} f_0 &= \frac{c\sigma^2}{\Theta(0,0)}, \\ f_1(\lambda,\mu) &= \frac{\left(1 + cf_0 e^{-i(l_1\lambda + l_2\mu}\right)}{\Theta(\lambda,\mu)}, \\ f_r\left(\lambda_{(r)},\mu_{(r)}\right) &= \frac{ce^{-i\sum_{j=1}^{r}\left(l_1\lambda_j + l_2\mu_j\right)}}{\Theta\left(\sum_{j=1}^{r}\lambda_j, \sum_{j=1}^{r}\mu_j\right)} f_{r-1}\left(\lambda_{(r-1)},\mu_{(r-1)}\right), \\ &= \prod_{s=1}^{r} \frac{\left(1 + cf_0 e^{-i(l_1\lambda + l_2\mu}\right)}{\Theta\left(\sum_{j=1}^{s}\lambda_j, \sum_{j=1}^{s}\mu_j\right)} c^{r-1} e^{-i\sum\limits_{j=2}^{r}(r-j+1)\left(l_1\lambda_j + l_2\mu_j\right) - i(r-1)(l_1\lambda_1 + l_2\mu_1)}, \end{aligned}$$

with $\Theta(\lambda,\mu) = \left(1 - a_1 e^{-i\lambda}\right)\left(1 - a_2 e^{-i\mu}\right)$. *Hence, from Corollary 1.2, the necessary and sufficient condition become* $a_1^2 + a_2^2 + \sigma^2 c^2 - a_1^2 a_2^2 < 1$.

For spatial super-diagonal model in the sense of Hannan (1982) for which $c_{\mathbf{i},\mathbf{j}} = 0$ for $\mathbf{i} \preceq \mathbf{j}$ in (1.3.2) their transfer functions (1.3.3) becomes quite simpler. Indeed,

$$f_r\left(\boldsymbol{\lambda}_{(r)}\right) = \begin{cases} 0, \text{ if } r = 0, \\ \dfrac{\Phi(\boldsymbol{\lambda})}{\Theta(\boldsymbol{\lambda})}, \text{ if } r = 1, \\ \dfrac{\Psi\left(\sum_{j=1}^{r-1}\boldsymbol{\lambda}_j, \boldsymbol{\lambda}_r\right)}{\Theta\left(\sum_{j=1}^{r}\boldsymbol{\lambda}_j\right)} f_{r-1}\left(\boldsymbol{\lambda}_{(r-1)}\right), \text{ if } r \geq 2, \end{cases}$$

so, we obtain $f_r\left(\boldsymbol{\lambda}_{(r)}\right) = \prod_{s=2}^{r} \frac{\Psi\left(\sum_{j=1}^{s-1}\boldsymbol{\lambda}_j, \boldsymbol{\lambda}_s\right)}{\Theta\left(\sum_{j=1}^{s}\boldsymbol{\lambda}_j\right)} \Psi_0\left(\boldsymbol{\lambda}_1\right)$. It is evident that this gives an unique solution of spatial super-diagonal model in the form (1.2.1) with $f_0 = 0$. Indeed, using the last expression of $f_r\left(\boldsymbol{\lambda}_{(r)}\right)$, the condition of stationarity is

$$\sum_{r=1}^{\infty} \left(\frac{\sigma^2}{(2\pi)^d}\right)^r \int_{\boldsymbol{\pi}^r} \left|f_r\left(\boldsymbol{\lambda}_{(r)}\right)\right|^2 d\boldsymbol{\lambda}_{(r)} < +\infty. \tag{1.3.6}$$

The following theorem gives a simple sufficient condition for the existence of a regular second order stationary solution

Theorem 1.3 *A sufficient condition for the spatial super-diagonal model to has a regular stationary solution is that*

$$\frac{\sigma^2}{(2\pi)^d}\int\limits_{\boldsymbol{\pi}}\left|\frac{\Psi(\boldsymbol{\lambda},\boldsymbol{\mu})}{\Theta(\boldsymbol{\lambda}+\boldsymbol{\mu})}\right|^2 d\boldsymbol{\lambda}\leq K<1, \boldsymbol{\mu}\in\boldsymbol{\pi},$$

Proof. It is easy to see that the series (1.3.6) is dominated by a geometrically converged series.

Remark 1.4 *Under the Conditions of Theorem 1.3, the Series* (1.2.2) *corresponding to the super-diagonal model, converges a.s. Indeed, in this case* $X(\mathbf{t})=\sum_{r\geq 1}\xi_r(\mathbf{t})$ *where* $\xi_r(\mathbf{t})=\sum\limits_{\mathbf{k}_{(r)}\in\mathbb{N}^r}\sum\limits_{\mathbf{s}_{(r)}\in(\mathbb{Z}^d_{\prec})^r} g_{\mathbf{k}_{(r)}}\left(\mathbf{s}_{(r)}\right)\prod\limits_{j=1}^{r}e(\mathbf{t}-\mathbf{s}_j)$. *Then under the conditions of the Theorem 1.3, the series* $\sum\limits_{r\geq 1}\xi_r(\mathbf{t})$ *converges a.s, since* $E\left\{|\xi_r(\mathbf{t})|\right\}\leq\sqrt{E\left\{\xi_r^2(\mathbf{t})\right\}}$ *and thus* $\sum\limits_{r\geq 1}E\left\{|\xi_r(\mathbf{t})|\right\}$ *is dominated by a geometrically convergent series.*

Remark 1.5 *If the assumption that the model is super-diagonal is eliminated, then the results of Theorem 1.3 still holds (see Terdik (2000)).*

As already mentioned by Wang and Wei (2004), that it is rather difficult to check the condition (1.3.4) in Theorem 1.2 because the calculation of $\int\limits_{\boldsymbol{\pi}^r}\left|f_r\left(\boldsymbol{\lambda}_{(r)}\right)\right|^2 dF(\boldsymbol{\lambda}_{(r)})$ is tedious when r is too large. To remedy this difficulty, Wang and Wei (2004) introduce a separable subdiagonal model in the sense that $\Psi(\boldsymbol{\lambda},\boldsymbol{\mu})=\Psi_1(\boldsymbol{\lambda}+\boldsymbol{\mu})\Psi_2(\boldsymbol{\lambda})$ where $\Psi_1(\boldsymbol{\lambda})=\sum\limits_{\mathbf{i}\in\Gamma]\mathbf{Q}]}c_{\mathbf{i}}^{(1)}e^{-i\mathbf{i}.\boldsymbol{\lambda}}$ and $\Psi_2(\boldsymbol{\lambda})=\sum\limits_{\mathbf{j}\in\Gamma[\mathbf{P}]}c_{\mathbf{j}}^{(2)}e^{-i\mathbf{j}.\boldsymbol{\lambda}}$. In this case $c_{\mathbf{i},\mathbf{j}}=c_{\mathbf{i}}^{(1)}c_{\mathbf{j}}^{(2)}$ with $\mathbf{i}\in\Gamma\,]\mathbf{Q}]$, $\mathbf{j}\in\Gamma\,[\mathbf{P}]$ and the Equation (1.3.2) become

$$X(\mathbf{t})=\sum_{\mathbf{i}\in\Gamma]\mathbf{p}]}a_{\mathbf{i}}X(\mathbf{t}-\mathbf{i})+\sum_{\mathbf{j}\in\Gamma[\mathbf{q}]}b_{\mathbf{j}}e(\mathbf{t}-\mathbf{j})+\sum_{\mathbf{i}\in\Gamma]\mathbf{Q}]}c_{\mathbf{i}}^{(1)}e(\mathbf{t}-\mathbf{i})\sum_{\mathbf{j}\in\Gamma[\mathbf{P}]}c_{\mathbf{j}}^{(2)}X(\mathbf{t}-\mathbf{i}-\mathbf{j}), \tag{1.3.7}$$

and hence $f_0=\sigma^2c_{\mathbf{0}}^{(2)}\Theta^{-1}(\mathbf{0})\,\Psi_1(\mathbf{0})$, $\Psi_0(\mathbf{0})=c_{\mathbf{0}}^{(2)}\,\Psi_1(\mathbf{0})$ and $\Phi^*(\boldsymbol{\lambda})=\Phi(\boldsymbol{\lambda})+f_0\Psi_2(\mathbf{0})\,\Psi_1(\boldsymbol{\lambda})$. For this class of models, we have

Theorem 1.4 *A necessary and sufficient condition for the existence of regular second order stationary solution of the process* $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ *generated by the separable* $SBL_d(\mathbf{p},\mathbf{q},\mathbf{P},\mathbf{Q})$ *model* (1.3.7) *is that*

$$\frac{\sigma^2}{(2\pi)^d}\int\limits_{\boldsymbol{\pi}}\left|\frac{\Psi_1(\boldsymbol{\lambda})\,\Psi_2(\boldsymbol{\lambda})}{\Theta(\boldsymbol{\lambda})}\right|^2 d\boldsymbol{\lambda}<1.$$

Proof. W*e use a similar approach to that used by Wang and Wei (2004), Theorem 1.* Indeed, for any $r \geq 2$ we have

$$\int_{\boldsymbol{\pi}^r} \left|f_r\left(\boldsymbol{\lambda}_{(r)}\right)\right|^2 dF\left(\boldsymbol{\lambda}_{(r)}\right)$$

$$= \int_{\boldsymbol{\pi}^r} \left|\prod_{l=1}^{r} \Theta^{-1}\left(\sum_{j=1}^{l} \boldsymbol{\lambda}_j\right) \Psi\left(\sum_{j=1}^{l-1} \boldsymbol{\lambda}_j, \boldsymbol{\lambda}_l\right)\right|^2 \left|\Theta^{-1}\left(\boldsymbol{\lambda}_1\right) \Phi^*\left(\boldsymbol{\lambda}_1\right)\right| dF\left(\boldsymbol{\lambda}_{(r)}\right)$$

$$= \int_{\boldsymbol{\pi}^r} \left|\prod_{l=1}^{r} \Theta^{-1}\left(\sum_{j=1}^{l} \boldsymbol{\lambda}_j\right) \Psi_1\left(\sum_{j=1}^{l} \boldsymbol{\lambda}_j\right) \Psi_2\left(\sum_{j=1}^{l-1} \boldsymbol{\lambda}_j\right)\right|^2 \left|\Theta^{-1}\left(\boldsymbol{\lambda}_1\right) \Phi^*\left(\boldsymbol{\lambda}_1\right)\right| dF\left(\boldsymbol{\lambda}_{(r)}\right)$$

$$= \int_{\boldsymbol{\pi}} \left|\Theta^{-1}\left(\boldsymbol{\lambda}\right) \Psi_1\left(\boldsymbol{\lambda}\right)\right|^2 dF(\boldsymbol{\lambda}) \left[\int_{\boldsymbol{\pi}} \left|\Theta^{-1}\left(\boldsymbol{\lambda}\right) \Psi_1\left(\boldsymbol{\lambda}\right) \Psi_2\left(\boldsymbol{\lambda}\right)\right|^2 dF\left(\boldsymbol{\lambda}\right)\right]^{r-2}$$

$$\times \int_{\boldsymbol{\pi}} \left|\Theta^{-1}\left(\boldsymbol{\lambda}_1\right) \Phi^*\left(\boldsymbol{\lambda}_1\right) \Psi_2\left(\boldsymbol{\lambda}\right)\right|^2 dF\left(\boldsymbol{\lambda}\right).$$

The result fellows by Theorem 1.2 if and only if $\int_{\boldsymbol{\pi}} \left|\Theta^{-1}\left(\boldsymbol{\lambda}\right) \Psi_1\left(\boldsymbol{\lambda}\right) \Psi_2\left(\boldsymbol{\lambda}\right)\right|^2 dF\left(\boldsymbol{\lambda}\right) < 1.$

Corollary 1.3 *Consider the model*

$$X(\mathbf{t}) = \sum_{\mathbf{i}\in\Gamma]\mathbf{p}]} a_{\mathbf{i}} X(\mathbf{t}-\mathbf{i}) + \sum_{\mathbf{j}\in\Gamma[\mathbf{q}]} b_{\mathbf{j}} e(\mathbf{t}-\mathbf{j}) + \sum_{\mathbf{j}\in\Gamma]\mathbf{P}]} c_{\mathbf{j}} X(\mathbf{t}-\mathbf{j}-\mathbf{l}) e(\mathbf{t}-\mathbf{j}), \tag{1.3.8}$$

where $\mathbf{l}$ *is a known vector of nonnegative integers. Then the necessary and sufficient condition for the existence of regular second order stationary solution for* (1.3.8) *is that*

$$\frac{\sigma^2}{(2\pi)^d} \int_{\boldsymbol{\pi}} \left|\frac{\Psi_1\left(\boldsymbol{\lambda}\right)}{\Theta\left(\boldsymbol{\lambda}\right)}\right|^2 d\boldsymbol{\lambda} < 1, \tag{1.3.9}$$

Proof. In this case, $\Psi_1\left(\boldsymbol{\lambda}\right) = \sum_{\mathbf{i}\in\Gamma]\mathbf{Q}]} c_{\mathbf{i}} e^{-i\boldsymbol{\lambda}.\mathbf{i}}$, $\Psi_2\left(\boldsymbol{\lambda}\right) = e^{-i\boldsymbol{\lambda}.\mathbf{l}}$ and $\Psi_0\left(\boldsymbol{\lambda}\right) = \Psi_1\left(\boldsymbol{\lambda}\right) \delta_{\mathbf{0}}\left(\mathbf{l}\right)$. So the necessary and sufficient condition for the regular second-order stationary solution reduce to (1.3.9).

Corollary 1.4 *Consider the model*

$$X(\mathbf{t}) = \sum_{\mathbf{i}\in\Gamma]\mathbf{p}]} a_{\mathbf{i}} X(\mathbf{t}-\mathbf{i}) + \sum_{\mathbf{j}\in\Gamma[\mathbf{q}]} b_{\mathbf{j}} e(\mathbf{t}-\mathbf{j}) + \sum_{\mathbf{j}\in\Gamma]\mathbf{P}]} c_{\mathbf{j}} X(\mathbf{t}-\mathbf{j}-\mathbf{l}) e(\mathbf{t}-\mathbf{l}), \tag{1.3.10}$$

where $\mathbf{l}$ *is a known vector of nonnegative integers. Then the necessary and sufficient condition for the existence of regular second order stationary solution for* (1.3.10) *is that*

$$\frac{\sigma^2}{(2\pi)^d}\int_{\boldsymbol{\pi}}\left|\frac{\Psi_2(\boldsymbol{\lambda})}{\Theta(\boldsymbol{\lambda})}\right|^2 d\boldsymbol{\lambda}<1. \tag{1.3.11}$$

Proof. In this case, $\Psi_1(\boldsymbol{\lambda})=e^{-i\mathbf{l}.\boldsymbol{\lambda}}$ and $\Psi_2(\boldsymbol{\lambda})=\sum\limits_{\mathbf{i}\in\Gamma]\mathbf{Q}]} c_{\mathbf{i}}e^{-i\mathbf{i}.\boldsymbol{\lambda}}$, so that the necessary and sufficient condition for the regular second order stationary solution reduce to (1.3.11).

Remark 1.6 *As already pointed by Wang and Wei (2004), the separable spatial models is a rather general class of spatial bilinear models, it is includes several subclass of popular spatial models specially the spatial GARCH models (c.f. Terdik (2000)).*

Corollary 1.5 [*The SGARCH*] *Consider the spatial GARCH* $(\mathbf{p},\mathbf{q})$ *models defined by*

$$\begin{cases} X(\mathbf{t})=\eta(\mathbf{t})\sqrt{h(\mathbf{t})}, \\ h(\mathbf{t})=c_{\mathbf{0}}+\sum\limits_{\mathbf{i}\in\Gamma]\mathbf{p}]} c_{\mathbf{i}}X^2(\mathbf{t}-\mathbf{i})+\sum\limits_{\mathbf{j}\in\Gamma]\mathbf{q}]} a_{\mathbf{j}}h(\mathbf{t}-\mathbf{j}), \end{cases} \tag{1.3.12}$$

where $(c_{\mathbf{i}},\mathbf{i}\in\Gamma[\mathbf{p}])$ *and* $(a_{\mathbf{i}},\mathbf{i}\in\Gamma]\mathbf{q}])$ *are nonnegative constants with* $c_{\mathbf{0}}>0$ *and* $(\eta(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ *is a Gaussian white noise field with zero mean and variance* 1. *Then the Model* (1.3.12) *has a regular second order stationary solution if and only if*

$$\sum_{\mathbf{i}\in\Gamma]\mathbf{p}]} a_{\mathbf{i}}+\sum_{\mathbf{j}\in\Gamma]\mathbf{q}]} c_{\mathbf{j}}<1.$$

Moreover

$$E\{X(\mathbf{t})\}=0, Cov\{X(\mathbf{t})X(\mathbf{s})\}=\begin{cases} 0 \text{ if } \mathbf{t}\neq\mathbf{s}, \\ a_{\mathbf{0}}\delta_{\mathbf{s}}^{\mathbf{t}}\left(1-\sum\limits_{\mathbf{i}\in\Gamma]\mathbf{p}]} a_{\mathbf{i}}-\sum\limits_{\mathbf{j}\in\Gamma]\mathbf{q}]} c_{\mathbf{j}}\right)^{-1} & \text{otherwise.}\end{cases}$$

Proof. Since the second equation in (1.3.12) can be regarded as a special case of diagonal model (1.3.5), then the proof follows thus from the Corollary 1.4 and the positivity of the coefficients.

1.4 Covariance structure and spectral density function

We assume in this section, that the field process $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ generated by (1.3.2) admits a regular second order stationary solution, and obtain its covariance function and its spectral density function.

For this purpose, we note that for any $r \geq 2$, the rth transfer function can be decomposed into two orthogonal parts, i.e.,

$$\begin{aligned}
f_r\left(\boldsymbol{\lambda}_{(r)}\right) &= \left[\prod_{k=2}^{r} \frac{\Psi\left(\sum_{j=1}^{k-1} \boldsymbol{\lambda}_j, \boldsymbol{\lambda}_k\right)}{\Theta\left(\sum_{j=1}^{k} \boldsymbol{\lambda}_j\right)}\right] \frac{\Phi^*\left(\boldsymbol{\lambda}_1\right)}{\Theta\left(\boldsymbol{\lambda}_1\right)} \\
&= \left[\prod_{k=3}^{r} \frac{\Psi\left(\sum_{j=1}^{k-1} \boldsymbol{\lambda}_j, \boldsymbol{\lambda}_k\right)}{\Theta\left(\sum_{j=1}^{k} \boldsymbol{\lambda}_j\right)}\right] \left[\frac{\Psi\left(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2\right)}{\Theta\left(\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2\right)} \frac{\Phi^*\left(\boldsymbol{\lambda}_1\right)}{\Theta\left(\boldsymbol{\lambda}_1\right)} - \frac{\Psi_0\left(\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2\right)}{\Theta\left(\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2\right)}\right] \\
&\quad + \left[\prod_{k=3}^{r} \frac{\Psi\left(\sum_{j=1}^{k-1} \boldsymbol{\lambda}_j, \boldsymbol{\lambda}_k\right)}{\Theta\left(\sum_{j=1}^{k} \boldsymbol{\lambda}_j\right)}\right] \left[\frac{\Psi_0\left(\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2\right)}{\Theta\left(\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2\right)}\right] \\
&= f_r^{(1)}\left(\boldsymbol{\lambda}_{(r)}\right) + f_r^{(2)}\left(\boldsymbol{\lambda}_{(r)}\right).
\end{aligned}$$

The following lemma is an extension of the result obtained by Wang and Wei (2004), Lemma 2.

Lemma 1.3 *Let* $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ *be a spatial second order stationary bilinear model satisfying* (1.3.2) *and let* $C(\mathbf{h}) := Cov\left\{X(\mathbf{t})X(\mathbf{t}+\mathbf{h})\right\}$. *Then for any* $\mathbf{h} \in\mathbb{Z}^d$

$$C(\mathbf{h}) = \frac{\sigma^2}{(2\pi)^d} \int_{\boldsymbol{\pi}} \left\{ \left|\frac{\Phi^*\left(\boldsymbol{\lambda}\right)}{\Theta\left(\boldsymbol{\lambda}\right)}\right|^2 + \sigma^2 \left|\frac{\Psi_0\left(\boldsymbol{\lambda}\right)}{\Theta\left(\boldsymbol{\lambda}\right)}\right|^2 + \frac{\sigma^2}{(2\pi)^d} \int_{\boldsymbol{\pi}} \left|\frac{\Psi\left(\boldsymbol{\theta}, \boldsymbol{\lambda}-\boldsymbol{\theta}\right)}{\Theta\left(\boldsymbol{\lambda}\right)}\right|^2 W(\boldsymbol{\theta}) d\boldsymbol{\theta} \right\} e^{-i\mathbf{h}.\boldsymbol{\lambda}} d\boldsymbol{\lambda}, \quad (1.4.1)$$

and the spatial spectral density function $f_X(\boldsymbol{\lambda})$ *is given by*

$$f_X(\boldsymbol{\lambda}) = \frac{\sigma^2}{(2\pi)^d} \left[\left|\frac{\Phi^*(\boldsymbol{\lambda})}{\Theta\left(\boldsymbol{\lambda}\right)}\right|^2 + \sigma^2 \left|\frac{\Psi_0\left(\boldsymbol{\lambda}\right)}{\Theta\left(\boldsymbol{\lambda}\right)}\right|^2 + \frac{\sigma^2}{(2\pi)^d} \int_{\boldsymbol{\pi}} \left|\frac{\Psi\left(\boldsymbol{\theta}, \boldsymbol{\lambda}-\boldsymbol{\theta}\right)}{\Theta\left(\boldsymbol{\lambda}\right)}\right|^2 W\left(\boldsymbol{\theta}\right) d\boldsymbol{\theta} \right],$$

where $W(\boldsymbol{\theta}) = \sum_{j=0}^{\infty} W_j(\boldsymbol{\theta})$ *with* $W_j(\boldsymbol{\theta})$ *can be computed recursively by*

$$W_j(\boldsymbol{\theta}) := \begin{cases} \left|\dfrac{\Phi^*\left(\boldsymbol{\theta}\right)}{\Theta\left(\boldsymbol{\theta}\right)}\right|^2 + \sigma^2 \left|\dfrac{\Psi_0\left(\boldsymbol{\theta}\right)}{\Theta\left(\boldsymbol{\theta}\right)}\right|^2, \ \text{if } j=0, \\ \dfrac{\sigma^2}{(2\pi)^d} \displaystyle\int_{\boldsymbol{\pi}} \left|\dfrac{\Psi\left(\boldsymbol{\lambda}, \boldsymbol{\theta}-\boldsymbol{\lambda}\right)}{\Theta\left(\boldsymbol{\theta}\right)}\right|^2 W_{j-1}(\boldsymbol{\lambda}) d\boldsymbol{\lambda}, \ \text{if } j \geq 1. \end{cases}$$

The Lemma 1.3, shows that the second order properties of $SBL_d(\mathbf{p}, \mathbf{q}, \mathbf{P}, \mathbf{Q})$ generated by the difference Equation (1.3.2) are similar to a linear spatial $ARMA$. More precisely, there exists an uncorrelated sequence of random variables $(\xi(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ with zero mean and finite variance such that the process $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ satisfies the stochastic difference equation $\Theta(B)X(\mathbf{t}) = \Lambda(B)\xi(\mathbf{t})$. The sequence $(\xi(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ is not Gaussian nor a martingale difference sequence when the $c_{\mathbf{ij}}$'s are not equal to zero. We draw the conclusion that first and second order properties of *superdiagonal spatial bilinear* model can not be distinguished from a linear spatial $ARMA$ models. Specific tools should be developed. We leave this important issue for future researches.

Some elegant expressions for spatial superdiagonal and separable subdiagonal bilinear models can be derived.

Theorem 1.5 *Let $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ be a second order stationary spatial bilinear model satisfying* (1.3.7). *Then*

$$\begin{aligned} C(\mathbf{h}) &= \frac{\sigma^2}{(2\pi)^d}\int\limits_{\boldsymbol{\pi}}\left\{\left|\frac{\Phi^*(\boldsymbol{\lambda})}{\Theta(\boldsymbol{\lambda})}\right|^2+\left|\frac{\Psi_1(\boldsymbol{\lambda})}{\Theta(\boldsymbol{\lambda})}\right|^2 W\right\}e^{-i\mathbf{h}.\boldsymbol{\lambda}}d\boldsymbol{\lambda}, \\ f_X(\boldsymbol{\lambda}) &= \frac{\sigma^2}{(2\pi)^d}\left\{\left|\frac{\Phi^*(\boldsymbol{\lambda})}{\Theta(\boldsymbol{\lambda})}\right|^2+\left|\frac{\Psi_1(\boldsymbol{\lambda})}{\Theta(\boldsymbol{\lambda})}\right|^2 W\right\}, \end{aligned}$$

where

$$W=\frac{\sigma^2}{(2\pi)^d}\int_{\boldsymbol{\pi}}\left\{\left|\frac{\Phi^*(\boldsymbol{\lambda})}{\Theta(\boldsymbol{\lambda})}\Psi_2(\boldsymbol{\lambda})\right|^2+\left(c_{\mathbf{0}}^{(2)}\right)^2\right\}\frac{d\boldsymbol{\lambda}}{1-S},\ S=\frac{\sigma^2}{(2\pi)^d}\int_{\boldsymbol{\pi}}\left|\frac{\Psi(\boldsymbol{\theta},\mathbf{0})}{\Theta(\boldsymbol{\theta})}\right|^2 d\boldsymbol{\theta}.$$

Proof. The proof follows essentially the same as Theorem 2 in Wang and Wei (2004).

Finally, we give the spectral densities for the models generated by (1.3.8) and (1.3.10).

Corollary 1.6 *The spectral density function for the spatial model generated by* (1.3.8) *is given by*

$$f_X(\boldsymbol{\lambda})=\frac{\sigma^2}{(2\pi)^d}\left\{\left|\frac{\Phi^*(\boldsymbol{\lambda})}{\Theta(\boldsymbol{\lambda})}\right|^2+\left|\frac{\Psi_1(\boldsymbol{\lambda})}{\Theta(\boldsymbol{\lambda})}\right|^2 W\right\},$$

where

$$W=\frac{\sigma^2}{(2\pi)^d}\int_{\boldsymbol{\pi}}\left\{\delta_{\mathbf{0}}^{\mathbf{1}}+\left|\frac{\Phi^*(\boldsymbol{\lambda})}{\Theta(\boldsymbol{\lambda})}\right|^2\right\}\frac{d\boldsymbol{\lambda}}{1-S},\ S=\frac{\sigma^2}{(2\pi)^d}\int_{\boldsymbol{\pi}}\left|\frac{\Psi_1(\boldsymbol{\theta})}{\Theta(\boldsymbol{\theta})}\right|^2 d\boldsymbol{\theta}.$$

Corollary 1.7 *The spectral density function for the spatial model generated by* (1.3.10) *is given by*

$$f_X(\boldsymbol{\lambda})=\frac{\sigma^2}{(2\pi)^d}\left\{\left|\frac{\Phi^*(\boldsymbol{\lambda})}{\Theta(\boldsymbol{\lambda})}\right|^2+\frac{W}{|\Theta(\boldsymbol{\lambda})|^2}\right\},$$

where

$$W=\frac{\sigma^2}{(2\pi)^d}\int_{\boldsymbol{\pi}}\left\{\left|\frac{\Phi^*(\boldsymbol{\lambda})}{\Theta(\boldsymbol{\lambda})}\right|^2+\sigma^2 c_{\mathbf{0}}^2\right\}\frac{d\boldsymbol{\lambda}}{1-S},\ S=\frac{\sigma^2}{(2\pi)^d}\int_{\boldsymbol{\pi}}\left|\frac{\Psi_2(\boldsymbol{\theta})}{\Theta(\boldsymbol{\theta})}\right|^2 d\boldsymbol{\theta}.$$

Example 1.3 *Consider the super-diagonal process*

$$X\left(\mathbf{t}\right)=a_{1}X(\mathbf{t}-\mathbf{e}_{1})+a_{2}X\left(\mathbf{t}-\mathbf{e}_{2}\right)-a_{1}a_{2}X\left(\mathbf{t}-\mathbf{1}\right)+cX\left(\mathbf{t}-\mathbf{2}\right)e\left(\mathbf{t}-\mathbf{1}\right)+e\left(\mathbf{t}\right), \qquad (1.4.2)$$

where $(e(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^2}$ *is a Gaussian white noise with zero mean and variance* $\sigma^2=1$ *and Max* $\{|a_1|,|a_2|\}<1$. *Then the Model* (1.4.2) *has a regular second order stationary solution if and only if* $c^2<(1-a_1^2)(1-a_2^2)$. *Under these conditions,*

$$\Theta\left(\lambda,\mu\right)=\left(1-a_{1}e^{-i\lambda}\right)\left(1-a_{2}e^{-i\mu}\right), \Phi^{*}\left(\lambda,\mu\right)=1, \Psi_{1}\left(\lambda,\mu\right)=ce^{-i(\lambda+\mu)}, \Psi_{2}\left(\lambda,\mu\right)=e^{-i(\lambda+\mu)}.$$

So

$$\begin{aligned} C(h,l) &= \frac{\sigma^2}{a_1^{h-2}a_2^{l-2}}\left\{(h-1)(l-1)+\frac{c^2\sigma^2(h+1)(l+1)}{a_1^2a_2^2}\left[\delta_{\mathbf{0}}^{\mathbf{l}}+(1-a_1^2)(1-a_2^2)\right]\right\}, \\ f_X\left(\lambda,\mu\right) &= \frac{\sigma^2}{(2\pi)^2\left|(1-a_1e^{-i\lambda})\left(1-a_2e^{-i\mu}\right)\right|^2}\left\{1+c^2\sigma^2\left[\delta_{\mathbf{0}}^{\mathbf{l}}+(1-a_1^2)(1-a_2^2)\right]\left|e^{-i(\lambda+\mu)}\right|^2\right\}. \end{aligned}$$

1.4.1 Applications

Theorem 1.6 [*ARMA representation*] *Assume that the field* $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ *defined by* (1.3.2) *is stationary, there exists an uncorrelated sequence of random fields* $(\xi(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ *with zero mean and finite variance such that*

$$X(\mathbf{t})=a_{\mathbf{0}}+\sum_{\mathbf{i}\in\Gamma]\mathbf{p}]}a_{\mathbf{i}}X(\mathbf{t}-\mathbf{i})+\sum_{\mathbf{j}\in\Gamma[\mathbf{q}^*]}b_{\mathbf{j}}^{*}\xi(\mathbf{t}-\mathbf{j}),\ b_{\mathbf{0}}^{*}=b_{\mathbf{0}}=1, \qquad (1.4.3)$$

where the coefficients $\left(b_{\mathbf{j}}^{*},\mathbf{j}\in\Gamma\left[\mathbf{q}^{*}\right]\right)$ *are functions of* $(a_{\mathbf{j}},\mathbf{j}\in\Gamma]\mathbf{p}])$, $(b_{\mathbf{j}},\mathbf{j}\in\Gamma|\mathbf{q}])$ *and* $(c_{\mathbf{ji}},\mathbf{j}\in\Gamma]\mathbf{Q}],\mathbf{i}\in\Gamma[\mathbf{P}])$. *The field* $(\xi(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ *is not Gaussian nor a martingale difference sequence when the* $c_{\mathbf{ij}}$*'s are not equal to zero.*

Proof. The proof follows essentially the same as that of Theorem 2 in Bibi (2003).
The above theorem implies that the spectral density of the field $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ is given by

$$f(\boldsymbol{\lambda})=\frac{\sigma^2}{(2\pi)^d}\frac{\left|\widetilde{\Phi}\left(\boldsymbol{\lambda}\right)\right|^2}{\left|\Theta\left(\boldsymbol{\lambda}\right)\right|^2}, \qquad (1.4.4)$$

where $\widetilde{\Phi}\left(\boldsymbol{\lambda}\right)=\sum_{\mathbf{j}\in\Gamma[\mathbf{q}^*]}b_{\mathbf{j}}^{*}e^{-i\mathbf{j}.\boldsymbol{\lambda}}$ such that $\left|\widetilde{\Phi}\left(\boldsymbol{\lambda}\right)\right|^2=\left|\Phi\left(\boldsymbol{\lambda}\right)\right|^2+\sigma^2\left|\Psi_0\left(\boldsymbol{\lambda}\right)\right|^2+\left|D\left(\boldsymbol{\lambda}\right)\right|^2$ for some transfer function $D\left(\boldsymbol{\lambda}\right)$. Hence, the second order properties of every bilinear random field $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ satisfying the Equation (1.3.2) are similar to an $ARMA\left(\mathbf{p},\mathbf{q}^*\right)$. So, one has to look to higher order moments and higher-order cumulant spectra for further information on the process. The best linear predictor of $X(\mathbf{t}+\mathbf{h})$ given $\{X(\mathbf{s}),\mathbf{s}\preceq\mathbf{t}\}$ where $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ satisfies (1.4.3) is now given

Theorem 1.7 *Let* $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ *be a stationary random field satisfying* (1.4.3) *and assume that the polynomial* $\widetilde{\Phi}(\mathbf{z}) = \sum_{\mathbf{j}\in\Gamma[\mathbf{q}^*]} b^*_{\mathbf{j}}\mathbf{z}^{\mathbf{j}} \neq 0$ *for all* $\mathbf{z}\in\mathbb{C}^d : |z_i| \leq 1,\ i = 1, ..., d$. *Let* $\widehat{X}_{\mathbf{h}}(\mathbf{t})$ *be the best linear predictor of* $X(\mathbf{t}+\mathbf{h})$, $\mathbf{0}\preceq\mathbf{h}\preceq\mathbf{1}$ *and* $\mathbf{h}\neq\mathbf{0}$ *when* $\{X(\mathbf{s}), \mathbf{s}\preceq\mathbf{t}\}$ *is given. Then*

$$\widehat{X}_{\mathbf{h}}(\mathbf{t}) = \left(1 - \frac{\Theta(\mathbf{B})}{\widetilde{\Phi}(\mathbf{B})}\right) X(\mathbf{t}+\mathbf{h}),$$

where $\mathbf{B}$ *denotes the backward shift operator, i.e.,* $\mathbf{B}^{\mathbf{i}}X(\mathbf{t}) = X(\mathbf{t}-\mathbf{i})$ *and* $\sigma^2_\xi = Var\{\xi(\mathbf{t})\} > Var\{e(\mathbf{t})\} = \sigma^2$.

Proof. The first assertion rests standard. For the second we have from (1.4.4) and since $\sigma^2 |\Psi_0(\boldsymbol{\lambda})|^2 + |D(\boldsymbol{\lambda})|^2 > 0$

$$\begin{aligned} E\left\{\left(X(\mathbf{t}+\mathbf{h}) - \widehat{X}_{\mathbf{h}}(\mathbf{t})\right)^2\right\} &= \sigma^2_\xi = \exp\left\{\frac{1}{(2\pi)^d}\int_{\boldsymbol{\pi}} \log(2\pi)^d f(\boldsymbol{\lambda})d\boldsymbol{\lambda}\right\} \\ &= \sigma^2 \exp\left\{\frac{1}{(2\pi)^d}\int_{\boldsymbol{\pi}} \log(|\Phi(\boldsymbol{\lambda})|^2 + \sigma^2|\Psi_0(\boldsymbol{\lambda})|^2 + |D(\boldsymbol{\lambda})|^2)d\boldsymbol{\lambda}\right\} > \sigma^2. \end{aligned}$$

Hence the variance of the prediction error is always greater than the optimal prediction error variance obtained from the bilinear field model.

1.5 Yule-Walker type difference equations for $SBL_d(\mathbf{p}, \mathbf{q}, \mathbf{P}, \mathbf{Q})$ model

In order to understand the second-order probabilistic structure for spatial bilinear model better, we can construct the Yule-Walker-type difference equations for autocovariance functions which are based on the spectral density function. We have $\Phi^*(\boldsymbol{\lambda}) = \Phi(\boldsymbol{\lambda}) + \mu\Psi_2(\mathbf{0})\Psi_1(\boldsymbol{\lambda})$ and define

$$\begin{aligned} d_{\mathbf{k}} &= \frac{1}{(2\pi)^d}\int_{\boldsymbol{\pi}} \frac{\Phi(\boldsymbol{\lambda})}{\Theta(\boldsymbol{\lambda})} e^{i\mathbf{k}.\lambda} d\boldsymbol{\lambda}, \\ e_{\mathbf{k}} &= \frac{1}{(2\pi)^d}\int_{\boldsymbol{\pi}} \frac{\Psi_1(\boldsymbol{\lambda})}{\Theta(\boldsymbol{\lambda})} e^{i\mathbf{k}.\lambda} d\boldsymbol{\lambda}. \end{aligned} \tag{1.5.1}$$

Then

$$C(\mathbf{h})-\sum_{\mathbf{i}\in\Gamma]\mathbf{p}]}a_{\mathbf{i}}C(\mathbf{h}-\mathbf{i}) = \int_{\boldsymbol{\pi}}\Theta\left(-\boldsymbol{\lambda}\right)f_X(\boldsymbol{\lambda})e^{-i\mathbf{h}.\lambda}d\boldsymbol{\lambda} \tag{1.5.2}$$

$$= \sigma^2\{\sum_{\mathbf{j}\in\Gamma[\mathbf{q}]}b_{\mathbf{j}}d_{\mathbf{j}-\mathbf{h}}+\mu\Psi_2\left(\mathbf{0}\right)\left[\sum_{\mathbf{i}\in\Gamma]\mathbf{Q}]}c_{\mathbf{i}}^{(1)}d_{\mathbf{i}-\mathbf{h}}+\sum_{\mathbf{j}\in\Gamma[\mathbf{q}]}b_{\mathbf{j}}e_{\mathbf{j}-\mathbf{h}}\right]$$
$$+[W+\mu^2(\Psi_2\left(\mathbf{0}\right))^2]\sum_{\mathbf{m}\in\Gamma]\mathbf{Q}]}c_{\mathbf{i}}^{(1)}e_{\mathbf{i}-\mathbf{h}}\}.$$

Let $\mathbf{q}^*=\max(\mathbf{q},\mathbf{Q}-\mathbf{1})$ if $c_{\mathbf{0}}^{(2)}=0$ and $\max(\mathbf{q},\mathbf{Q})$ otherwise. Obviously, if $\mathbf{h}\succ\mathbf{q}^*$, then the autocovariance functions satisfy the equation $C(\mathbf{h})-\sum_{\mathbf{i}\in\Gamma]\mathbf{p}]}a_{\mathbf{i}}C(\mathbf{h}-\mathbf{i})=0$. The process $(Y(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ obtained from $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ by

$$Y(\mathbf{t})=X(\mathbf{t})-\sum_{\mathbf{i}\in\Gamma]\mathbf{p}]}a_{\mathbf{i}}X(\mathbf{t}-\mathbf{i}), \tag{1.5.3}$$

is second order stationary too. We consider the autocovariance function for $(Y(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}, C_Y(\mathbf{h}):=$ $Cov\left\{Y(\mathbf{t})Y(\mathbf{t}+\mathbf{h})\right\}$ (only the case of $\mathbf{h}\succeq\mathbf{0}$ is discussed here because of the symmetric relation $C_Y(-\mathbf{h})=C_Y(\mathbf{h})$). We have $C_Y(\mathbf{h})=\int_{\boldsymbol{\pi}}\left|\Theta\left(\boldsymbol{\lambda}\right)\right|^2f_X(\boldsymbol{\lambda})e^{-i\mathbf{h}.\lambda}d\boldsymbol{\lambda}$, then

$$C_Y(\mathbf{h})=\begin{cases}\sigma^2\{\sum_{\mathbf{j}\in\Gamma[\mathbf{q}]}b_{\mathbf{j}}b_{\mathbf{j}+\mathbf{h}}+\mu\Psi_2\left(\mathbf{0}\right)\left[\sum_{\mathbf{i}\in\Gamma]\mathbf{Q}]}c_{\mathbf{i}}^{(1)}b_{\mathbf{i}+\mathbf{h}}+\sum_{\mathbf{j}\in\Gamma]\mathbf{q}]}b_{\mathbf{j}}c_{\mathbf{j}+\mathbf{h}}^{(1)}\right] & \\ \quad +[W+\mu^2(\Psi_2\left(\mathbf{0}\right))^2]\sum_{\mathbf{i}\in\Gamma]\mathbf{Q}]}c_{\mathbf{i}}^{(1)}c_{\mathbf{i}+\mathbf{h}}^{(1)}\} & ,\text{ if }\mathbf{h}\succeq\mathbf{0},\\ 0, & \text{if }\mathbf{h}\succ\mathbf{q}^*,\end{cases} \tag{1.5.4}$$

where $c_{\mathbf{i}}^{(1)}=0$ for $\mathbf{i}\succ\mathbf{Q}$ and $b_{\mathbf{j}}=0$ for $\mathbf{j}\succ\mathbf{q}$.

By (1.5.2) and (1.5.4), as far as the second-order structure is concerned, the $SBL_d(\mathbf{p},\mathbf{q},\mathbf{P},\mathbf{Q})$ model is similar to spatial $ARMA(\mathbf{p},\mathbf{q}^*)$ model.

For the two special cases given in section 1.3, for the spatial model generated by (1.3.8), We have $\Phi^*\left(\boldsymbol{\lambda}\right)=\Phi\left(\boldsymbol{\lambda}\right)+\mu\Psi_1\left(\boldsymbol{\lambda}\right)$. Then $C(\mathbf{0})=\int_{\boldsymbol{\pi}}f_X(\boldsymbol{\lambda})d\boldsymbol{\lambda}=W-\sigma^2\delta_{\mathbf{l}}^{\mathbf{0}}$. Therefore, using (1.5.2), we get

$$C(\mathbf{h})-\sum_{\mathbf{i}\in\Gamma]\mathbf{p}]}a_{\mathbf{i}}C(\mathbf{h}-\mathbf{i})=\begin{cases}\sigma^2[\sum_{\mathbf{j}\in\Gamma]\mathbf{q}]}b_{\mathbf{j}}d_{\mathbf{j}-\mathbf{h}}+\mu\sum_{\mathbf{i}\in\Gamma]\mathbf{Q}]}c_{\mathbf{i}}d_{\mathbf{i}-\mathbf{h}}+\mu\sum_{\mathbf{m}\in\Gamma]\mathbf{Q}]}b_{\mathbf{m}}e_{\mathbf{m}-\mathbf{k}} & \\ \quad +(C(\mathbf{0})+\mu^2+\sigma^2\delta_{\mathbf{l}}^{\mathbf{0}})\sum_{\mathbf{i}\in\Gamma]\mathbf{Q}]}c_{\mathbf{i}}e_{\mathbf{i}-\mathbf{h}}] & ,\text{ if }\mathbf{h}\succeq 0,\\ 0, & \text{if }\mathbf{h}\succ\max\left(\mathbf{q},\mathbf{Q}-\mathbf{1}+\delta_{\mathbf{l}}^{\mathbf{0}}\right),\end{cases}$$

where $d_{\mathbf{k}}$ and $e_{\mathbf{k}}$ are given in (1.5.1).

By (1.5.4), the autocovariance function for the process $(Y(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$, derived by (1.5.3), are given by

$$C_Y(\mathbf{h}) = \begin{cases} \sigma^2\left[\sum\limits_{\mathbf{j}\in\Gamma]\mathbf{q}]} b_{\mathbf{j}}b_{\mathbf{j}+\mathbf{h}} + \mu\sum\limits_{\mathbf{i}\in\Gamma]\mathbf{Q}]} c_{\mathbf{i}}b_{\mathbf{i}+\mathbf{h}} + \mu\sum\limits_{\mathbf{j}\in\Gamma]\mathbf{q}]} b_{\mathbf{j}}c_{\mathbf{j}+\mathbf{h}} + (C(\mathbf{0})+\mu^2+\sigma^2\delta_{\mathbf{l}}^0)\sum\limits_{\mathbf{i}\in\Gamma]\mathbf{Q}]} c_{\mathbf{i}}c_{\mathbf{i}+\mathbf{h}}\right], & \text{if } \mathbf{h}\succeq\mathbf{0}, \\ 0, \qquad \text{if } \mathbf{k}\succ\max(\mathbf{q},\mathbf{Q}-\mathbf{1}+\delta_{\mathbf{l}}^{\mathbf{0}}), \end{cases}$$

where $c_{\mathbf{i}}=0, \mathbf{i}\succ\mathbf{Q}$, and $b_{\mathbf{j}}=0, \mathbf{j}\succ\mathbf{q}$.

In the case for the spatial model generated by (1.3.10), we have $\Phi^*(\boldsymbol{\lambda}) = \Phi(\boldsymbol{\lambda}) + \mu\Psi_2(\mathbf{0})\,e^{-i\mathbf{l}.\boldsymbol{\lambda}}$ and let $d_{\mathbf{k}} = \frac{1}{2\pi}\int\limits_{\boldsymbol{\pi}}\frac{e^{i\mathbf{k}.\boldsymbol{\lambda}}}{\Theta(\boldsymbol{\lambda})}d\boldsymbol{\lambda}$. By (1.5.2), we have

$$C(\mathbf{h}) - \sum_{\mathbf{i}\in\Gamma]\mathbf{p}]} a_{\mathbf{i}}C(\mathbf{h}-\mathbf{i}) = \begin{cases} \begin{matrix} \sigma^2[\sum\limits_{\mathbf{j},\mathbf{i}\in\Gamma]\mathbf{q}]} b_{\mathbf{j}}b_{\mathbf{i}}d_{\mathbf{i}-\mathbf{j}-\mathbf{h}} + \mu\Psi_2(\mathbf{0})\sum\limits_{\mathbf{j}\in\Gamma]\mathbf{q}]} b_{\mathbf{j}}d_{\mathbf{k}-\mathbf{h}-\mathbf{j}} \\ +\mu\Psi_2(\mathbf{0})\sum\limits_{\mathbf{j}\in\Gamma]\mathbf{Q}]} b_{\mathbf{j}}d_{\mathbf{j}-\mathbf{k}-\mathbf{h}} + (W+\mu^2(\Psi_2(\mathbf{0}))^2)\delta_{\mathbf{k}}^{\mathbf{0}}] \end{matrix}, & \text{if } \mathbf{h}\succeq\mathbf{0}, \\ 0, \qquad \text{if } \mathbf{h}\succ\max(\mathbf{q},\mathbf{k}), \end{cases}$$

and the autocovariance functions for the process $(Y(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ are

$$C_Y(\mathbf{h}) = \begin{cases} \sigma^2[\sum\limits_{\mathbf{j}\in\Gamma]\mathbf{q}]} b_{\mathbf{j}}b_{\mathbf{j}+\mathbf{h}} + \mu\Psi_2(\mathbf{0})(b_{\mathbf{k}-\mathbf{h}}+b_{\mathbf{k}+\mathbf{h}}) + (W+\mu^2(\Psi_2(\mathbf{0}))^2)\delta_{\mathbf{h}}^{\mathbf{0}}], & \text{if } \mathbf{h}\succeq\mathbf{0}, \\ 0, \qquad \text{if } \mathbf{h}\succ\max(\mathbf{q},\mathbf{k}). \end{cases}$$

where $b_{\mathbf{j}}=0$ if $\mathbf{j}\succ\mathbf{q}$ or $\mathbf{j}\prec\mathbf{0}$.

1.5.1 The third-order probabilistic structure for the $SBL_d(\mathbf{p},\mathbf{q},\mathbf{P},\mathbf{Q})$ model

Based on our analysis above, it is shown that the second-order probabilistic structure of the $SBL_d(\mathbf{p},\mathbf{q},\mathbf{P},\mathbf{Q})$ model is similar to that of a spatial $ARMA$ model. This means that the $SBL_d(\mathbf{p},\mathbf{q},\mathbf{P},\mathbf{Q})$ model cannot be distinguished from the linear model only according to its autocovariance and spectral density. So we assume that $SBL_d(\mathbf{p},\mathbf{q},\mathbf{P},\mathbf{Q})$ model is third-order stationary and we investigate its third-order cumulants $C_3(\mathbf{s}_1,\mathbf{s}_2) = E\{(X(\mathbf{t})-\mu)(X(\mathbf{t}+\mathbf{s}_1)-\mu)(X(\mathbf{t}+\mathbf{s}_2)-\mu)\}$, and bispectral density $f_3(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2) = \frac{1}{(2\pi)^{2d}}\sum_{\mathbf{s}_1,\mathbf{s}_2} C_3(\mathbf{s}_1,\mathbf{s}_2)e^{-i(\mathbf{s}_1.\boldsymbol{\lambda}_1+\mathbf{s}_2\boldsymbol{\lambda}_2)}$. The approximate formulae to evaluate the third-order cumulants is given in the following theorem.

Theorem 1.8 *Let* $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ *be a stationary process generated by* $SBL_d(\mathbf{p},\mathbf{q},\mathbf{P},\mathbf{Q})$ *model* (1.3.7). *Then*

$$C_3(\mathbf{s}_1,\mathbf{s}_2)=\left(\frac{\sigma^2}{(2\pi)^d}\right)^2\int_{\boldsymbol{\pi}^2}\varphi(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)e^{i(\mathbf{s}_1.\boldsymbol{\lambda}_1+\mathbf{s}_2\boldsymbol{\lambda}_2)}d\boldsymbol{\lambda}_1 d\boldsymbol{\lambda}_2,$$

where $\varphi(\boldsymbol{\lambda}_{(2)})=6sym\phi(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_3)$ *and the function* $\phi(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_3)$ *is defined by*

$$\begin{aligned}
&\phi(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_3)\\
=\;&\frac{\Psi_1(\boldsymbol{\lambda}_1)\Phi^*(\boldsymbol{\lambda}_2)}{\Theta(\boldsymbol{\lambda}_1)\Theta(\boldsymbol{\lambda}_2)}\left|\frac{\Phi^*(\boldsymbol{\lambda}_3)}{\Theta(\boldsymbol{\lambda}_3)}\right|^2\Psi_2(-\boldsymbol{\lambda}_3)\\
&+\frac{\Psi_1(\boldsymbol{\lambda}_1)\Psi_1(\boldsymbol{\lambda}_2)}{\Theta(\boldsymbol{\lambda}_1)\Theta(\boldsymbol{\lambda}_2)}\left|\frac{\Phi^*(\boldsymbol{\lambda}_3)}{\Theta(\boldsymbol{\lambda}_3)}\right|^2\Psi_2(-\boldsymbol{\lambda}_3)\left[\frac{\sigma^2}{(2\pi)^d}\int_{\boldsymbol{\pi}}\frac{\Phi^*(\boldsymbol{\theta})}{\Theta(-\boldsymbol{\theta})}\Psi_2(\boldsymbol{\theta})\frac{\Psi(\boldsymbol{\theta}-\boldsymbol{\lambda}_3,0)}{\Theta(\boldsymbol{\theta}-\boldsymbol{\lambda}_3)}d\boldsymbol{\theta}\right]\\
&+\frac{\Psi_1(\boldsymbol{\lambda}_1)\Psi_1(\boldsymbol{\lambda}_2)\Phi^*(\boldsymbol{\lambda}_3)}{\Theta(\boldsymbol{\lambda}_1)\Theta(\boldsymbol{\lambda}_2)\Theta(\boldsymbol{\lambda}_3)}\left[\frac{\sigma^2}{(2\pi)^d}\int_{\boldsymbol{\pi}}\left|\frac{\Phi^*(\boldsymbol{\theta})}{\Theta(\boldsymbol{\theta})}\Psi_2(\boldsymbol{\theta})\right|^2\frac{\Psi(\boldsymbol{\theta}-\boldsymbol{\lambda}_3,0)}{\Theta(\boldsymbol{\theta}-\boldsymbol{\lambda}_3)}d\boldsymbol{\theta}\right]\\
&+\frac{\Psi_1(\boldsymbol{\lambda}_1)\Phi^*(\boldsymbol{\lambda}_2)}{\Theta(\boldsymbol{\lambda}_1)\Theta(\boldsymbol{\lambda}_2)}\left|\frac{\Psi_1(\boldsymbol{\lambda}_3)}{\Theta(\boldsymbol{\lambda}_3)}\right|^2\Psi_2(-\boldsymbol{\lambda}_3)\left[\frac{\sigma^2}{(2\pi)^d}\int_{\boldsymbol{\pi}}((c_0^{(2)})^2+\left|\frac{\Phi^*(\boldsymbol{\theta})}{\Theta(\boldsymbol{\theta})}\Psi_2(\boldsymbol{\theta})\right|^2)d\boldsymbol{\theta}\right]\\
&+\frac{\Psi_1(\boldsymbol{\lambda}_1)\Psi_1(\boldsymbol{\lambda}_2)\Psi_1(\boldsymbol{\lambda}_3)}{\Theta(\boldsymbol{\lambda}_1)\Theta(\boldsymbol{\lambda}_2)\Theta(\boldsymbol{\lambda}_3)}\left[\frac{\sigma^2}{(2\pi)^d}\int_{\boldsymbol{\pi}}(\frac{(c_0^{(2)})^3}{3}+\left|\frac{\Phi^*(\boldsymbol{\theta})}{\Theta(\boldsymbol{\theta})}\Psi_2(\boldsymbol{\theta})\right|^2\frac{\Phi^*(\boldsymbol{\theta}-\boldsymbol{\lambda}_3)}{\Theta(\boldsymbol{\theta}-\boldsymbol{\lambda}_3)}\Psi_2(\boldsymbol{\theta}-\boldsymbol{\lambda}_3))d\boldsymbol{\theta}\right].
\end{aligned}$$

Proof. The proof is similar as that of Theorem 3 in Wang and Wei (2004).

Furthermore, the bispectral density can also be approximated by $f_3(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)=\left(\frac{\sigma^2}{(2\pi)^d}\right)^2\varphi(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)$ where the function $\varphi(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)$ is defined in the above theorem.

Note that the third-order cumulants fulfils the following symmetric relations:

$$\begin{aligned}
C_3(\mathbf{s}_1,\mathbf{s}_2) &= C_3(\mathbf{s}_2,\mathbf{s}_1)=C_3(-\mathbf{s}_1,\mathbf{s}_2-\mathbf{s}_1)=C_3(\mathbf{s}_2-\mathbf{s}_1,-\mathbf{s}_1) \\
&= C_3(-\mathbf{s}_2,\mathbf{s}_1-\mathbf{s}_2)=C_3(\mathbf{s}_1-\mathbf{s}_2,-\mathbf{s}_2).
\end{aligned}\qquad(1.5.5)$$

Under the single additional assumption that $\mathbf{q}\preceq\mathbf{Q}$, we discover a group of Yule Walker type difference equations for the third order cumulants. For the $SBL_d(\mathbf{p},\mathbf{q},\mathbf{P},\mathbf{Q})$ model, it is reasonable since we are concerned with bilinear terms more than moving-average terms.

Remark 1.7 *Undoubtedly, all of the Yule Walker type difference equations for the third order cumulants can be approximately derived from the bispectral density.*

We only need to consider the case of $\mathbf{s}_2\succeq\mathbf{s}_1\succeq 0$ to get the following result.

Theorem 1.9 *Let* $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ *be a stationary process generated by* $SBL_d(\mathbf{p},\mathbf{q},\mathbf{P},\mathbf{Q})$ *model* (1.3.7)*. Then the third order cumulants satisfy the following difference equation: if* $\mathbf{h}=\mathbf{0}$*,*

$$C_3(\mathbf{k},\mathbf{Q}+\mathbf{k})-\sum_{\mathbf{i}\in\Gamma]\mathbf{p}]}a_{\mathbf{i}}C_3(\mathbf{k},\mathbf{Q}+\mathbf{k}-\mathbf{i})=\begin{cases}2\sigma^2c^{(1)}_{\mathbf{Q}}\sum_{\mathbf{j}\in\Gamma]\mathbf{P}]}c^{(2)}_{\mathbf{j}}C(\mathbf{j}),\mathbf{k}=\mathbf{0},\\ \sigma^2c^{(1)}_{\mathbf{Q}}\sum_{\mathbf{j}\in\Gamma]\mathbf{P}]}c^{(2)}_{\mathbf{j}}C(\mathbf{k}-\mathbf{j}),\mathbf{k}\succ\mathbf{0},\end{cases}$$

and, if $\mathbf{h}\succ\mathbf{0}$

$$C_3(\mathbf{k},\mathbf{Q}+\mathbf{h}+\mathbf{k})-\sum_{\mathbf{i}\in\Gamma]\mathbf{p}]}a_{\mathbf{i}}C_3(\mathbf{k},\mathbf{Q}+\mathbf{h}+\mathbf{k})=0,\mathbf{k}\succeq\mathbf{0}.$$

Proof. Firstly, centering all of the terms in (1.3.7), we get

$$X(\mathbf{t})-\sum_{\mathbf{i}\in\Gamma]\mathbf{p}]}a_{\mathbf{i}}X(\mathbf{t}-\mathbf{i})=-\Psi_0(\mathbf{0})\sigma^2+\sum_{\mathbf{j}\in\Gamma[\mathbf{Q}]}b^*_{\mathbf{j}}e(\mathbf{t}-\mathbf{j})+\sum_{\mathbf{i}\in\Gamma]\mathbf{Q}]}c^{(1)}_{\mathbf{i}}e\left(\mathbf{t}-\mathbf{i}\right)\sum_{\mathbf{j}\in\Gamma[\mathbf{P}]}c^{(2)}_{\mathbf{j}}X(\mathbf{t}-\mathbf{i}-\mathbf{j}),\quad(1.5.6)$$

where

$$b^*_{\mathbf{j}}=\begin{cases}b_{\mathbf{j}}+c^{(1)}_{\mathbf{j}}\mu\Psi_2(\mathbf{0})\text{ for }\mathbf{1}\preceq\mathbf{j}\preceq\mathbf{q},\\ c^{(1)}_{\mathbf{j}}\mu\Psi_2(0)\text{ for }\mathbf{q}\preceq\mathbf{j}\preceq\mathbf{Q}.\end{cases}$$

Let $X(\mathbf{t})=Z(\mathbf{t})+e(\mathbf{t})$. It is not difficult to see that $Z(\mathbf{t})$ is independent of $e(\mathbf{t})$. From this, we multiply both sides of (1.5.6) by $X(\mathbf{t}-\mathbf{Q}-\mathbf{h})X(\mathbf{t}-\mathbf{Q}-\mathbf{h}-\mathbf{k})$ for all $\mathbf{h},\mathbf{k}\succeq\mathbf{0}$ and take expectation. We compute all terms on the right-hand side,

If $\mathbf{0}\preceq\mathbf{i}\preceq\mathbf{Q}$, then, for all $\mathbf{h}\succeq\mathbf{0},\mathbf{k}\succeq\mathbf{0}$,

$$E\left\{e(\mathbf{t}-\mathbf{j})X(\mathbf{t}-\mathbf{Q}-\mathbf{h})X(\mathbf{t}-\mathbf{Q}-\mathbf{h}-\mathbf{k})\right\}=0.$$

If $\mathbf{0}\preceq\mathbf{i}\prec\mathbf{Q}$, then, for all $\mathbf{h}\succeq\mathbf{0},\mathbf{k}\succeq\mathbf{0}$,

$$E\left\{X(\mathbf{t}-\mathbf{i}-\mathbf{j})e(\mathbf{t}-\mathbf{j})X(\mathbf{t}-\mathbf{Q}-\mathbf{h})X(\mathbf{t}-\mathbf{Q}-\mathbf{h}-\mathbf{k})\right\}=\begin{cases}\sigma^2C(\mathbf{k}),\text{ if }\mathbf{j}=\mathbf{0},\\ 0,\qquad\text{ if }\mathbf{j}\succ\mathbf{0}.\end{cases}$$

If $\mathbf{h}=\mathbf{0},\mathbf{k}=\mathbf{0}$, then

$$E\left\{X(\mathbf{t}-\mathbf{Q}-\mathbf{j})e(\mathbf{t}-\mathbf{Q})X^2(\mathbf{t}-\mathbf{Q})\right\}=\begin{cases}3\sigma^2C(\mathbf{0}),\text{ if }\mathbf{j}=\mathbf{0},\\ 2\sigma^2C(\mathbf{j}),\text{ if }\mathbf{j}\succ\mathbf{0}.\end{cases}$$

If $\mathbf{h}=\mathbf{0},\mathbf{k}\succ\mathbf{0}$, then

$$E\left\{X(\mathbf{t}-\mathbf{Q}-\mathbf{j})e(\mathbf{t}-\mathbf{Q})X(\mathbf{t}-\mathbf{Q})X(\mathbf{t}-\mathbf{Q}-\mathbf{k})\right\}=\begin{cases}2\sigma^2C(\mathbf{k}),\text{ if }\mathbf{j}=\mathbf{0},\\ \sigma^2C(\mathbf{k}-\mathbf{j}),\text{ if }\mathbf{j}\succ\mathbf{0}.\end{cases}$$

If $\mathbf{h} \succ \mathbf{0}, \mathbf{k} \succeq \mathbf{0}$, then

$$E\left\{X(\mathbf{t}-\mathbf{Q}-\mathbf{j})e(\mathbf{t}-\mathbf{Q})X(\mathbf{t}-\mathbf{Q}-\mathbf{h})X(\mathbf{t}-\mathbf{Q}-\mathbf{h}-\mathbf{k})\right\} = \begin{cases} \sigma^2 C(\mathbf{k}), \text{ if } \mathbf{j} = \mathbf{0}, \\ 0, \qquad\qquad \text{if } \mathbf{j} \succ \mathbf{0}. \end{cases}$$

We finally obtain that for $\mathbf{k} \succeq \mathbf{0}$

$$\begin{aligned} & C_3(\mathbf{k}, \mathbf{Q}+\mathbf{h}+\mathbf{k}) - \sum_{\mathbf{i}\in\Gamma]\mathbf{p}]} a_{\mathbf{i}} C_3(\mathbf{k}, \mathbf{Q}+\mathbf{h}+\mathbf{k}-\mathbf{i}) \\ = & \sum_{\mathbf{i}\in\Gamma[\mathbf{p}]} a_{\mathbf{i}} E\left\{X(\mathbf{t}-\mathbf{i})X(\mathbf{t}-\mathbf{Q}-\mathbf{h})X(\mathbf{t}-\mathbf{Q}-\mathbf{h}-\mathbf{k})\right\} \\ = & \begin{cases} \sigma^2(1+\delta_{\mathbf{l}}^{\mathbf{0}})c_{\mathbf{Q}}^{(1)} \sum_{\mathbf{j}\in\Gamma[\mathbf{P}]} c_{\mathbf{j}}^{(2)} C(\mathbf{k}-\mathbf{j}), \text{ if } \mathbf{h} = \mathbf{0}, \\ \qquad\qquad 0, \qquad\qquad \text{if } \mathbf{h} \succ \mathbf{0}. \end{cases} \end{aligned}$$

Those difference equations can be used to identify the $SBL_d(\mathbf{p}, \mathbf{q}, \mathbf{P}, \mathbf{Q})$ model.

From Theorem 1.9, we can easily derive the Yule Walker type difference equations for the third order cumulants for the two special cases as

Corollary 1.8 *For the spatial model generated by* (1.3.8)*, the third order cumulants satisfy the following difference equations: for* $\mathbf{h} = \mathbf{0}$,

$$C_3(\mathbf{k}, \mathbf{Q}+\mathbf{k}) - \sum_{\mathbf{i}\in\Gamma]\mathbf{p}]} a_{\mathbf{i}} C_3(\mathbf{k}, \mathbf{Q}+\mathbf{k}-\mathbf{i}) = \begin{cases} 2\sigma^2 c_{\mathbf{Q}} C(\mathbf{k}), \ \mathbf{k} = \mathbf{0}, \\ \sigma^2 c_{\mathbf{Q}} C(\mathbf{k}-\mathbf{l}), \ \mathbf{k} \succ \mathbf{0}, \end{cases}$$

and for $\mathbf{h} \succ \mathbf{0}$,

$$C_3(\mathbf{k}, \mathbf{Q}+\mathbf{h}+\mathbf{k}) - \sum_{\mathbf{i}\in\Gamma]\mathbf{p}]} a_{\mathbf{i}} C_3(\mathbf{k}, \mathbf{Q}+\mathbf{h}+\mathbf{k}-\mathbf{i}) = 0, \ \mathbf{k} \succeq \mathbf{0}$$

Corollary 1.9 *The third order cumulants of the process* $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ *generated by the model* (1.3.10) *satisfy the following difference equations: for* $\mathbf{h} = \mathbf{0}$,

$$C_3(\mathbf{k}, \mathbf{l}+\mathbf{k}) - \sum_{\mathbf{i}\in\Gamma]\mathbf{p}]} a_{\mathbf{i}} C_3(\mathbf{k}, \mathbf{l}+\mathbf{k}-\mathbf{i}) = \begin{cases} 2\sigma^2 \sum_{\mathbf{i}\in\Gamma]\mathbf{Q}]} c_{\mathbf{i}} C(\mathbf{i}), \ \mathbf{k} = \mathbf{0}, \\ \sigma^2 \sum_{\mathbf{i}\in\Gamma]\mathbf{Q}]} c_{\mathbf{i}} C(\mathbf{k}-\mathbf{i}), \ \mathbf{k} \succ \mathbf{0}, \end{cases}$$

and, if $\mathbf{h} \succ \mathbf{0}$,

$$C_3(\mathbf{k}, \mathbf{l}+\mathbf{h}+\mathbf{k}) - \sum_{\mathbf{i}\in\Gamma]\mathbf{p}]} a_{\mathbf{i}} C_3(\mathbf{k}, \mathbf{l}+\mathbf{h}+\mathbf{k}-\mathbf{i}) = 0, \ \mathbf{k} \succeq \mathbf{0}.$$

Example 1.4 *Consider the spatial diagonal bilinear models*

$$X(\mathbf{t}) = \beta X(\mathbf{t}-\mathbf{k})e(\mathbf{t}-\mathbf{k}) + e(\mathbf{t}),$$

where $\mathbf{k} = (k_1, k_2),\ k_1 \geq 1, k_2 \geq 2, E\{e(\mathbf{t})\} = 0, \sigma^2 = 1, E\{e^4(\mathbf{t})\} = 3$ *and* $E\{e^6(\mathbf{t})\} = 15$. *We have*

$$\begin{aligned} \mu &= E\{X(\mathbf{t})\} = \beta, \\ \mu_2 &= E\left\{X^2(\mathbf{t})\right\} = \frac{1+2\beta^2}{1-\beta^2}, \\ var\{X(\mathbf{t})\} &= E\left\{(X(\mathbf{t})-\mu)^2\right\} = \frac{1+\beta^2+\beta^4}{1-\beta^2}, \end{aligned}$$

and

$$C(\mathbf{l}) = \begin{cases} \beta^2 \ \textit{if}\ \mathbf{l} = \mathbf{k}, \\ 0 \ \textit{if}\ \mathbf{l} \neq \mathbf{k}. \end{cases}$$

Then

$$C_3(\mathbf{s},\mathbf{h}) = \begin{cases} 2\beta^2(4+5\beta^2)/(1-\beta^2) & \text{if } \mathbf{s} = \mathbf{h} = \mathbf{0}, \\ 2\beta(1+\beta^2+\beta^4)/(1-\beta^2) & \text{if } \mathbf{s} = \mathbf{h} = \mathbf{k}, \\ 3\beta[1+6\beta^{2n_1n_2+1}(1+\beta^2+2\beta^4)]/(1-\beta^2) & \text{if } \mathbf{s} = \mathbf{0}, h_i = n_i k_i, i = 1, 2, \\ 4\beta^3(1+2\beta^2+3\beta^4)/(1-\beta^2) & \text{if } \mathbf{s} = \mathbf{0}, \mathbf{h} = \mathbf{k}, \\ \beta^3 & \text{if } \mathbf{s} = \mathbf{k}, \mathbf{h} = 2\mathbf{k}, \\ 0 & \text{otherwise.} \end{cases}$$

and the bispectral density function

$$\begin{aligned} f_3(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2) &= \frac{1}{(2\pi)^4}\{C_3(\mathbf{0},\mathbf{0}) + C_3(\mathbf{k},\mathbf{k})[e^{i\mathbf{k}.(\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2)} + e^{-i\mathbf{k}.\boldsymbol{\lambda}_1} + e^{-i\mathbf{k}.\boldsymbol{\lambda}_2}] \\ &\quad +\beta^3[\cos(\mathbf{k}.\boldsymbol{\lambda}_1 + 2\mathbf{k}.\boldsymbol{\lambda}_2) + \cos(2\mathbf{k}.\boldsymbol{\lambda}_1 + \mathbf{k}.\boldsymbol{\lambda}_2) + \cos(\mathbf{k}.\boldsymbol{\lambda}_1 - \mathbf{k}.\boldsymbol{\lambda}_2)] \\ &\quad +4\beta^3(1+2\beta^2+3\beta^4)/(1-\beta^2)[e^{i\mathbf{k}.\boldsymbol{\lambda}_1} + e^{i\mathbf{k}.\boldsymbol{\lambda}_2} + e^{-i\mathbf{k}.(\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2)}]\}. \end{aligned}$$

Chapter 2

Higher-order spectral density estimation

2.1 Introduction

In recent years, spectral analysis based on higher order statistics has received great attention, and constituted a significant part of modern signal processing and digital image processing. It is used in a variety of applications, e.g. sonar, radar, plasma physics, image reconstruction, array processing, seismic data processing, harmonic retrieval, system identification (see Li and Cheng (1998)).

Spectral density estimation is an important problem and there is a rich literature (see for example Rosenblatt (1985), Guyon (1995), Yao and Brockwell (2006), Subba Rao and Gabr (1984)). If a random field is Gaussian, then its statistical properties are completely determined by its second order spectrum, otherwise we have to resort to higher order spectra. The idea of estimating the second and higher order spectral density of a random field is readily extendible from times series analysis (see Rosenblatt (1985)). However, the asymptotic cumulant properties of the spectral estimates for random fields have been given in Yuan and Subba Rao (1993) and Rosenblatt (1985). In this chapter, we obtained asymptotic normality of spectral density for a class of spatial nonlinear processes in section 3; section 4 is concerned with the asymptotic distribution of certain estimates of the bispectral density, this estimate would have distribution which tend to complex normal distributions under certain conditions. Estimator of the fourth-order cumulant spectral density is proposed, this result is sufficiently complete to indicate what happens in general, study in section 5 and 6.

The following notation is used throughout. Let $\mathbf{a} = (a_1, ..., a_d)$ and $\mathbf{b} = (b_1, ..., b_d)$ two vectors of non negative integers, we have $\mathbf{a}.\mathbf{b} = a_1b_1 + ... + a_db_d, \mathbf{a} \odot \mathbf{b} = (a_1b_1, ..., a_db_d)$, $\frac{\mathbf{a}}{\mathbf{b}} = (\frac{a_1}{b_1}, ..., \frac{a_d}{b_d})$ if $b_1, ..., b_d \neq 0, \mathbf{a} \preceq \mathbf{b}$ means that $a_i \leq b_i, i = 1, ...d$. The sample size is $\mathbf{N} = (N_1, ..., N_d)$,i.e. we observe $X(\mathbf{t}), \mathbf{t} =(t_1, ..., t_d)$ for $t_i = 1, ..., N_i, i = 1, ..., d$, but the number of observation is $|\mathbf{N}| = \prod_{i=1}^{d} N_i, |\underline{\mathbf{N}}| = (|N_1|, |N_2|, ..., |N_d|)$. For brevity, we write $\mathbf{t} = \mathbf{1}, ..., \mathbf{N}$ and $\mathbf{N} \longrightarrow \infty$ means that $N_i \longrightarrow \infty, i = 1, ..., d$. Then define the multi-index sum as $\sum_{t_1=1}^{N_1} ... \sum_{t_d=1}^{N_d} = \sum_{\mathbf{t}=\mathbf{1}}^{\mathbf{N}}$.

2.2 Cumulant spectra and their estimates

The use of either nonparametric or parametric polyspectral methods need to calculate higher order moments and cumulants which depend on lower-order product moments and the cumulant spectral density agrees with the Fourier transform of the same order product moment.

Let $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ be a weakly stationary real random field with a zero mean and finite pth-order moments on $\mathbb{Z}^d, d \geq 2$. For all $\mathbf{t} \in \mathbb{Z}^d$, we have

$$\begin{aligned} E\left\{X\left(\mathbf{t}\right)X\left(\mathbf{t}+\mathbf{h}_1\right)\dots X\left(\mathbf{t}+\mathbf{h}_{p-1}\right)\right\} &= m_p(\mathbf{t}, \mathbf{t}+\mathbf{h}_1, \dots, \mathbf{t}+\mathbf{h}_{p-1}) \\ &= r_p(\mathbf{h}_1, \dots, \mathbf{h}_{p-1}), \end{aligned} \tag{2.2.1}$$

and

$$cum\left\{X\left(\mathbf{t}\right), X\left(\mathbf{t}+\mathbf{h}_1\right), \dots, X\left(\mathbf{t}+\mathbf{h}_{p-1}\right)\right\} = C_p(\mathbf{h}_1, ..., \mathbf{h}_{p-1}).$$

Let $\nu = \{\nu_1, ..., \nu_k\}$ be a partition of the set $\{\mathbf{0}, \mathbf{h}_1, \cdots, \mathbf{h}_{p-1}\}$ into k subsets where $0 \leq k \leq p-1$. Then

$$C_p(\mathbf{h}_1, ..., \mathbf{h}_{p-1}) = \sum_{\nu} (-1)^{m(\nu)-1} \left[m(\nu)-1\right]! \prod_{j=1}^{m(\nu)} E\left\{\prod_{\mathbf{h}_u\in\nu_j} X\left(\mathbf{h}_u\right)\right\}, \tag{2.2.2}$$

where $m(\nu)$ is the number of nonvacuous sets in the partition ν and the outer sum is over all partitions ν of $\{\mathbf{0}, \mathbf{h}_1, \cdots, \mathbf{h}_{p-1}\}$.

The pth order cumulant spectrum (or polyspectrum) is defined by

$$f_p(\boldsymbol{\lambda}_1, ..., \boldsymbol{\lambda}_{p-1}) = \frac{1}{(2\pi)^{d(p-1)}} \sum_{\mathbf{h}_1\in\mathbb{Z}^d} ... \sum_{\mathbf{h}_{p-1}\in\mathbb{Z}^d} C_p(\mathbf{h}_1, ..., \mathbf{h}_{p-1}) e^{-i\sum_{j=1}^{p-1}\mathbf{h}_j.\boldsymbol{\lambda}_j},$$

where $\mathbf{h}.\boldsymbol{\lambda} = \sum_{i=1}^{d} h_i\lambda_i$, $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_d) \in \boldsymbol{\pi}, \boldsymbol{\pi} = [-\pi, \pi[\times ... \times [-\pi, \pi[$, $d-$times, provided that

$$\sum_{\mathbf{h}_j\in\mathbb{Z}^d} \left|C_p(\mathbf{h}_1, ..., \mathbf{h}_{p-1})\right| < \infty, j = 1, ..., p-1,$$

where the pth order cumulant function of the random fields satisfies the inverse relation

$$C_p(\mathbf{h}_1, \dots, \mathbf{h}_{p-1}) = \int_{\boldsymbol{\pi}} ... \int_{\boldsymbol{\pi}} \sum_{\mathbf{h}_1\in\mathbb{Z}^d} ... \sum_{\mathbf{h}_{p-1}\in\mathbb{Z}^d} e^{i\sum_{j=1}^{p-1}\mathbf{h}_j.\boldsymbol{\lambda}_j} f_p(\boldsymbol{\lambda}_1, ..., \boldsymbol{\lambda}_{p-1}) d\boldsymbol{\lambda}_1 ... d\boldsymbol{\lambda}_{p-1}.$$

Define the moment estimate as

$$\widehat{r}_p(\mathbf{h}_1,\ldots,\mathbf{h}_{p-1}) = |\mathbf{N}-2l_{\mathbf{N}}|^{-1}\sum_{\mathbf{t}=l_{\mathbf{N}}+\mathbf{1}}^{\mathbf{N}-l_{\mathbf{N}}} X(\mathbf{t})\,X(\mathbf{t}+\mathbf{h}_1)\ldots X(\mathbf{t}+\mathbf{h}_{p-1}),\ \left|\underline{\mathbf{h}}_j\right| \preceq l_{\mathbf{N}}, j=1,\ldots,p-1. \tag{2.2.3}$$

and the cumulant estimate $\widehat{C}_p(\mathbf{h}_1,\ldots,\mathbf{h}_{p-1})$ are obtained by replacing the moment by their estimates in (2.2.3) into formula (2.2.2).

The conventional estimate $f_{p\mathbf{N}}(\boldsymbol{\lambda}_1,\ldots,\boldsymbol{\lambda}_{p-1})$ of the pth order polyspectrum takes the form

$$f_{p\mathbf{N}}(\boldsymbol{\lambda}_1,\ldots,\boldsymbol{\lambda}_{p-1}) = \frac{1}{(2\pi)^{d(p-1)}}\sum_{|\underline{\mathbf{h}}_1|\preceq l_{\mathbf{N}}}\cdots\sum_{\left|\underline{\mathbf{h}}_{p-1}\right|\preceq l_{\mathbf{N}}}\widehat{C}_p(\mathbf{h}_1,\ldots,\mathbf{h}_{p-1})w(b_{\mathbf{N}}\odot\mathbf{h}_1,\ldots,b_{\mathbf{N}}\odot\mathbf{h}_{p-1})e^{-i\sum_{j=1}^{p-1}\mathbf{h}_j.\boldsymbol{\lambda}_j}, \tag{2.2.4}$$

where $b_{\mathbf{N}}$ is a vector of the bandwidth parameter, $w(\mathbf{x}_1,\ldots,\mathbf{x}_{p-1})$ is the weight function and $\widehat{C}_p(\mathbf{h}_1,\ldots,\mathbf{h}_{p-1})$ is a cumulant estimate of $C_p(\mathbf{h}_1,\ldots,\mathbf{h}_{p-1})$ based on a realization $\{X_{\mathbf{1}},\ldots,X_{\mathbf{N}}\}$ from the process $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$. Consider now the following condition

Condition 2.1 *For an integer $p \geq 2$,*

1. $E\left\{|X(\mathbf{t})|^{4p}\right\} < \infty$.
2. $\sum_{\mathbf{h}_j\in\mathbb{Z}^d}\left|C_j(\mathbf{h}_1,\ldots,\mathbf{h}_{j-1})\right||\mathbf{h}_j|^q < \infty$ *for some* $q \geq 1, j = 1,\ldots 4p$.
3. *the weight function* $w(\mathbf{x}_1,\ldots,\mathbf{x}_{p-1})$ *is of bounded support and continuous.*
4. $\lim_{\mathbf{h}_j/|\mathbf{h}_j|\longrightarrow\theta}\frac{w(b_{\mathbf{N}}\odot\mathbf{h}_1,\ldots,b_{\mathbf{N}}\odot\mathbf{h}_{p-1})-1}{\prod_{j=1}^{p-1}|\mathbf{h}_j|^q|b_{\mathbf{N}}|^q} = \alpha(\theta)\neq 0$ *as* $b_{\mathbf{N}}\longrightarrow\mathbf{0}$ *and* $|\alpha(\theta)|<\infty$.
5. $\mathbf{N}\odot b_{\mathbf{N}}^{p-1}\longrightarrow\infty$, $b_{\mathbf{N}}\searrow\mathbf{0}$ *as* $\mathbf{N}\longrightarrow\infty$.
6. *The lag in* (2.2.3) *is* $l_{N_i} = O\left(b_{N_i}^{-1}\right)$*, and usually we take* $l_{N_i} = b_{N_i}^{-1}$.

2.3 Spectral density estimate

A random field which is not linear is always non Gaussian and hence the analysis of its higher order spectra can be used to study departure from linearity. But so far, no significant effort has been made to investigate the sampling properties of the estimates of the second and higher order spectra from random fields which are nonlinear. In this section, we consider the spectral estimators based on a sample from a class of strictly stationary nonlinear spatial processes which include in particular the spatial bilinear and spatial Volterra processes and non linear random fields which satisfy the geometric-moment contraction condition, and we establish the asymptotic normality of the spectral density estimate.

2.3.1 Asymptotic normality of stationary nonlinear spatial processes

We have, instead limited ourselves to nonlinear processes $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ which can be defined as

$$X(\mathbf{t}) = e(\mathbf{t}) + \sum_{\mathbf{r}\succ\mathbf{0}} W_{\mathbf{r}}(\mathbf{t}), \mathbf{t}\in\mathbb{Z}^d, \tag{2.3.1}$$

where $(e(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ is an independly and identically distributed random fields, $W_{\mathbf{r}}(\mathbf{t})\,(\mathbf{r}\succeq\mathbf{1})$ is a function $f_{\mathbf{r}}(e(\mathbf{t}-\mathbf{1}),...,e(\mathbf{t}-\mathbf{r}-\mathbf{v}))$ of $(e(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$, $\mathbf{v}$ is an arbitrary vector but all component are a fixed integer ≥ 0. We assume that $E\{e(\mathbf{t})\}=0, var\{e(\mathbf{t})\}=\sigma_e^2\ (0<\sigma_e<\infty), E\{|e(\mathbf{t})|^p\}<\infty$ for some $p\geq 2, E\{W_{\mathbf{r}}(\mathbf{t})\}=0\ (\mathbf{r}\succeq\mathbf{0})$, and that there exists a vectors of sequence $\{g_{\mathbf{r}},\mathbf{r}\succeq\mathbf{1}\}$ of real numbers such that

$$\sum_{\mathbf{r}\succ\mathbf{0}} |\mathbf{r}|^3|g_{\mathbf{r}}|<\infty \text{ and } E\left\{|W_{\mathbf{r}}(\mathbf{t})|^4\right\}\leq M|g_{\mathbf{r}}|^4, \mathbf{r}\succeq\mathbf{1},$$

where M is denotes a finite positive constant, independent of r_i and $t_i, i=1,...,d$.

Spatial bilinear processes

In two dimensions, we shall focus our attention on $(SSBL_d)$ models defined by matrix form as

$$\underline{X}(\mathbf{t}) = B(\mathbf{t})\underline{e}(\mathbf{t}) + \mathcal{A}(\mathbf{t})\underline{X}(\mathbf{t}-\mathbf{e}_1) + \mathcal{B}(\mathbf{t})\underline{X}(\mathbf{t}-\mathbf{e}_2). \tag{2.3.3}$$

where $\underline{X}(\mathbf{t})$, $\mathcal{A}(\mathbf{t})$, $\mathcal{B}(\mathbf{t})$ and $\underline{e}(\mathbf{t})$ are defined in the Appendix 2.1 and $\mathbf{e}_1=(1,0)$ and $\mathbf{e}_2=(0,1)$.

Then, we can write (2.3.3) in representation form as

$$\underline{X}(\mathbf{t}) = \underline{e}(\mathbf{t}) + \sum_{\mathbf{r}\succ\mathbf{0}}^{\infty} W_{\mathbf{r}}(\mathbf{t}),$$

where

$$W_{\mathbf{r}}(\mathbf{t}) = \eta^T T^{r_1,r_2}(\mathbf{t})\, B\underline{e}(\mathbf{t}-\mathbf{r}),$$

and $\eta=(1,0,...,0)'$. It is also shown that

$$|W_{\mathbf{r}}(\mathbf{t})| \leq M(\mathcal{C}(\mathbf{t})\left\|T^{r_1,r_2-1}(\mathbf{t})\right\| + \mathcal{D}(\mathbf{t})\left\|T^{r_1-1,r_2}(\mathbf{t})\right\|)\left\|e(\mathbf{t}-\mathbf{r}_1)\right\|_{\infty},$$

where $\mathcal{C}(\mathbf{t})$, $\mathcal{D}(\mathbf{t})$ and $T^{r_1,r_2}(\mathbf{t})$ are defined in the Appendix 2.2, from which by a suitable choice of $\{g_{\mathbf{r}},(\mathbf{r}\succ\mathbf{0})\}$, we can establish the conditions in (2.3.1).

Spatial Volterra processes

Let $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^2}$ be defined by

$$X(\mathbf{t}) = \sum_{k=1}^{q}\left\{\sum\nolimits_{(k)} V_k\left(\mathbf{u}_1,...,\mathbf{u}_k\right)\prod_{j=1}^{k} e(\mathbf{t}-\mathbf{u}_j)\right\}, \tag{2.3.4}$$

where $V_k\left(\mathbf{u}_1,...,\mathbf{u}_k\right)$ are the kth-order Volterra kernels and $\sum_{(k)}$ is over all $\mathbf{u}_j \in S\left[\mathbf{0},\infty\right[$, where the coefficients V_k are symmetric functions of their arguments, q is an arbitrary but fixed integer ≥ 2, and $\left(e\left(\mathbf{t}\right)\right)_{\mathbf{t}\in\mathbb{Z}^2}$ is an i.i.d random field. Without any loss of generality, we can assume that in relation (2.3.4).

Then we can write $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^2}$ in the form (2.3.1) with

$$W_{\mathbf{r}}\left(\mathbf{t}\right) = \sum_{k=1}^{q}\left\{\sum\nolimits_{(k)} V_k^{(\mathbf{r})}\left(\mathbf{u}_1,...,\mathbf{u}_k\right)\prod_{j=1}^{k-1} e(\mathbf{t}-\mathbf{u}_j)\right\},$$

where $W_{\mathbf{r}}\left(\mathbf{t}\right)$ is a function of $\left\{e\left(\mathbf{t}-\mathbf{1}\right),...,e\left(\mathbf{t}-\mathbf{r}\right)\right\}$ and $\sum_{(k)}$ is over all $\mathbf{0}\prec\mathbf{u}_1\preceq...\preceq\mathbf{u}_{k-1}\preceq\mathbf{r}$, and

$$E\left\{W_{\mathbf{r}}\left(\mathbf{t}\right)\right\} \leq M g_{\mathbf{r}}^4 \text{ where } g_{\mathbf{r}} = \sum_{k=1}^{q}\left\{\sum\nolimits_{(k)}\left|V_k^{(\mathbf{r})}\left(\mathbf{u}_1,...,\mathbf{u}_k\right)\right|\right\}.$$

In fact, if we assume that $E\left\{\left|e(\mathbf{1})\right|^{4q}\right\}<\infty$, then $E\left\{\left(W_{\mathbf{r}}\left(\mathbf{t}\right)\right)^4\right\}\leq E\left\{\left|e(\mathbf{1})\right|^{4q}\right\}g_{\mathbf{r}}^4$ and we need to specify that for $\mathbf{r}\succeq\mathbf{1}, \sum_{\mathbf{r}}(r_1r_2)^3 g_{\mathbf{r}}^4<\infty$.

Remark 2.1 *Note that if in general, we can assume that relation* (2.3.1) *for* $d=2$ *holds with* $E\left\{\left|W_{\mathbf{r}}\left(\mathbf{t}\right)\right|^p\right\}\leq M g_{\mathbf{r}}^p$ *for some* $p\geq 1$ *such that* $\sum_{\mathbf{r}\succeq\mathbf{1}}(r_1r_2)^2 g_{\mathbf{r}}<\infty$, *then one can show that the right side of* (2.3.1) *converges a.s. This result follows from the fact that if we write*

$$A_{\mathbf{N}} = \left\{\left|\sum_{\mathbf{r}\succeq\mathbf{N}} W_{\mathbf{r}}\left(\mathbf{t}\right)\right| > \epsilon\right\},$$

where ϵ *is an arbitrary number* >0 *then for* $\mathbf{N}=(N_1,N_2)$

$$\begin{aligned} P\left(\bigcup_{\mathbf{N}=\mathbf{m}}^{\infty} A_{\mathbf{N}}\right) &\leq \sum_{\mathbf{N}=\mathbf{m}}^{\infty} p(A_{\mathbf{N}}) \\ &\leq M\sum_{\mathbf{N}=\mathbf{m}}^{\infty}\sum_{\mathbf{r}=\mathbf{N}}^{\infty}\frac{\left|\mathbf{r}\right|^2 g_{\mathbf{r}}}{\epsilon} \longrightarrow 0 \textit{ as } \mathbf{m}\longrightarrow\infty. \end{aligned}$$

Asymptotic normality

For the sake of simplicity, we restrict ourselves in this section to the study of the sampling properties of $f_{\mathbf{N}}(\boldsymbol{\lambda})$ (i.e. $p = 2$) as

$$f_{\mathbf{N}}(\boldsymbol{\lambda}) = \frac{1}{(2\pi)^2} \sum_{|\underline{\mathbf{h}}| \preceq l_{\mathbf{N}}} \widehat{C}(\mathbf{h}) w(b_{\mathbf{N}} \odot \mathbf{h}) \cos(\mathbf{h}.\boldsymbol{\lambda}), \tag{2.3.5}$$

where $\mathbf{h}.\boldsymbol{\lambda} = h_1.\lambda_1 + h_2.\lambda_2$, $w(b_{\mathbf{N}} \odot \mathbf{h}) = w(b_{N_1}.h_1, b_{N_2}.h_2)$ for every $(\mathbf{h} \succeq \mathbf{0})$ and

$$\widehat{C}(\mathbf{h}) = |\mathbf{N}|^{-1} \sum_{\mathbf{t}=1}^{\mathbf{N}-\mathbf{h}} X(\mathbf{t}) X(\mathbf{t}+\mathbf{h}) \text{ and } \widehat{C}(\mathbf{h}) = \widehat{C}(-\mathbf{h}), \tag{2.3.6}$$

Remark 2.2 *If* $E\{X(\mathbf{t})\} \neq 0$ *we replace* $X(\mathbf{t})$ *by* $X(\mathbf{t}) - \overline{X(\mathbf{t})}$ *where*

$$\overline{X(\mathbf{t})} = |\mathbf{N}|^{-1} \sum_{\mathbf{t}=1}^{\mathbf{N}-\mathbf{h}} X(\mathbf{t}),$$

and the corresponding analysis of $f_{\mathbf{N}}(\boldsymbol{\lambda})$ *remains unchanged, asymptotically.*

Theorem 2.1 *Let* $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^2}$ *be a nonlinear strictly stationary spatial processes satisfying the relation* (2.3.1) $(d = 2)$, *and* $w(\mathbf{x})$ *is continuous function in* $[-1, 1]^2$. *Then*

$$\mathcal{L}\left((|\mathbf{N}|\,|b_{\mathbf{N}}|)^{1/2}[f_{\mathbf{N}}(\boldsymbol{\lambda}) - E\{f_{\mathbf{N}}(\boldsymbol{\lambda})\}]\right) \to \mathcal{N}(0, \sigma^2_{\boldsymbol{\lambda}}) \text{ as } \mathbf{N} \longrightarrow \infty, \tag{2.3.7}$$

where

$$\sigma^2_{\boldsymbol{\lambda}} = K f^2(\boldsymbol{\lambda}), K = \int w^2(\mathbf{x}) d\mathbf{x} \text{ if } \boldsymbol{\lambda} \neq \mathbf{0}, \pm\boldsymbol{\pi},$$

and

$$K = 2\int w^2(\mathbf{x}) d\mathbf{x} \text{ if } \boldsymbol{\lambda} = \mathbf{0}, \pm\boldsymbol{\pi},$$

Corollary 2.1 *Let the condition in Theorem 2.1 hold. Also let for some* $q > 0, k_q > 0, \lim_{\mathbf{x}\longrightarrow\mathbf{0}}(1 - w(\mathbf{x}))/|\mathbf{x}|^q = k_q$, *and assume that* $\sum_{\mathbf{h}=\mathbf{1}}^{\infty} |\mathbf{h}|^p |C(\mathbf{h})| < \infty$ *for some* $p > 0$, *if we choose* $w(\mathbf{x}) \approx |\mathbf{N}|^{1/(2q+1)} \log(|\mathbf{N}|)$, *where* $p \geq q$ *and* $w(\mathbf{x}) \approx |\mathbf{N}|^{1/(2p+1)}$ *when* $p < q$, *then*

$$\mathcal{L}\left((|\mathbf{N}|\,|b_{\mathbf{N}}|)^{1/2} (f_{\mathbf{N}}(\boldsymbol{\lambda}) - f(\boldsymbol{\lambda}))\right) \to \mathcal{N}(0, \sigma^2_{\boldsymbol{\lambda}}) \text{ as } \mathbf{N} \longrightarrow \infty,$$

where $\sigma^2_{\boldsymbol{\lambda}}$ *is as defined in* (2.3.7).

Corollary 2.2 *Let* $f_{\mathbf{N}}(\boldsymbol{\lambda})$ *be as defined in* (2.3.5) *and let the condition of Theorem 2.1 and Corollary 2.1 hold. Then for any* $(\boldsymbol{\lambda})$ *and* $(\boldsymbol{\nu})$ *where* $(\lambda_i \neq \nu_i, i = 1, 2)$, *then* $(|\mathbf{N}|\,|b_{\mathbf{N}}|)^{1/2} [f_{\mathbf{N}}(\boldsymbol{\lambda}) - f(\boldsymbol{\lambda})]$ *and* $(|\mathbf{N}|\,|b_{\mathbf{N}}|)^{1/2} [f_{\mathbf{N}}(\boldsymbol{\nu}) - f(\boldsymbol{\nu})]$ *are asymptotically independent, with zero mean and asymptotic variances* $\sigma^2_{\boldsymbol{\lambda}}$ *and* $\sigma^2_{\boldsymbol{\nu}}$ *respectively.*

2.3.2 Asymptotic normality for random fields under GMC condition

Most of the asymptotic results developed in the literature are for strong mixing random fields and random fields with quite restrictive summability conditions on joint cumulants (see Rosenblatt (1985)). Such conditions seem restrictive and they are not easily verifiable. In this fact, we employ the GMC as an underlying assumption for our asymptotic theory of spectral density estimates.

Let $(e(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ be an i.i.d random fields and let $X(\mathbf{t}) = G(..., e(\mathbf{t}-\mathbf{1}), e(\mathbf{t}))$, where G is a measurable function such that $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ is a proper random field. Then the process $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ is causal in the sense that it only depends on $\Im_{\mathbf{t}} = (..., e(\mathbf{t}-\mathbf{1}), e(\mathbf{t}))$, not on the future innovations. To establish an asymptotic theory for $f_{\mathbf{N}}(\boldsymbol{\lambda})$ define as

$$f_{\mathbf{N}}(\boldsymbol{\lambda}) = \frac{1}{(2\pi)^d} \sum_{|\underline{\mathbf{h}}|\preceq l_{\mathbf{N}}} \widehat{C}(\mathbf{h}) w(b_{\mathbf{N}} \odot \mathbf{h}) e^{-i\mathbf{h}.\boldsymbol{\lambda}}, \tag{2.3.8}$$

where $\widehat{C}(\mathbf{h})$ satisfied (2.3.6) , we shall adopt the geometric-moment contraction (GMC) condition. From a finite realization $\{X(\mathbf{t}), \mathbf{t} = \mathbf{1}, ..., \mathbf{N}\}$ of a random fields $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$, let $X'(\mathbf{N}) = G\{..., e'(-\mathbf{1}), e'(\mathbf{0}), e(\mathbf{1}), ..., e(\mathbf{N})\}$ be a coupled version of $X(\mathbf{N})$ and $(e'(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ be an i.i.d copy of $(e(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$. We say that $X(\mathbf{N})$ is GMC(α), $\alpha > 0$, if there exist $C > 0$ and $0 < \rho = \rho(\alpha) < 1$ such that, for all $\mathbf{N} \in \mathbb{N}^d$,

$$E\left\{|X'(\mathbf{N}) - X(\mathbf{N})|^{\alpha}\right\} \leq C\rho^{|\mathbf{N}|}. \tag{2.3.9}$$

Note that under GMC(2), $|C(\mathbf{h})| = O(\rho^{|\mathbf{h}|})$ for some $\rho \in (0, 1)$ and hence the spectral density function is infinitely many random differentiable.

Lemma 2.1 *Assume* (2.3.9) *with $\alpha = p$ for some $p \in \mathbb{N}$. Then there exists a constant $C > 0$ such that for all $\mathbf{0} \preceq \mathbf{t}_1 \preceq ... \preceq \mathbf{t}_{p-1}$,*

$$|cum(X(\mathbf{0}), X(\mathbf{t}_1), ..., X(\mathbf{t}_{p-1}))| \leq C\rho^{|\mathbf{t}_{p-1}|/[p(p-1)]}.$$

Lemma 2.2 *Let $s_{\mathbf{N}} \in \mathbb{N}^d$ a vector of sequence satisfy $s_{\mathbf{N}} \preceq \mathbf{N}$ and $b_{N_i} = o(s_{N_i}), i = 1, ..., d$ and*

$$Y_{\mathbf{u}}(\boldsymbol{\lambda}) = \frac{1}{(2\pi)^d} \sum_{|\underline{\mathbf{h}}|\prec l_{\mathbf{N}}} w(\mathbf{b}_{\mathbf{N}} \odot \mathbf{h}) \cos(\mathbf{h}.\boldsymbol{\lambda}) X(\mathbf{u}) X(\mathbf{u}+\mathbf{h}). \tag{2.3.10}$$

Then under GMC(4) *we have*

$$\left\| \sum_{\mathbf{u}=\mathbf{1}}^{s_{\mathbf{N}}} [Y_{\mathbf{u}}(\boldsymbol{\lambda}) - E\{Y_{\mathbf{u}}(\boldsymbol{\lambda})\}] \right\|^2 \sim |s_{\mathbf{N}}|\, |b_{\mathbf{N}}|\, \sigma^2.$$

Theorem 2.2 *Consider* $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d} \in \mathcal{L}^{4+\delta}$ *for some* $\delta > 0$ *satisfies* $(2.3.9)$, $b_{N_i} = o[(\log N_i)^{2+8/\delta}/N_i]$. *Then*

$$(|\mathbf{N}|\,|b_{\mathbf{N}}|)^{1/2}[f_{\mathbf{N}}(\boldsymbol{\lambda}) - E\{f_{\mathbf{N}}(\boldsymbol{\lambda})\}] \longrightarrow \mathcal{N}\left(0, \sigma^2(\boldsymbol{\lambda})\right) \tag{2.3.11}$$

where

$$\sigma^2(\boldsymbol{\lambda}) = \{1 + \eta(2\lambda_1)... + \eta(2\lambda_d)\} f^2(\boldsymbol{\lambda}) \int W^2(\boldsymbol{\alpha}) d\boldsymbol{\alpha}$$

and

$$W(\boldsymbol{\alpha}) = \frac{1}{(2\pi)^d} \int w(\mathbf{x}) e^{-i\mathbf{x}.\boldsymbol{\alpha}} d\mathbf{x}$$

$$\eta(\boldsymbol{\lambda}) = \begin{cases} 1, & \text{if } \boldsymbol{\lambda} \equiv 2\pi\mathbf{k}, \\ 0, & \text{otherwise.} \end{cases}$$

2.4 Bispectral density estimates

In this section, we study the asymptotic distribution of certain estimates of the bispectrum. This estimate would have distribution which tend to complex normal distributions under certain conditions. The first condition involves a uniform summability condition on the first six cumulants of a random field obtained from the original random field by projecting on a Borel fields. The second condition and much more intuitively meaningful, involves the strong mixing condition.

We define the bispectral density function as

$$f_3(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) = \frac{1}{(2\pi)^{2d}} \sum_{\mathbf{h}_1, \mathbf{h}_2 \in \mathbb{Z}^d} C_3(\mathbf{h}_1, \mathbf{h}_2) e^{-i(\mathbf{h}_1.\boldsymbol{\lambda}_1 + \mathbf{h}_2.\boldsymbol{\lambda}_2)}, \tag{2.4.1}$$

$$C_3(\mathbf{h}_1, \mathbf{h}_2) = \int_{\boldsymbol{\pi}} \int_{\boldsymbol{\pi}} e^{i(\mathbf{h}_1.\boldsymbol{\lambda}_1 + \mathbf{h}_2.\boldsymbol{\lambda}_2)} f_3(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) d\boldsymbol{\lambda}_1 d\boldsymbol{\lambda}_2.$$

where $C_3(\mathbf{h}_1, \mathbf{h}_2)$ fulfils the symmetry relation $(1.5.5)$.

It is easy to see that a natural estimate of $f_3(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)$ is not consistent. In order to solve this problem, we have:

Definition 2.1 *A real function $w(\mathbf{x}_1, \mathbf{x}_2)$, is called a symmetric bispectral estimating kernel if*[1]

(i) for any $\epsilon > 0$, there is an $M_1(\epsilon)$ such that for all $M > M_1$ and uniformly in $N_i > M$,

$$|b_{\mathbf{N}}|^2 \sum\sum_{(\mathbf{h}_1,\mathbf{h}_2)\in C_M'^d} w^2(b_{\mathbf{N}} \odot \mathbf{h}_1, b_{\mathbf{N}} \odot \mathbf{h}_2) < \epsilon,$$

(ii) $w(\mathbf{x}_1, \mathbf{x}_2) \le M_1 < \infty$ for all $-\infty \prec \mathbf{x}_1, \mathbf{x}_2 \prec \infty$,

(iii) $w(\mathbf{x}_1, \mathbf{x}_2) = w(\mathbf{x}_2, \mathbf{x}_1) = w(-\mathbf{x}_1, \mathbf{x}_2 - \mathbf{x}_1)$,

(iv) for any $\epsilon > 0$, there is an $M_2(\epsilon)$ such that for all $M > M_2$ and uniformly in $N_i > M, i = 1, ..., d$, and in $\mathbf{h}_1$,

$$|b_{\mathbf{N}}| \sum_{|\underline{\mathbf{h}}_2|\succ Ml_{\mathbf{N}}} |w(\mathbf{h}_1, b_{\mathbf{N}} \odot \mathbf{h}_2)| < \epsilon,$$

(v) for all fixed numbers a and c, and any fixed $M > 0$, and for any $\epsilon > 0$ there is an $N_0(\epsilon, M, a, c)$ such that for all $\mathbf{N} \succ \mathbf{N}_0$,

$$|b_{\mathbf{N}}|^2 \left| \sum_{|\underline{\mathbf{h}}_1|,|\underline{\mathbf{h}}_2|\succeq Ml_{\mathbf{N}}} w(b_{\mathbf{N}} \odot \mathbf{h}_1 + ab_{\mathbf{N}}, b_{\mathbf{N}} \odot \mathbf{h}_2) w(b_{\mathbf{N}} \odot \mathbf{h}_1, b_{\mathbf{N}} \odot \mathbf{h}_2 + cb_{\mathbf{N}}) \right.$$
$$\left. - \sum_{|\underline{\mathbf{h}}_1|,|\underline{\mathbf{h}}_2|\preceq Ml_{\mathbf{N}}} w^2(b_{\mathbf{N}} \odot \mathbf{h}_1, b_{\mathbf{N}} \odot \mathbf{h}_2) \right| < \epsilon,$$

and

$$|b_{\mathbf{N}}| \left| \sum_{|\underline{\mathbf{h}}_1|\succeq Ml_{\mathbf{N}}} w(b_{\mathbf{N}} \odot \mathbf{h}_1, ab_{\mathbf{N}}) - \sum_{|\underline{\mathbf{h}}_1|\preceq Ml_{\mathbf{N}}} w(b_{\mathbf{N}} \odot \mathbf{h}_1, \mathbf{0}) \right| < \epsilon,$$

From the above definition we can define the estimate $f_{3\mathbf{N}}(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)$ based on bispectral estimating kernel as follows:

$$f_{3\mathbf{N}}(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) = \frac{1}{(2\pi)^{2d}} \sum_{|\underline{\mathbf{h}}_1|,|\underline{\mathbf{h}}_2|\preceq \mathbf{N}} \widehat{C}_3(\mathbf{h}_1, \mathbf{h}_2) w(b_{\mathbf{N}} \odot \mathbf{h}, b_{\mathbf{N}} \odot \mathbf{h}_2) e^{-i(\mathbf{h}_1.\boldsymbol{\lambda}_1 + \mathbf{h}_2.\boldsymbol{\lambda}_2)}, \tag{2.4.2}$$

where

$$\widehat{C}_3(\mathbf{h}_1, \mathbf{h}_2) = \frac{1}{|\mathbf{N}|} \sum_{\mathbf{t}\in D_{\mathbf{N}}} X(\mathbf{t}) X(\mathbf{t} + \mathbf{h}_1) X(\mathbf{t} + \mathbf{h}_2), \tag{2.4.3}$$

and $D_{N_i} = \left[-\min\left(0, h_i^{(1)}, h_i^{(2)}\right), N_i - \max\left(0, h_i^{(1)}, h_i^{(2)}\right)\right]$, and $\lambda_i^{(1)}$, $\lambda_i^{(2)}, i = 1, ..., d$ are on the triangle with vertices $(0, 0), (\pi, 0), \left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$ (see Terdik (2000)).

[1] *C_M : the n-dimensional hypercube centered at the origin with sides of length $2M$ parallel to the n axes. The dimension n, will be obvious from the context. Also let C_M' denote the complement of C_M in R_n*

2.4.1 Asymptotic normality under a uniform summability of cumulants condition

Let $(e(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ be an i.i.d random field, T be the back shift operator defined as $T^{-\mathbf{1}}e(\mathbf{t}) = e(\mathbf{t}+\mathbf{1})$, and define a real strictly stationary random field $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ as $X(\mathbf{t}) = G(T^{-\mathbf{1}}e(\mathbf{t})), \mathbf{t} \in \mathbb{Z}^d$ where G is a Borel measurable function. Let

$$X_{\mathbf{k}}(\mathbf{t}) = E\left\{X(\mathbf{t})\,|e(\mathbf{t}-\mathbf{k}), ..., e(\mathbf{t}+\mathbf{k})\right\},$$

where $X_{\mathbf{k}}(\mathbf{t})$ is the projection of $X(\mathbf{t})$ on to the Borel filed $B_{\mathbf{t}-\mathbf{k}}^{\mathbf{t}+\mathbf{k}}$ generated by $(e(\mathbf{t}-\mathbf{k}), ..., e(\mathbf{t}+\mathbf{k}))$ and define

$$\begin{aligned}
r^{(\infty,\mathbf{k})}(\mathbf{h}) &= E\left\{X(\mathbf{t})\,X_{\mathbf{k}}(\mathbf{t}+\mathbf{h})\right\}, \\
r^{(\mathbf{k},\mathbf{k})}(\mathbf{h}) &= E\left\{X_{\mathbf{k}}(\mathbf{t})\,X_{\mathbf{k}}(\mathbf{t}+\mathbf{h})\right\}, \\
&\vdots \\
r_6^{(\mathbf{k},\dots,\mathbf{k})}(\mathbf{h}_1, ..., \mathbf{h}_5) &= E\left\{X_{\mathbf{k}}(\mathbf{t})\,X_{\mathbf{k}}(\mathbf{t}+\mathbf{h}_1)\,...X_{\mathbf{k}}(\mathbf{t}+\mathbf{h}_5)\right\}.
\end{aligned}$$

and similarly define $C^{(\infty,\mathbf{k})}(\mathbf{h}), ...,C_6^{(\mathbf{k},\dots,\mathbf{k})}(\mathbf{h}_1, ..., \mathbf{h}_5)$.

Theorem 2.3 *Let $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ be a strictly stationary random fields with $E\left\{X(\mathbf{t})\right\} = 0, E\left\{X^{12}(\mathbf{t})\right\} < \infty$, cumulants up to sixth order absolutely summable and $w(\mathbf{x}_1, \mathbf{x}_2)$ is a symmetric bispectral estimating kernel. Then*

$$|\mathbf{N}|^{\frac{1}{2}}\,|b_{\mathbf{N}}|\left[f_{3\mathbf{N}}(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) - E\left\{f_{3\mathbf{N}}(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)\right\}\right] \xrightarrow{d} X + iY,$$

where X and Y have zero mean, and the following variances:

i)

$$\sigma_X^2 = \sigma_Y^2 = \frac{1}{2}\frac{w_2}{(2\pi)^d}f(\boldsymbol{\lambda}_1)f(\boldsymbol{\lambda}_2)f(\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2),$$

if $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)$ lies inside the region one and not on its boundaries.

ii) *and if we include the boundaries*

$$\begin{aligned}
\sigma_X^2 &= \frac{w_1}{(2\pi)^d}f(\boldsymbol{\lambda}_1)f(\boldsymbol{\lambda}_2)f(\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2)[8\delta_{\boldsymbol{\lambda}_1} + \delta_{\boldsymbol{\lambda}_2}] + A + B, \\
\sigma_Y^2 &= A - B,
\end{aligned}$$

where

$$\begin{aligned} A &= \frac{1}{2}\frac{w_1}{(2\pi)^d} f(\boldsymbol{\lambda}_1)f(\boldsymbol{\lambda}_2)f(\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2)[(1+\delta_{\boldsymbol{\lambda}_1-\boldsymbol{\lambda}_2})(1+\delta_{\boldsymbol{\lambda}_1+2\boldsymbol{\lambda}_2-2\underline{\boldsymbol{\pi}}}+\delta_{2\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2-2\underline{\boldsymbol{\pi}}})+4\delta_{\boldsymbol{\lambda}_1}], \\ B &= \frac{1}{2}\frac{w_2}{(2\pi)^d} f(\boldsymbol{\lambda}_1)f(\boldsymbol{\lambda}_2)f(\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2)[5\delta_{\boldsymbol{\lambda}_1}+\delta_{\boldsymbol{\lambda}_2}(1+\delta_{\boldsymbol{\lambda}_1-\underline{\boldsymbol{\pi}}})], \end{aligned}$$

and

$$\begin{aligned} w_1 &= [\int w(\mathbf{0},\mathbf{x})d\mathbf{x}]^2, w_2 = \int\int w(\mathbf{x}_1,\mathbf{x}_2)d\mathbf{x}_1 d\mathbf{x}_2, \\ \delta_{\mathbf{x}} &= \begin{cases} 1, & \mathbf{x}=\mathbf{0} \\ 0, & otherwise \end{cases} \\ \underline{\boldsymbol{\pi}} &= (\pi,...,\pi), d-dimension. \end{aligned}$$

2.4.2 Asymptotic normality under the strong mixing condition

Let $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ be a real 6th-order weakly stationary random field, and S, S' be two sets of indices. The Borel fields $\mathcal{B}(S)=\mathcal{B}(X(\mathbf{t}), \mathbf{t}\in S)$ and $\mathcal{B}(S')=\mathcal{B}(X(\mathbf{t}), \mathbf{t}\in S')$ as usual are the σ-fields generated by the random field $X(\mathbf{t})$. Consider the distance $d(S,S')$ between the set of indices S and S'. The random field $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ is said to be strong mixing if

$$\sup_{A\in\mathcal{B}(S),B\in\mathcal{B}(S')} |P(AB)-P(A)P(B)| \le \varphi(d(S,S')),$$

for any two sets of indices S and S' with φ a function such that $\varphi(d)\longrightarrow 0$ as $d\longrightarrow\infty$.

Theorem 2.4 *Let $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ be a strong mixing weakly real random field with $E\{X(\mathbf{t})\}=0$, $E\{X^{12}(\mathbf{t})\}<\infty$, cumulants up to sixth order absolutely summable, $w(\mathbf{x}_1,\mathbf{x}_2)$is a symmetric bispectral estimating kernel, there is some $\delta>0$ such that for $\alpha_{\mathbf{N}},\beta_{\mathbf{N}}$ and $\sigma_{\boldsymbol{\lambda}}$*

$$\left\{\left[(|\alpha_{\mathbf{N}}|\,.\,|\beta_{\mathbf{N}}|)^{1/2}\sigma_{\boldsymbol{\lambda}}\right]^{2+\delta}\right\}^{-1}\sum_{\mathbf{j}=\mathbf{1}}^{\alpha_{\mathbf{N}}} E\left\{\left|U_{\mathbf{j}}^{(\mathbf{N})}\right|^{2+\delta}\right\}\longrightarrow 0, \tag{2.4.5}$$

where $U_{\mathbf{j}}^{(\mathbf{N})}$ defined by (2.7.25)*. Then the result of Theorem* 2.3 *holds.*

Remark 2.3 *If $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ be a stationary random field, then* (2.4.5) *becomes*

$$\left[(|\alpha_{\mathbf{N}}|^{\delta/2}\,.\,|\beta_{\mathbf{N}}|)^{1+\delta/2}\right]^{-1} E\left\{\left|U_{\mathbf{1}}^{(\mathbf{N})}\right|^{2+\delta}\right\}\longrightarrow 0.$$

2.5 The fourth-order cumulant spectral density estimation

Let $f_4(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3)$ the trispectral density function defined as

$$f_4(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) = \frac{1}{(2\pi)^{3d}} \sum_{\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3 \in \mathbb{Z}^d} C_4(\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3) e^{-i \sum_{j=1}^{3} \mathbf{h}_j . \boldsymbol{\lambda}_j}, \tag{2.5.1}$$

where

$$C_4(\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3) = \int_{\boldsymbol{\pi}} \int_{\boldsymbol{\pi}} \int_{\boldsymbol{\pi}} e^{i(\mathbf{h}_1.\boldsymbol{\lambda}_1 + \mathbf{h}_2.\boldsymbol{\lambda}_2 + \mathbf{h}_3.\boldsymbol{\lambda}_3)} f_4(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) d\boldsymbol{\lambda}_1 d\boldsymbol{\lambda}_2 d\boldsymbol{\lambda}_3.$$

We construct an estimate of the fourth-order cumulant as follows (see Rosenblatt (1985)):

$$\widehat{c}_4(\mathbf{h}_1, \mathbf{h_2}, \mathbf{h}_3) = \widehat{r}_4(\mathbf{h}_1, \mathbf{h_2}, \mathbf{h}_3) - \widehat{r}(\mathbf{h}_1)\widehat{r}(\mathbf{h_2} - \mathbf{h}_3) - \widehat{r}(\mathbf{h}_2)\widehat{r}(\mathbf{h_1} - \mathbf{h}_3) - \widehat{r}(\mathbf{h}_3)\widehat{r}(\mathbf{h}_1 - \mathbf{h_2}), \tag{2.5.2}$$

and we can shown that (2.5.2) is an asymptotically unbiased and consistent estimator of $c_4(\mathbf{h}_1, \mathbf{h_2}, \mathbf{h}_3)$.

Then we define an estimator for (2.5.1) as

$$f_{4\mathbf{N}}(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) = \frac{1}{(2\pi)^{3d}} \sum_{\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3} \widehat{C}_4(\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3) w_{\mathbf{N}}(\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3) e^{-i \sum_{j=1}^{3} \mathbf{h}_j . \boldsymbol{\lambda}_j}, \tag{2.5.3}$$

where $w_{\mathbf{N}}(\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3) = w(b_{\mathbf{N}} \odot \mathbf{h}_1, b_{\mathbf{N}} \odot \mathbf{h}_2, b_{\mathbf{N}} \odot \mathbf{h}_3)$ and $w(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = w_1(\mathbf{x}_1) w_2(\mathbf{x}_2) w_3(\mathbf{x}_3)$ and $w_i(\mathbf{x}), i = 1, 2, 3$ be a bounded continuous function defined on $[-1, 1]^d$ with $w_i(\mathbf{0}) = 1$.

2.5.1 Asymptotic properties under a uniform summability of cumulants condition

Let

$$Y_{\mathbf{u}}^{\mathbf{N}}(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) = \sum_{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3} X(\mathbf{u}) X(\mathbf{u} + \mathbf{s}_1) X(\mathbf{u} + \mathbf{s}_2) X(\mathbf{u} + \mathbf{s}_3) w_{\mathbf{N}}(\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3) e^{-i \sum_{j=1}^{3} \mathbf{s}_j . \boldsymbol{\lambda}_j}, \tag{2.5.4}$$

and

$$g(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) = |\mathbf{N}^*|^{-1} \sum_{\mathbf{u}=\mathbf{1}}^{\mathbf{N}^*} Y_{\mathbf{u}}^{\mathbf{N}}(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3), \tag{2.5.5}$$

where $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) \in \boldsymbol{\pi}^3, \mathbf{N}^* \longrightarrow \infty, N_i^* = o(N_i)$ as $N_i \longrightarrow \infty$.

Consider ν the collection of two partitions A and B defined as

$$A = \{(\mathbf{u} + \mathbf{s}_j, \mathbf{v} + \mathbf{h}_j), j = 0, ..., 3\},$$

$$B = \{(\mathbf{u} + \mathbf{s}_0, \mathbf{u} + \mathbf{s}_1), (\mathbf{v} + \mathbf{h}_0, \mathbf{v} + \mathbf{h}_1), (\mathbf{u} + \mathbf{s}_2, \mathbf{v} + \mathbf{h}_2), (\mathbf{u} + \mathbf{s}_3, \mathbf{v} + \mathbf{h}_3)\},$$

where $\mathbf{s}_0 = \mathbf{h}_0 = 0$, and define

$$D = \{\mathbf{u}, \mathbf{v}, \mathbf{s}_j, \mathbf{h}_j; \mathbf{u}, \mathbf{v} = \mathbf{1}, ..., \mathbf{N}^*, \left|\underline{\mathbf{s}}_j\right|, \left|\underline{\mathbf{h}}_j\right| \preceq l_{\mathbf{N}}, j = 1, 2, 3\}. \tag{2.5.6}$$

Lemma 2.3 *Suppose that* $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ *be a weakly stationary real random field with a zero mean and cumulants up to eighth order absolutely summable. Then for all*

$$var\{g(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_3)\}\sim|\mathbf{N}^*|^{-2}\sum_{\nu}\sum_{D}C\left(\nu_1\right)...C\left(\nu_4\right)w_{\mathbf{N}}(\mathbf{s}_1,\mathbf{s}_2,\mathbf{s}_3)w_{\mathbf{N}}(\mathbf{h}_1,\mathbf{h}_2,\mathbf{h}_3)e^{-i\sum_{j=1}^{3}(\mathbf{s}_j-\mathbf{h}_j).\boldsymbol{\lambda}_j},$$

as $\mathbf{N}\longrightarrow\infty$.

Proof. The proof follows from Lemma 3.1 in Kim (1988).

By permuting the indice of $\mathbf{h}$ in A partition, and for a fixed $\omega\in S_4$, where S_4 is the permutation group on four letters, let

$$\nu(\omega)=\{(\mathbf{u}+\mathbf{s}_j,\mathbf{v}+\mathbf{h}_{\omega(j)}),j=0,...,3,\mathbf{s}_0=\mathbf{h}_{\omega(0)}=0\},\tag{2.5.7}$$

then a direct calculation gives us

$$\begin{aligned}&\sum_{D}\prod_{j=0}^{4}C(\mathbf{u}-\mathbf{v}+\mathbf{s}_j-\mathbf{h}_{\omega(j)})w_{\mathbf{N}}(\mathbf{s}_1,\mathbf{s}_2,\mathbf{s}_3)w_{\mathbf{N}}(\mathbf{h}_1,\mathbf{h}_2,\mathbf{h}_3)e^{-i\sum_{j=1}^{3}(\mathbf{s}_j-\mathbf{h}_j).\boldsymbol{\lambda}_j}\\ \sim\ &\frac{(2\pi)^{4d}\left|\mathbf{N}^*\right|\upsilon_A(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_3,\omega)}{\left|b_N\right|^3},\end{aligned}\tag{2.5.8}$$

where

$$\upsilon_A(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_3,\omega)=\begin{cases}f(\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2+\boldsymbol{\lambda}_3)f(\boldsymbol{\lambda}_1)f(\boldsymbol{\lambda}_2)f(\boldsymbol{\lambda}_3)W_1, & \text{if }\ \boldsymbol{\lambda}_l\equiv\boldsymbol{\lambda}_{\omega(l)}(l=1,2,3)\\ f(\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2+\boldsymbol{\lambda}_3)f(\boldsymbol{\lambda}_1)f(\boldsymbol{\lambda}_2)f(\boldsymbol{\lambda}_3)W_1, & \text{if }\omega_j(0)=0\ (j=1,2,3)\\ & \text{and }\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2+\boldsymbol{\lambda}_3+\boldsymbol{\lambda}_j\equiv\mathbf{0},\\ & \boldsymbol{\lambda}_l\equiv\boldsymbol{\lambda}_{\omega(l)}(l\neq j,l=1,2,3)\\ 0 & \text{otherwise,}\end{cases}$$

and

$$W_1=\int\int\int w^2(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)d\mathbf{x}_1d\mathbf{x}_2d\mathbf{x}_3.$$

Finally, define

$$\upsilon_A(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_3)=\sum_{\omega\in S_4}\upsilon_A(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_3,\omega),\tag{2.5.9}$$

We can realize by B partition, $\nu(\alpha,\beta)$ in the following way

$$\nu(\alpha,\beta)=\{(\mathbf{u}+\mathbf{s}_{\alpha_{(0)}},\mathbf{u}+\mathbf{s}_{\alpha_{(1)}}),(\mathbf{v}+\mathbf{h}_{\beta(0)},\mathbf{v}+\mathbf{h}_{\beta(1)}),(\mathbf{u}+\mathbf{s}_{\alpha(2)},\mathbf{v}+\mathbf{h}_{\beta(2)}),(\mathbf{u}+\mathbf{s}_{\alpha(3)},\mathbf{v}+\mathbf{h}_{\beta(3)})\},\tag{2.5.10}$$

where $(\alpha, \beta) \in S_4 \times S_4$. By a direct calculation we can show that

$$\sum_{D} C(\mathbf{s}_{\alpha_{(0)}} - \mathbf{s}_{\alpha_{(1)}}) C(\mathbf{h}_{\beta(0)} - \mathbf{h}_{\beta(1)}) C(\mathbf{u} - \mathbf{v} + \mathbf{s}_{\alpha(2)} - \mathbf{h}_{\beta(2)}) C(\mathbf{u} - \mathbf{v} + \mathbf{s}_{\alpha(3)} - \mathbf{h}_{\beta(3)}) \quad (2.5.11)$$
$$\times w_{\mathbf{N}}(\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3) w_{\mathbf{N}}(\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3) e^{-i \sum_{j=1}^{3} (\mathbf{s}_j - \mathbf{h}_j) . \boldsymbol{\lambda}_j}$$
$$\sim \quad \frac{(2\pi)^{4d} |\mathbf{N}^*| \, \upsilon_B(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3, \alpha, \beta)}{|b_N|^3},$$

where $(\alpha, \beta) \in S_4 \times S_4$, $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) \in \boldsymbol{\pi}^3$, and the values of $\upsilon_B(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3, \alpha, \beta)$ are given similarly in Table I in Kim (1988).

Let $(\alpha, \beta) \cong (\alpha', \beta')$ iff $\nu(\alpha, \beta) = \nu(\alpha', \beta')$ ($\cong$ is an equivalence relation) and let $G = S_4 \times S_4$ be the collection of equivalence classes. As a consequence define

$$\upsilon_B(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) = \sum_{(\alpha,\beta) \in G} \upsilon_B(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3, \alpha, \beta), \quad (2.5.12)$$

where the summation is performed over any representative of each element in G.

Define

$$\widehat{r}_4(\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3) = \frac{1}{|\mathbf{N}^*|} \sum_{\mathbf{t} \in D_{\mathbf{n}}} X(\mathbf{t}) \, X(\mathbf{t} + \mathbf{h}_1) \, X(\mathbf{t} + \mathbf{h}_2) \, X(\mathbf{t} + \mathbf{h}_3), \quad (2.5.13)$$

where $D_{n_i} = \left[1 - \min(0, h_i^{(1)}, h_i^{(2)}, h_i^{(3)}), ..., n_i - \max(0, h_i^{(1)}, h_i^{(2)}, h_i^{(3)})\right]$, $n_i = \min(N_i, N_i^*)$ and $N_i^* \longrightarrow \infty$, $N_i^* = o(N_i)$ as $N_i \longrightarrow \infty$. By the absolute summability of cumulants up to 8th order, we obtain

$$E\left\{\widehat{r}_4(\mathbf{r}, \mathbf{s}, \mathbf{t})\right\} - r_4(\mathbf{r}, \mathbf{s}, \mathbf{t}) = O\left(|\mathbf{N}^*|^{-1}\right), E\left\{\widehat{r}(\mathbf{r})\widehat{r}(\mathbf{s} - \mathbf{t})\right\} = O\left(|\mathbf{N}|^{-1}\right), \mathbf{r}, \mathbf{s}, \mathbf{t} \in \mathbb{Z}^d,$$

and

$$var\left\{\widehat{r}_4(\mathbf{r}, \mathbf{s}, \mathbf{t})\right\} = O\left(|\mathbf{N}^*|^{-1}\right), var\left\{\widehat{r}(\mathbf{r})\widehat{r}(\mathbf{s} - \mathbf{t})\right\} = O\left(|\mathbf{N}|^{-1/2}\right),$$

where $\widehat{r}(.)$ define similarly as (2.3.6). Indeed define

$$\begin{aligned}
\widehat{g}_{\mathbf{1}}(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) &= \sum_{\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3} \widehat{r}_4(\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3) w_{\mathbf{N}}(\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3) e^{-i \sum_{j=1}^{3} \mathbf{h}_j . \boldsymbol{\lambda}_j}, \\
\widehat{g}_{\mathbf{2}}(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) &= -\sum_{\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3} \widehat{r}_2(\mathbf{h}_1) \widehat{r}_2(\mathbf{h}_2 - \mathbf{h}_3) w_{\mathbf{N}}(\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3) e^{-i \sum_{j=1}^{3} \mathbf{h}_j . \boldsymbol{\lambda}_j}, \\
\widehat{g}_{\mathbf{3}}(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) &= -\sum_{\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3} \widehat{r}_2(\mathbf{h}_2) \widehat{r}_2(\mathbf{h}_1 - \mathbf{h}_3) w_{\mathbf{N}}(\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3) e^{-i \sum_{j=1}^{3} \mathbf{h}_j . \boldsymbol{\lambda}_j}, \\
\widehat{g}_{\mathbf{4}}(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) &= -\sum_{\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3} \widehat{r}_2(\mathbf{h}_3) \widehat{r}_2(\mathbf{h}_1 - \mathbf{h}_2) w_{\mathbf{N}}(\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3) e^{-i \sum_{j=1}^{3} \mathbf{h}_j . \boldsymbol{\lambda}_j},
\end{aligned}$$

then we can write (2.5.3) as

$$f_{4\mathbf{N}}(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) = \frac{1}{(2\pi)^{3d}} \sum_{j=1}^{4} \widehat{g}_j(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3), (\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) \in \boldsymbol{\pi}^3. \tag{2.5.14}$$

Lemma 2.4 *Suppose that* $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ *be a weakly stationary real random field with a zero mean and cumulants up to eighth order absolutely summable. Then*

$$var\{\widehat{g}_1(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3)\} \sim var\{g(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3)\},$$

and

$$var\{\widehat{g}_j(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3)\} = o\left(|\mathbf{N}^*|^{-1}\,|b_{\mathbf{N}}|^{-3}\right),\ (j = 2, 3, 4)\ as\ \mathbf{N} \longrightarrow \infty$$

uniformly for all $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) \in \boldsymbol{\pi}^3$.

Theorem 2.5 *Suppose that* $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ *be a weakly stationary real random field with a zero mean and cumulants up to eighth order absolutely summable. Then, for all* $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) \in \boldsymbol{\pi}^3$

$$E\left\{f_{4\mathbf{N}}(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3)\right\} \longrightarrow f_{\mathbf{4}}(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3)\ as\ \mathbf{N} \longrightarrow \infty,$$

and there exist bounded functions $\upsilon_A(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3)$ *and* $\upsilon_B(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3)$ *defined on* $\boldsymbol{\pi}^3$ *so that*

$$(2\pi)^{2d} var\{f_{4\mathbf{N}}(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3)\} \sim \frac{\upsilon(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3)}{|\mathbf{N}^*|\,|b_{\mathbf{N}}|^3}\ as\ \mathbf{N} \longrightarrow \infty,$$

where $\upsilon(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) = \upsilon_A(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) + \upsilon_B(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3)$ *and* $\upsilon_B(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) = 0$ *if* $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) \in \boldsymbol{\pi}^3$ *have no submanifolds*[2].

Corollary 2.3 *Under the conditions of Theorem 2.5,* $f_{4\mathbf{N}}(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3)$ *is a consistent estimator of* $f_{\mathbf{4}}(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3)$ *for all* $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) \in \boldsymbol{\pi}^3$.

2.5.2 Asymptotic normality under the strong mixing condition

In this section, we obtained the asymptotic normality of fourth-order cumulant spectral density estimates under the strong mixing condition for all frequencies including those lying on what have been called submanifolds. This result is sufficiently complete to indicate what happens in general, in the following section.

[2] *For* $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) \in \boldsymbol{\pi}^3$ *a submanifold is defined to be any subset* $\{j_1, ..., j_s\}$ *of* $\{1, 2, 3\}$ *so that* $\sum_{k-1}^{s} \boldsymbol{\lambda}_{j_k} \equiv \mathbf{0}$ *for* $1 \le s \le 3$, *where* $\mathbf{x} \equiv \mathbf{y}$ *means* $x_i = y_i \,(\operatorname{mod} 2\pi)\,, i = 1, ..., d$.

Theorem 2.6 *Let* $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ *be a strictly stationary strong mixing random field with zero mean, and assume that condition 2.1 is satisfied for* $p=4$. *Then*

$(|\mathbf{N}|\,|b_{\mathbf{N}}|^3)^{1/2}\,[f_{4\mathbf{N}}(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_3)-E\{f_{4\mathbf{N}}(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_3)\}]$ *are jointly asymptotic normal with zero mean and covariance given by*

$$\begin{aligned}&(2\pi)^{-2d}f(\boldsymbol{\mu}_1)\,f(\boldsymbol{\mu}_2)\,f(\boldsymbol{\mu}_3)\,f(\boldsymbol{\mu}_1+\boldsymbol{\mu}_2+\boldsymbol{\mu}_3)\\ &\sum_T[\eta(\boldsymbol{\mu}_1-\boldsymbol{\lambda}_{T_1})\eta(\boldsymbol{\mu}_2-\boldsymbol{\lambda}_{T_2})\eta(\boldsymbol{\mu}_3-\boldsymbol{\lambda}_{T_3})\\ &\times\int w(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)w(\mathbf{x}_{T_1}-\mathbf{x}_{T_4},\mathbf{x}_{T_2}-\mathbf{x}_{T_4},\mathbf{x}_{T_3}-\mathbf{x}_{T_4})d\mathbf{x}_1d\mathbf{x}_2d\mathbf{x}_3],\end{aligned}$$

where the sum is over all $4!$ *permutation* $T=(T_1,T_2,T_3,T_4)$ *of* $(1,2,3,4)$ *with the convention that* $\boldsymbol{\mu}_4=-\boldsymbol{\mu}_1-\boldsymbol{\mu}_2-\boldsymbol{\mu}_3$, $\boldsymbol{\lambda}_4=-\boldsymbol{\lambda}_1-\boldsymbol{\lambda}_2-\boldsymbol{\lambda}_3$, $\mathbf{x}_4=\mathbf{0}$, *and*

$$\eta(\boldsymbol{\lambda})=\begin{cases}0 & \text{if } \lambda_i\neq 2\pi k \text{ with } k \text{ an integer }, i=1,...d\\ 1 & \text{otherwise.}\end{cases}$$

Remark 2.4 *we can show that for a weight function* $w(\mathbf{x})$ *satisfying* $3-4$ *in condition 2.1, there exists a sequence of weight functions* $w_l^{(s_i)}$ *satisfying* 3 *in condition 2.1 such that*

$$\sum_{\mathbf{s}=\mathbf{1}}^{\mathbf{N}}\alpha_{\mathbf{s}}\prod_{l=1}^{p-1}w_l^{(s_i)}(\mathbf{j}\odot\mathbf{h}_l)\xrightarrow{u}w(\mathbf{h}_1,...,\mathbf{h}_{p-1}),\ \text{as } \mathbf{N}\longrightarrow\infty.$$

2.6 General p-order case

The asymptotic normality of the general p-order spectral density estimate under a limited number of cumulant summability assumptions and the strong mixing condition for all frequencies have been a submanifolds given in the following theorem

Theorem 2.7 *Let* $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ *be a strictly stationary strong mixing random fields with zero mean, and assume that condition 2.1 is satisfied for* $p\geq 4$. *Then*

$\left(|\mathbf{N}|\,|b_{\mathbf{N}}|^{p-1}\right)^{1/2}[f_{p\mathbf{N}}(\boldsymbol{\lambda}_1,...,\boldsymbol{\lambda}_{p-1})-E\{f_{p\mathbf{N}}(\boldsymbol{\lambda}_1,...,\boldsymbol{\lambda}_{p-1})\}]$ *are asymptotically jointly normal with zero mean and covariance of the submanifold is given by*

$$\begin{aligned}&(2\pi)^{d(2-p)}(|\mathbf{N}|\,|b_{\mathbf{N}}|^{p-1})^{-1}\prod_{j=1}^{p}f(\boldsymbol{\lambda}_j)\sum_T\prod_{j=1}^{p-1}[\eta(\boldsymbol{\mu}_j-\boldsymbol{\lambda}_{T_j})\\ &\times\int w(\mathbf{x}_1,...,\mathbf{x}_{p-1})w(\mathbf{x}_{T_1}-\mathbf{x}_{T_p},...,\mathbf{x}_{T_{p-1}}-\mathbf{x}_{T_p})d\mathbf{x}_1...d\mathbf{x}_{p-1}],\end{aligned}$$

where the sum is over all $p!$ *permutation* $T=(T_1,...,T_p)$ *of* $(1,2,...,p)$ *with the convention that* $\boldsymbol{\lambda}_p=-\sum_{j=1}^{p-1}\boldsymbol{\lambda}_j$, $\boldsymbol{\mu}_p=-\sum_{j=1}^{p-1}\boldsymbol{\mu}_j$, $\mathbf{x}_p=\mathbf{0}$, *and* η *is the Kronecker delta function.*

2.7 Proof

Proof. Theorem 2.1

Write

$$T_{\mathbf{N}} = \pi^{-2}\left(\frac{|\mathbf{N}|}{|b_{\mathbf{N}}|}\right)^{-1/2}\sum_{\mathbf{h}=\mathbf{1}}^{l_{\mathbf{N}}} w(b_{\mathbf{N}}\odot\mathbf{h})\cos(\mathbf{h}.\boldsymbol{\lambda})\sum_{\mathbf{t}=1}^{\mathbf{N}-\mathbf{h}} X(\mathbf{t})X(\mathbf{t}+\mathbf{h}). \tag{2.7.1}$$

Then it is easy to see that

$$\mathcal{L}\left((|\mathbf{N}|\,|b_{\mathbf{N}}|)^{1/2}\left[f_{\mathbf{N}}(\boldsymbol{\lambda}) - E\left\{f_{\mathbf{N}}(\boldsymbol{\lambda})\right\}\right]\right) \approx \mathcal{L}(T_{\mathbf{N}} - E\left\{T_{\mathbf{N}}\right\}). \tag{2.7.2}$$

Moreover, if we define

$$Y_{\mathbf{N}}(\mathbf{t}) = \pi^{-2}\,|b_{\mathbf{N}}|^{1/2}\sum_{\mathbf{h}=\mathbf{1}}^{l_{\mathbf{N}}} w(b_{\mathbf{N}}\odot\mathbf{h})\cos(\mathbf{h}.\boldsymbol{\lambda})[X(\mathbf{t})X(\mathbf{t}+\mathbf{h}) - C(\mathbf{h})], \tag{2.7.3}$$

$$U_{\mathbf{N}} = \pi^{-2}\,|b_{\mathbf{N}}|^{1/2}\sum_{\mathbf{h}=\mathbf{1}}^{l_{\mathbf{N}}} w(b_{\mathbf{N}}\odot\mathbf{h})\cos(\mathbf{h}.\boldsymbol{\lambda})\sum_{\mathbf{t}=\mathbf{N}-\mathbf{h}-\mathbf{1}}^{\mathbf{N}}[X(\mathbf{t})X(\mathbf{t}+\mathbf{h}) - C(\mathbf{h})],$$

Then we can show that

$$E\left\{U_{\mathbf{N}}^2\right\} \longrightarrow 0, \text{ as } \mathbf{N} \longrightarrow \infty.$$

Therefore, we conclude that

$$\mathcal{L}(T_{\mathbf{N}} - E\left\{T_{\mathbf{N}}\right\}) \approx \mathcal{L}\left(|\mathbf{N}|^{-1/2}\sum_{\mathbf{t}=1}^{\mathbf{N}} Y_{\mathbf{N}}(\mathbf{t})\right).$$

From condition of Theorem 2.1 and lemma 1 in Chanda (2005), we show that

$$var\left\{|\mathbf{N}|^{-1/2}\sum_{\mathbf{t}=\mathbf{1}}^{\mathbf{N}} Y_{\mathbf{N}}(\mathbf{t})\right\} \longrightarrow \sigma_{\boldsymbol{\lambda}}^2, \tag{2.7.4}$$

where $\sigma_{\boldsymbol{\lambda}}^2$ is as defined in (2.3.7).

Write

$$Y_{\mathbf{N}}^{\mathbf{m}}(\mathbf{t}) = \pi^{-2}\,|b_{\mathbf{N}}|^{1/2}\sum_{\mathbf{h}=\mathbf{1}}^{l_{\mathbf{N}}} w(b_{\mathbf{N}}\odot\mathbf{h})\cos(\mathbf{h}.\boldsymbol{\lambda})[X_{\mathbf{m}}(\mathbf{t})X_{\mathbf{m}}(\mathbf{t}+\mathbf{h}) - C_{\mathbf{m}}(\mathbf{h})], \tag{2.7.5}$$

where for $\mathbf{m} = (m_1, m_2), \mathbf{m} = m_{\mathbf{N}} \longrightarrow \infty$, but $\frac{m_{\mathbf{N}}}{b_{\mathbf{N}}} \longrightarrow \mathbf{0}$ as $\mathbf{N} \longrightarrow \infty$, $X_{\mathbf{m}}(\mathbf{t}) = \sum_{\mathbf{j}=\mathbf{0}}^{\mathbf{m}} W_{\mathbf{j}}(\mathbf{t})$ and $C_{\mathbf{m}}(\mathbf{h}) = E\left\{X_{\mathbf{m}}(\mathbf{t})X_{\mathbf{m}}(\mathbf{t}+\mathbf{h})\right\}$. Write

$$T_{\mathbf{N}}^{\mathbf{m}} = |\mathbf{N}|^{-1/2}\sum_{\mathbf{t}=\mathbf{1}}^{\mathbf{N}} Y_{\mathbf{N}}^{\mathbf{m}}(\mathbf{t}),$$

then

$$E\left\{T_{\mathbf{N}}-T_{\mathbf{N}}^{\mathbf{m}}\right\} \leq 2(var(S_1)+var(S_2)), \tag{2.7.6}$$

where

$$S_1=\pi^{-2}\left(\frac{|\mathbf{N}|}{|b_{\mathbf{N}}|}\right)^{1/2}\sum_{\mathbf{t}=\mathbf{1}}^{\mathbf{N}}\sum_{\mathbf{h}=\mathbf{1}}^{l_{\mathbf{N}}} w(b_{\mathbf{N}}\odot\mathbf{h})\cos(\mathbf{h}.\boldsymbol{\lambda})[X_{\mathbf{m}}(\mathbf{t})X_{\mathbf{m}}^{*}(\mathbf{t}+\mathbf{h})],$$

$$S_2=\pi^{-2}\left(\frac{|\mathbf{N}|}{|b_{\mathbf{N}}|}\right)^{1/2}\sum_{\mathbf{t}=\mathbf{1}}^{\mathbf{N}}\sum_{\mathbf{h}=\mathbf{1}}^{l_{\mathbf{N}}} w(b_{\mathbf{N}}\odot\mathbf{h})\cos(\mathbf{h}.\boldsymbol{\lambda})[X_{\mathbf{m}}^{*}(\mathbf{t})X_{\mathbf{m}}(\mathbf{t}+\mathbf{h})],$$

and

$$X_{\mathbf{m}}^{*}(\mathbf{t})=X(\mathbf{t})-X_{\mathbf{m}}(\mathbf{t})=\sum_{\mathbf{j}\succeq\mathbf{m}+\mathbf{1}} W_{\mathbf{j}}\left(\mathbf{t}\right),$$

we can establish that $var(S_1)\longrightarrow 0, var(S_1)\longrightarrow 0$ as $\mathbf{N}\longrightarrow\infty$ (see Chanda (2005)). From (2.7.6), we conclude that

$$\mathcal{L}(T_{\mathbf{N}}-E\left\{T_{\mathbf{N}}\right\})\approx\mathcal{L}(T_{\mathbf{N}}^{\mathbf{m}}-E\left\{T_{\mathbf{N}}^{\mathbf{m}}\right\}), \tag{2.7.7}$$

In other words,

$$\lim_{\mathbf{N}\longrightarrow\infty} var\left\{T_{\mathbf{N}}^{\mathbf{m}}\right\}=\lim_{\mathbf{N}\longrightarrow\infty} var\left\{T_{\mathbf{N}}\right\}=\sigma_{\boldsymbol{\lambda}}^{2},$$

First note that $(Y_{\mathbf{N}}^{\mathbf{m}}(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^2}$ is a $|k_{\mathbf{N}}|$ $-$dependent strictly stationary random fields. Choosing the vector of sequences $\{p_{\mathbf{N}};\mathbf{N}\succeq 1\}$ of integers such that $p_{\mathbf{N}}\succ 2k_{\mathbf{N}}\longrightarrow\infty$, $\frac{p_{\mathbf{N}}}{\mathbf{N}}\longrightarrow\mathbf{0}$ and $\frac{k_{\mathbf{N}}}{p_{\mathbf{N}}}\longrightarrow\mathbf{0}$ as $\mathbf{N}\longrightarrow\infty$. Let $\mathbf{N}=p_{\mathbf{N}}\odot t_{\mathbf{N}}+r_{\mathbf{N}}$, where $t_{N_i}=[N_i/p_{Ni}]<N_i/p_{Ni}$ and $\mathbf{0}\preceq r_{\mathbf{N}}\prec p_{\mathbf{N}}$. Set

$$Z_{\mathbf{s},\mathbf{N}}^{\mathbf{m}}=\sum_{\mathbf{t}=(\mathbf{s}-\mathbf{1})\odot p_{\mathbf{N}}+\mathbf{1}}^{\mathbf{s}\odot p_{\mathbf{N}}-k_{\mathbf{N}}} Y_{\mathbf{N}}^{\mathbf{m}}\left(\mathbf{t}\right),\mathbf{1}\preceq\mathbf{s}\preceq t_{\mathbf{N}},$$

$$V_{\mathbf{s},\mathbf{N}}^{\mathbf{m}}=\sum_{\mathbf{t}=\mathbf{s}\odot p_{\mathbf{N}}-k_{\mathbf{N}}+\mathbf{1}}^{\mathbf{s}\odot p_{\mathbf{N}}} Y_{\mathbf{N}}^{\mathbf{m}}\left(\mathbf{t}\right),$$

and

$$R_{\mathbf{N}}^{\mathbf{m}}=\sum_{\mathbf{t}=p_{\mathbf{N}}\odot t_{\mathbf{N}}+\mathbf{1}}^{\mathbf{N}} Y_{\mathbf{N}}^{\mathbf{m}}\left(\mathbf{t}\right).$$

Since $\frac{\mathbf{N}}{p_{\mathbf{N}}}=t_{\mathbf{N}}+\frac{r_{\mathbf{N}}}{p_{\mathbf{N}}}\prec t_{\mathbf{N}}+\mathbf{1}$, we have that $t_{\mathbf{N}}\longrightarrow\infty$ as $\mathbf{N}\longrightarrow\infty$, $p_{\mathbf{N}}\succ 2k_{\mathbf{N}},\{Z_{\mathbf{s},\mathbf{N}}^{\mathbf{m}};\mathbf{1}\preceq\mathbf{s}\preceq t_{\mathbf{N}}\}$ and $\{V_{\mathbf{s},\mathbf{N}}^{\mathbf{m}};\mathbf{1}\preceq\mathbf{s}\preceq t_{\mathbf{N}}\}$ are two i.i.d. sequences of random fields. Moreover

$$\begin{aligned} var\left\{|\mathbf{N}|^{-1/2}\sum_{\mathbf{s}=\mathbf{1}}^{t_{\mathbf{N}}} V_{\mathbf{s},\mathbf{N}}^{\mathbf{m}}\right\} &= |\mathbf{N}|^{-1}\left|t_{\mathbf{N}}\right| var\left\{V_{\mathbf{1},\mathbf{N}}^{\mathbf{m}}\right\} \\ &\leq M\left|\mathbf{N}\right|^{-1}\left|t_{\mathbf{N}}\right|\left|k_{\mathbf{N}}\right| \\ &\leq \frac{M\left|k_{\mathbf{N}}\right|}{\left|p_{\mathbf{N}}\right|}\longrightarrow 0 \text{ as } \mathbf{N}\longrightarrow\infty. \end{aligned} \tag{2.7.8}$$

The first inequality is the direct consequence of the fact that

$$var\left\{|k_{\mathbf{N}}|^{-1/2}V^{\mathbf{m}}_{\mathbf{1,N}}\right\}=var\left\{|k_{\mathbf{N}}|^{-1/2}\sum_{\mathbf{t=1}}^{k_{\mathbf{N}}}Y^{\mathbf{m}}_{\mathbf{N}}(\mathbf{t})\right\}\longrightarrow\sigma^2_{\boldsymbol{\lambda}}\text{ as }\mathbf{N}\longrightarrow\infty.$$

Similarly we can show that

$$var\left\{|\mathbf{N}|^{-1/2}R^{\mathbf{m}}_{\mathbf{N}}\right\}\longrightarrow 0\text{ as }\mathbf{N}\longrightarrow\infty, \tag{2.7.9}$$

and that

$$\begin{aligned} var\left\{|\mathbf{N}|^{-1/2}\sum_{\mathbf{s=1}}^{t_{\mathbf{N}}}Z^{\mathbf{m}}_{\mathbf{s,N}}\right\} &= |\mathbf{N}|^{-1}|k_{\mathbf{N}}|\,var\left\{Z^{\mathbf{m}}_{\mathbf{1,N}}\right\} \qquad (2.7.10)\\ &= |\mathbf{N}|^{-1}|t_{\mathbf{N}}|\,|n_{\mathbf{N}}|\,var\left(|n_{\mathbf{N}}|^{-1/2}\sum_{\mathbf{t=1}}^{n_{\mathbf{N}}}Y^{\mathbf{m}}_{\mathbf{N}}(\mathbf{t})\right)\\ &\longrightarrow \sigma^2_{\boldsymbol{\lambda}}\text{ as }\mathbf{N}\longrightarrow\infty, \end{aligned}$$

where we write $n_{\mathbf{N}}=p_{\mathbf{N}}-k_{\mathbf{N}}\longrightarrow\infty$ as $\mathbf{N}\longrightarrow\infty$,
$\{|\mathbf{N}|^{-1}|t_{\mathbf{N}}|\,|n_{\mathbf{N}}|=(1-|r_{\mathbf{N}}|\,/\,|\mathbf{N}|)\,(1-|k_{\mathbf{N}}|\,/\,|p_{\mathbf{N}}|)\longrightarrow 1$ because $\frac{r_{\mathbf{N}}}{n_{\mathbf{N}}}\prec t_{\mathbf{N}}^{-1}\longrightarrow 0$ and $\frac{k_{\mathbf{N}}}{p_{\mathbf{N}}}\longrightarrow 0$ as $\mathbf{N}\longrightarrow\infty\}$ Since $\{Z^{\mathbf{m}}_{\mathbf{s,N}};\mathbf{1}\preceq\mathbf{s}\preceq t_{\mathbf{N}}\}$ is an i.i.d. sequence of random fields, and (2.7.10) holds true, we must have that

$$\mathcal{L}\left(|\mathbf{N}|^{-1/2}\sum_{\mathbf{s=1}}^{t_{\mathbf{N}}}Z^{\mathbf{m}}_{\mathbf{s,N}}\right)\longrightarrow\mathcal{N}(0,\sigma^2_{\boldsymbol{\lambda}})\text{ as }\mathbf{N}\longrightarrow\infty,$$

then

$$\sum_{\mathbf{s=1}}^{t_{\mathbf{N}}}\left(Z^{\mathbf{m}}_{\mathbf{s,N}}+V^{\mathbf{m}}_{\mathbf{s,N}}\right)+R^{\mathbf{m}}_{\mathbf{N}}=\sum_{\mathbf{t=1}}^{\mathbf{N}}Y^{\mathbf{m}}_{\mathbf{N}}(\mathbf{t}),\,[t_{\mathbf{N}}\odot(p_{\mathbf{N}}-k_{\mathbf{N}})+t_{\mathbf{N}}\odot k_{\mathbf{N}}+r_{\mathbf{N}}=\mathbf{N}]$$

and relations (2.7.8) – (2.7.10) hold. Therefore, we finally conclude that

$$\mathcal{L}\left(|\mathbf{N}|^{-1/2}\sum_{\mathbf{t=1}}^{\mathbf{N}}Y^{\mathbf{m}}_{\mathbf{N}}(\mathbf{t})\right)\longrightarrow\mathcal{N}(0,\sigma^2_{\boldsymbol{\lambda}})\text{ as }\mathbf{N}\longrightarrow\infty, \tag{2.7.11}$$

The result of Theorem 2.1 will now follow immediately from (2.7.2), (2.7.7) and (2.7.11).
Proof. Theorem 2.2

Let $\rho=\rho(4), w_{\mathbf{h}}=w(b_{\mathbf{N}}\odot\mathbf{h})\cos(\mathbf{h}.\boldsymbol{\lambda})$ and

$$h_{\mathbf{N}}(\boldsymbol{\lambda})=\frac{(|\mathbf{N}|\,|b_{\mathbf{N}}|)^{-1/2}}{(2\pi)^d}\left(\sum_{\mathbf{h=0}}^{l_{\mathbf{N}}}\sum_{\mathbf{u=N-h+1}}^{\mathbf{N}}X(\mathbf{u})\,X(\mathbf{u}+\mathbf{h})\,w_{\mathbf{h}}+\sum_{\mathbf{h}=-l_{\mathbf{N}}}^{\mathbf{-1}}\sum_{\mathbf{u=N+h+1}}^{\mathbf{N}}X(\mathbf{u})\,X(\mathbf{u}+\mathbf{h})\,w_{\mathbf{h}}\right).$$

By the summability of cumulants of order 2 and 4, $\|h_{\mathbf{N}}(\boldsymbol{\lambda})\| = (|\mathbf{N}|\,|b_{\mathbf{N}}|)^{-1/2}O(|b_{\mathbf{N}}|)$. Let $g_{\mathbf{N}}(\boldsymbol{\lambda}) = \sum_{\mathbf{u}=\mathbf{1}}^{\mathbf{N}} Y_{\mathbf{u}}(\boldsymbol{\lambda})$. Then

$$(|\mathbf{N}|\,|b_{\mathbf{N}}|)^{-1/2}\left[f_{\mathbf{N}}(\boldsymbol{\lambda}) - E\left\{f_{\mathbf{N}}(\boldsymbol{\lambda})\right\}\right] = \frac{g_{\mathbf{N}}(\boldsymbol{\lambda}) - E\left\{g_{\mathbf{N}}(\boldsymbol{\lambda})\right\}}{(|\mathbf{N}|\,|b_{\mathbf{N}}|)^{1/2}} + h_{\mathbf{N}}(\boldsymbol{\lambda}) - E\left\{h_{\mathbf{N}}(\boldsymbol{\lambda})\right\}. \qquad (2.7.12)$$

For $\mathbf{h} \in \mathbb{Z}^d$, let $\widetilde{X}(\mathbf{h}) = E\left\{X(\mathbf{h})\,|e(\mathbf{h}-\mathbf{k}+\mathbf{1}),...,e(\mathbf{h})\right\}$, where $k_i = k_{N_i} = [c\log N_i]$ and $c = -8/\log\rho$. Let $\widetilde{Y}_{\mathbf{u}} := \widetilde{Y}_{\mathbf{u}}(\boldsymbol{\lambda})$ be the corresponding sum with $X(\mathbf{h})$ replaced by $\widetilde{X}(\mathbf{h})$. Observe that $\widetilde{X}(\mathbf{n})$ and $\widetilde{X}(\mathbf{m})$ are i.i.d if $[n_i - m_i] \geq k_i, i = 1,...,d.$ and $\widetilde{Y}_{\mathbf{u}}$ and $\widetilde{Y}_{\mathbf{v}}$ are i.i.d if $|u_i - v_i| \geq 2b_{N_i} + k_i$. The independence plays an important role in establishing the asymptotic normality of $\widetilde{g}_{\mathbf{N}}(\boldsymbol{\lambda}) = \sum_{\mathbf{u}=\mathbf{1}}^{\mathbf{N}} \widetilde{Y}_{\mathbf{u}}(\boldsymbol{\lambda})$. Then $\|g_{\mathbf{N}}(\boldsymbol{\lambda}) - \widetilde{g}_{\mathbf{N}}(\boldsymbol{\lambda})\| = o(1)$ since

$$\begin{aligned}\left\|Y_{\mathbf{u}}(\boldsymbol{\lambda}) - \widetilde{Y}_{\mathbf{u}}(\boldsymbol{\lambda})\right\| &\leq (2\pi)^{-d}\sum_{\underline{\mathbf{h}}\preceq b_{\mathbf{N}}} |w_{\mathbf{h}}|\left\|X(\mathbf{u})\,X(\mathbf{u}+\mathbf{h}) - \widetilde{X}(\mathbf{u})\,\widetilde{X}(\mathbf{u}+\mathbf{h})\right\| \qquad (2.7.13)\\ &= O(|b_{\mathbf{N}}|\,\rho^{|\mathbf{a}|/4}).\end{aligned}$$

Let

$$\psi_{N_i} = N_i/(\log N_i)^{2+8/\delta}, p_{N_i} = [\psi_{N_i}^{2/3} b_{N_i}^{1/3}] \text{ and } q_{N_i} = [\psi_{N_i}^{1/3} b_{N_i}^{2/3}].$$

Then

$$\begin{aligned} p_{\mathbf{N}}, q_{\mathbf{N}} &\longrightarrow \infty, q_{N_i} = o(p_{N_i}), i = 1,...,d. \qquad (2.7.14)\\ 2b_{N_i} + k_i &= o(q_{N_i}) \text{ and } k_{Ni} = [N_i/(p_{N_i} + q_{N_i})] \longrightarrow \infty.\end{aligned}$$

Define for $\mathbf{1} \preceq \mathbf{r} \preceq k_{\mathbf{N}} - \mathbf{1}$,

$$\begin{aligned}\mathcal{L}_{\mathbf{r}} &= \{\mathbf{j} \in \mathbb{N}^d : (\mathbf{r}-\mathbf{1})\odot(p_{\mathbf{N}} + q_{\mathbf{N}}) + \mathbf{1} \preceq \mathbf{j} \preceq \mathbf{r}\odot(p_{\mathbf{N}} + q_{\mathbf{N}}) - q_{\mathbf{N}}\},\\ S_{\mathbf{r}} &= \{\mathbf{j} \in \mathbb{N}^d : \mathbf{r}\odot(p_{\mathbf{N}} + q_{\mathbf{N}}) - q_{\mathbf{N}} + \mathbf{1} \preceq \mathbf{j} \preceq \mathbf{r}\odot(p_{\mathbf{N}} + q_{\mathbf{N}})\},\\ S_{k_{\mathbf{N}}} &= \{\mathbf{j} \in \mathbb{N}^d : k_{\mathbf{N}}\odot(p_{\mathbf{N}} + q_{\mathbf{N}}) - q_{\mathbf{N}} + \mathbf{1} \preceq \mathbf{j} \preceq \mathbf{N}\},\end{aligned}$$

and let $U_{\mathbf{r}} = \sum_{\mathbf{j}\in\mathcal{L}_{\mathbf{r}}} \widetilde{Y}_{\mathbf{j}}$ and $V_{\mathbf{r}} = \sum_{\mathbf{j}\in S_{\mathbf{r}}} \widetilde{Y}_{\mathbf{j}}$. Observe that $U_1,...,U_{k_{\mathbf{N}}}$ and $V_1,...,V_{k_{\mathbf{N}}-1}$ are i.i.d. By Lemma 2.1 and 2.2, we have

$$\begin{aligned}\|U_{\mathbf{1}} - E\{U_{\mathbf{1}}\}\| &= \left\|\sum_{\mathbf{j}=\mathbf{1}}^{p_{\mathbf{N}}}(Y_{\mathbf{j}} - E\{Y_{\mathbf{0}}\})\right\| + O(|p_{\mathbf{N}}|\left\|Y_0 - \widetilde{Y}_0\right\| \qquad (2.7.15)\\ &\sim \left(|p_{\mathbf{N}}|\,|b_{\mathbf{N}}|\,\sigma^2\right)^{1/2} + O(|p_{\mathbf{N}}|\,|b_{\mathbf{N}}|\,\rho^{|\mathbf{a}|/4})\\ &\sim \left(|p_{\mathbf{N}}|\,|b_{\mathbf{N}}|\,\sigma^2\right)^{1/2}.\end{aligned}$$

Similarly,

$$\|V_{\mathbf{1}}-E\{V_{\mathbf{1}}\}\|\sim\left(|q_{\mathbf{N}}|\,|b_{\mathbf{N}}|\,\sigma^2\right)^{1/2}+O(|q_{\mathbf{N}}|\,|b_{\mathbf{N}}|\,\rho^{|\mathbf{a}|/4}).$$

By (2.7.14) ,

$$\begin{aligned} var\{V_{\mathbf{1}}+...+V_{k_{\mathbf{N}}}\} &= (|k_{\mathbf{N}}|-1)\,\|V_{\mathbf{1}}-E\{V_{\mathbf{1}}\}\|^2+\|V_{k_{\mathbf{N}}}-E\{V_{k_{\mathbf{N}}}\}\|^2 \\ &= O(|k_{\mathbf{N}}|\,|q_{\mathbf{N}}|\,|b_{\mathbf{N}}|)+O[(|p_{\mathbf{N}}|+|q_{\mathbf{N}}|)\,|b_{\mathbf{N}}|] \\ &= o(|\mathbf{N}|\,|b_{\mathbf{N}}|). \end{aligned}$$

From Theorem 3.1 in Shao and Wu (2007), we can prove that

$$(|\mathbf{N}|\,|b_{\mathbf{N}}|)^{-1/2}[\widetilde{g}_{\mathbf{N}}(\boldsymbol{\lambda})-E\{\widetilde{g}_{\mathbf{N}}(\boldsymbol{\lambda})\}]\longrightarrow\mathcal{N}\left(0,\sigma^2(\boldsymbol{\lambda})\right).$$

if

$$(|\mathbf{N}|\,|b_{\mathbf{N}}|)^{-1/2}\sum_{\mathbf{r}=\mathbf{1}}^{k_{\mathbf{N}}}(U_{\mathbf{r}}-E\{U_{\mathbf{1}}\})\longrightarrow\mathcal{N}\left(0,\sigma^2(\boldsymbol{\lambda})\right), \tag{2.7.16}$$

where $\|U_{\mathbf{1}}-E(U_{\mathbf{1}})\|=o\left((|\mathbf{N}|\,|b_{\mathbf{N}}|)^{1/2}k_{\mathbf{N}}^{-1/\tau}\right)$ and $\tau=2+\delta/2$. So (2.3.11) follows from (2.7.12) .

Proof. Theorem 2.3

Define

$$V_{\mathbf{N}}=|\mathbf{N}|^{\frac{1}{2}}\,|b_{\mathbf{N}}|\,[f_{3\mathbf{N}}(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)-E\{f_{3\mathbf{N}}(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)\}],$$

$$\begin{aligned} V_{\mathbf{N}M} &= \left[|b_{\mathbf{N}}|\,/(2\pi)^{2d}\,|\mathbf{N}|^{1/2}\right]\sum_{|\underline{\mathbf{h}}_1|,|\underline{\mathbf{h}}_2|\preceq Ml_{\mathbf{N}}}e^{-i(\mathbf{h}_1.\boldsymbol{\lambda}_1+\mathbf{h}_2.\boldsymbol{\lambda}_2)}w(b_{\mathbf{N}}\odot\mathbf{h}_1,b_{\mathbf{N}}\odot\mathbf{h}_2) \\ & \sum_{\mathbf{t}=\mathbf{1}}^{\mathbf{N}}[X(\mathbf{t})\,X(\mathbf{t}+\mathbf{h}_1)\,X(\mathbf{t}+\mathbf{h}_2)-r_3(\mathbf{h}_1,\mathbf{h}_2)]\,, \end{aligned} \tag{2.7.17}$$

we can prove that for any $\epsilon>0$, there is an $M_0(\epsilon)$ such that for all $M>M_0, N_i>M, i=1,...,d$;

$$\sigma^2(V_{\mathbf{N}}-V_{\mathbf{N}M})<\epsilon,$$

and if we replace the $X(\mathbf{t})$ in $V_{\mathbf{N}M}$ by $X_{\mathbf{k}}(\mathbf{t})$ to get $V_{\mathbf{N}M}^{(\mathbf{k})}$, we show that $\sigma^2(V_{\mathbf{N}M}-V_{\mathbf{N}M}^{(\mathbf{k})})$ can be made smaller that any previously chosen $\epsilon>0$ uniformly in $\mathbf{N}$ for k sufficiently large (M being fixed) (c.f. Van ness (1966)).

Write

$$\begin{aligned} U_{\mathbf{N}}^{(R)} &= \operatorname{Re}V_{\mathbf{N}M}^{(\mathbf{k})}=|\mathbf{N}|^{-1/2}\sum_{\mathbf{t}=\mathbf{1}}^{\mathbf{N}}Y_{\mathbf{k}}^{(\mathbf{N},\mathbf{M})}(\mathbf{t})\,, \\ U_{\mathbf{N}}^{(I)} &= \operatorname{Im}V_{\mathbf{N}M}^{(\mathbf{k})}=|\mathbf{N}|^{-1/2}\sum_{\mathbf{t}=\mathbf{1}}^{\mathbf{N}}Z_{\mathbf{k}}^{(\mathbf{N},\mathbf{M})}(\mathbf{t})\,, \end{aligned}$$

where

$$Y_{\mathbf{k}}^{(\mathbf{N},\mathbf{M})}(\mathbf{t}) = \left(|b_{\mathbf{N}}|/(2\pi)^{2d}\right) \sum_{\underline{\mathbf{h}}_1,\underline{\mathbf{h}}_2 \preceq Ml_{\mathbf{N}}} \cos(\mathbf{h}_1.\boldsymbol{\lambda}_1 + \mathbf{h}_2.\boldsymbol{\lambda}_2) w(b_{\mathbf{N}} \odot \mathbf{h}_1, b_{\mathbf{N}} \odot \mathbf{h}_2) \times [X_{\mathbf{k}}(\mathbf{t}) X_{\mathbf{k}}(\mathbf{t}+\mathbf{h}_1) X_{\mathbf{k}}(\mathbf{t}+\mathbf{h}_2) - r_3^{(\mathbf{k},\mathbf{k},\mathbf{k})}(\mathbf{h}_1,\mathbf{h}_2)],$$

and $Z_{\mathbf{k}}^{(\mathbf{N},\mathbf{M})}(\mathbf{t})$ is as above except with a sine instead of cosine. For any two real parameters κ_1 and κ_2 we have

$$U_{\mathbf{N}}(\kappa_1,\kappa_2) = \kappa_1 U_{\mathbf{N}}^{(R)} + \kappa_2 U_{\mathbf{N}}^{(I)}, \tag{2.7.18}$$

with $U_{\mathbf{k}}^{(\mathbf{N},\mathbf{M})}(\mathbf{t}) = |\mathbf{N}|^{-1/2}(\kappa_1 Y_{\mathbf{k}}^{(\mathbf{N},\mathbf{M})}(\mathbf{t}) + \kappa_2 Z_{\mathbf{k}}^{(\mathbf{N},\mathbf{M})}(\mathbf{t}))$. Note that the $U_{\mathbf{k}}^{(\mathbf{N},\mathbf{M})}(\mathbf{t})$ is a $2M|A_{\mathbf{N}}|+2|\mathbf{k}|$ dependent random fields. This prompts one to use the following lemma

Lemma 2.5 *Let $\{V_{\mathbf{N}}(\mathbf{t})\}$ a sequence of $d(\mathbf{N})$-dependent strictly stationary random fields, and*

a) $d(\mathbf{N}) \longrightarrow \infty$ *as* $\mathbf{N} \to \infty$,

b) $d(\mathbf{N})/\mathbf{N} \longrightarrow 0$ *as* $\mathbf{N} \to \infty$,

c) $E\left\{|V_{\mathbf{N}}(\mathbf{t})|^{2+\delta}\right\} < \infty$, *for some* $\delta > 0$,

d) $t(\mathbf{N})$ *is an* integer-*valued function*

1. $t(\mathbf{N}) \longrightarrow \infty$,
2. $d(N_i) = o(t(\mathbf{N})), i = 1, ..., d$,
3. $t(\mathbf{N}) = o(N_i), i = 1, ..., d$.

e) *for* $C_{\mathbf{N}}(.)$ *the covariance of sequence* $V_{\mathbf{N}}(\mathbf{t})$, $\sum_{|\underline{\mathbf{h}}| \preceq t(\mathbf{N})_d} \prod_{i=1}^{d} |h_i| C_{\mathbf{N}}(\mathbf{h}) = o(\sum_{|\underline{\mathbf{h}}| \preceq t(\mathbf{N})_d} C_{\mathbf{N}}(\mathbf{h}) t(\mathbf{N}))$ *as* $\mathbf{N} \to \infty$,

f) $E\left\{\left|\sum_{\mathbf{t}=\mathbf{1}}^{t(\mathbf{N})_d} V_{\mathbf{N}}(\mathbf{t})\right|^{2+\delta}\right\} / |\mathbf{N}|^{\delta/2} t(\mathbf{N})(\sum_{\mathbf{h}=-d(\mathbf{N})}^{d(\mathbf{N})} C_{\mathbf{N}}(\mathbf{h}))(1+\delta/2) \longrightarrow 0$ *as* $\mathbf{N} \longrightarrow \infty$, (*i.e.* $t(\mathbf{N})_d = (t(\mathbf{N}), ..., t(\mathbf{N}))$ *d- dimension.*

Then $\sum_{\mathbf{t}=\mathbf{1}}^{\mathbf{N}} V_{\mathbf{N}}(\mathbf{t})$ *is asymptotically normally distributed with zero mean and variance* $(2\pi)^d |\mathbf{N}| f_{\mathbf{N}}(\mathbf{0})$, *where* $f_{\mathbf{N}}(\boldsymbol{\lambda})$ *is the spectral density of* $V_{\mathbf{N}}(\mathbf{t})$.

To apply this lemma to (2.7.18), put

$$\begin{aligned} V_{\mathbf{N}}(\mathbf{t}) &= U_{\mathbf{k}}^{(\mathbf{N},\mathbf{M})}(\mathbf{t}), \\ d(\mathbf{N}) &= 2MA_{\mathbf{N}} + 2\mathbf{k}, \\ t(\mathbf{N}) &= M\left|A_{\mathbf{N}}\right|^2, \\ \delta &= 2, \end{aligned}$$

conditions (a), (b), and (d) of Lemma 2.5 are certainly satisfied. Condition (c) is satisfied since there is a constant K so that

$$\begin{aligned} E\left\{\left|V_{\mathbf{N}}(\mathbf{t})\right|^4\right\} &\leq \left(\left|b_{\mathbf{N}}\right|^4 / \left|\mathbf{N}\right|^2\right) \sum_{\left|\underline{\mathbf{h}}_1\right|,\ldots,\left|\underline{\mathbf{h}}_8\right| \preceq Ml_{\mathbf{N}}} \left|w(b_{\mathbf{N}} \odot \mathbf{h}_1, b_{\mathbf{N}} \odot \mathbf{h}_2) \ldots w(b_{\mathbf{N}} \odot \mathbf{h}_7, b_{\mathbf{N}} \odot \mathbf{h}_8)\right| .K.E\left\{X^{12}(\mathbf{t})\right\} \\ &\leq 2.\widehat{w}^4.K.E\left\{X^{12}(\mathbf{t})\right\} / (\left|\mathbf{N}\right| \left|b_{\mathbf{N}}\right|^2)^2 \\ &< \infty. \end{aligned}$$

Condition (e) involves

$$\begin{aligned} &\sum_{\left|\underline{\mathbf{h}}\right| \leq Ml_{\mathbf{N}}^2} \prod_{i=1}^{d} \left|h_i\right| C_{\mathbf{N}}(\mathbf{h}) / \sum_{\left|\underline{\mathbf{h}}\right| \leq Ml_{\mathbf{N}}^2} C_{\mathbf{N}}(\mathbf{h})\, M \left|l_{\mathbf{N}}\right|^2 \qquad (2.7.19) \\ &= \left|\mathbf{N}\right| \sum_{\left|\underline{\mathbf{h}}\right| \leq Ml_{\mathbf{N}}^2} \left(\prod_{i=1}^{d} \left|h_i\right| / M. \left|l_{\mathbf{N}}\right|^2\right) C_{\mathbf{N}}(\mathbf{h}) / \left|\mathbf{N}\right| \sum_{\left|\underline{\mathbf{h}}\right| \leq Ml_{\mathbf{N}}^2} C_{\mathbf{N}}(\mathbf{h}). \end{aligned}$$

But

$$\begin{aligned} \left|\mathbf{N}\right| \sum_{\left|\underline{\mathbf{h}}\right| \leq Ml_{\mathbf{N}}^2} C_{\mathbf{N}}(\mathbf{h}) &= \left|\mathbf{N}\right| \sum_{\left|\underline{\mathbf{h}}\right| \leq Ml_{\mathbf{N}}^2} E\left\{V_{\mathbf{N}}(\mathbf{0})\, V_{\mathbf{N}}(\mathbf{h})\right\} \qquad (2.7.20) \\ &= \left(\left|b_{\mathbf{N}}\right|^2 / (2\pi)^{4d}\right) \sum_{\left|\underline{\mathbf{h}}\right| \leq Ml_{\mathbf{N}}^2} \sum_{\left|\underline{\mathbf{h}}_1\right|,\left|\underline{\mathbf{h}}_2\right| \preceq Ml_{\mathbf{N}}} \left[\kappa_1 \cos(\mathbf{h}_1.\boldsymbol{\lambda}_1 + \mathbf{h}_2.\boldsymbol{\lambda}_2) + \kappa_2 \sin(\boldsymbol{\lambda}_1.\mathbf{h}_1 + \boldsymbol{\lambda}_2.\mathbf{h}_2)\right] \\ &\quad .w(b_{\mathbf{N}} \odot \mathbf{h}_1, b_{\mathbf{N}} \odot \mathbf{h}_2) \sum_{\left|\underline{\mathbf{h}}_3\right|,\left|\underline{\mathbf{h}}_4\right| \preceq Ml_{\mathbf{N}}} \left[\kappa_1 \cos(\mathbf{h}_3.\boldsymbol{\lambda}_1 + \mathbf{h}_4.\boldsymbol{\lambda}_2) + \kappa_2 \sin(\mathbf{h}_3.\boldsymbol{\lambda}_1 + \mathbf{h}_4.\boldsymbol{\lambda}_2)\right] \\ &\quad .w(b_{\mathbf{N}} \odot \mathbf{h}_3, b_{\mathbf{N}} \odot \mathbf{h}_4)\left[r_6(\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}, \mathbf{h} + \mathbf{h}_3, \mathbf{h} + \mathbf{h}_4) - r_3(\mathbf{h}_1, \mathbf{h}_2) r_3(\mathbf{h}_3, \mathbf{h}_4)\right], \end{aligned}$$

and this from earlier results converges absolutely uniformly in $\mathbf{N}$. Therefore provided $\left|\mathbf{N}\right| f_{\mathbf{N}}(\mathbf{0}) \neq 0$ and since $\prod_{i=1}^{d} \left|h_i\right| / M \left|l_{\mathbf{N}}\right|^2$ converges to zero, (2.7.19) tends to zero. Finally condition (f) leads to

$$E\left\{\sum_{\mathbf{1} \preceq \mathbf{t} \preceq Ml_{\mathbf{N}}^2} V_{\mathbf{N}}(\mathbf{t})\right\}^4 / M \left|\mathbf{N}\right| \left|l_{\mathbf{N}}\right|^2 \Big(\sum_{\left|\underline{\mathbf{h}}\right| \leq Ml_{\mathbf{N}}^2} C_{\mathbf{N}}(\mathbf{h})\Big)^2, \qquad (2.7.21)$$

by $(2.7.20)$, $|\mathbf{N}|\, M\, |l_{\mathbf{N}}|^2 (\sum_{|\underline{\mathbf{h}}|\leq Ml^2_{\mathbf{N}}} C_{\mathbf{N}}(\mathbf{h}))^2 \sim |\mathbf{N}|^{-1} |b_{\mathbf{N}}|^{-2}$. Define

$$D_{\mathbf{j}} = \sum_{\mathbf{t}=(\mathbf{j}-\mathbf{1})\odot d(\mathbf{N})+\mathbf{1}}^{\mathbf{j}\odot d(\mathbf{N})} V_{\mathbf{N}}(\mathbf{t}), \mathbf{1} \preceq \mathbf{j} \preceq 2\mathbf{u}_0,$$

where $2\mathbf{u}_0 \preceq Ml^2_{\mathbf{N}}/d(\mathbf{N})$, and

$$D_{2\mathbf{j}+\mathbf{1}} = \begin{cases} 0 & \text{if } \mathbf{j} = 2\mathbf{u}_0 + \mathbf{1}, , Ml^2_{\mathbf{N}} = 2\mathbf{u}_0 \odot d(\mathbf{N}), \\ \sum_{2\mathbf{u}_0\odot d(\mathbf{N})+\mathbf{1}}^{\min[Ml^2_{\mathbf{N}},(2\mathbf{u}_0+\mathbf{1})\odot d(\mathbf{N})]} V_{\mathbf{N}}(\mathbf{t}) & \text{if } Ml^2_{\mathbf{N}} \succ 2\mathbf{u}_0 \odot d(\mathbf{N}), \end{cases}$$

and

$$D_{2\mathbf{j}+\mathbf{2}} = \begin{cases} 0 & \text{if } \mathbf{j} = 2\mathbf{u}_0 + \mathbf{2}, Ml^2_{\mathbf{N}} \preceq (2\mathbf{u}_0+\mathbf{1}) \odot d(\mathbf{N}), \\ \sum_{\mathbf{1}\preceq\mathbf{t}\preceq MA_{\mathbf{N}}} V_{\mathbf{N}}(\mathbf{t}) & \text{if } Ml^2_{\mathbf{N}} \succ (2\mathbf{u}_0+\mathbf{1}) d(\mathbf{N}), \end{cases}$$

then

$$\sum_{\mathbf{1}\preceq\mathbf{t}\preceq Ml^2_{\mathbf{N}}} V_{\mathbf{N}}(\mathbf{t}) = \sum_{\mathbf{1}\preceq\mathbf{j}\preceq\mathbf{u}_0+\mathbf{1}} D_{2\mathbf{j}-\mathbf{1}} + \sum_{\mathbf{1}\preceq\mathbf{j}\preceq\mathbf{u}_0+\mathbf{1}} D_{2\mathbf{j}},$$

By Minkowski's inequality we have

$$\left[E\left\{(\sum_{\mathbf{1}\preceq\mathbf{t}\preceq Ml^2_{\mathbf{N}}} V_{\mathbf{N}}(\mathbf{t}))^4\right\}\right]^{1/4} \leq \left[E\left\{(\sum_{\mathbf{1}\preceq\mathbf{j}\preceq\mathbf{u}_0+\mathbf{1}} D_{2\mathbf{j}-\mathbf{1}})^4\right\}\right]^{1/4} + \left[E\left\{(\sum_{\mathbf{1}\preceq\mathbf{j}\preceq\mathbf{u}_0+\mathbf{1}} D_{2\mathbf{j}})^4\right\}\right]^{1/4}. \tag{2.7.22}$$

From (2.7.21) and (2.7.22) and Lemma 4 in Van Ness (1966), that condition (f) satisfied.

Lemma 2.5 states that

$$\operatorname{Re} V^{(\mathbf{k})}_{\mathbf{N}M} + i \operatorname{Im} V^{(\mathbf{k})}_{\mathbf{N}M} \stackrel{d}{\longrightarrow} X^{(\mathbf{k})}_M + iY^{(\mathbf{k})}_M \text{ as } \mathbf{N} \longrightarrow \infty,$$

where $X^{(\mathbf{k})}_M$ and $Y^{(\mathbf{k})}_M$ are jointly normal with zero mean and

$$\begin{aligned} E\left\{(X^{(\mathbf{k})}_M)^2\right\} &= \sigma^2_{\mathbf{k}MR}, \\ E\left\{(Y^{(\mathbf{k})}_M)^2\right\} &= \sigma^2_{\mathbf{k}MI}, \\ E\left\{(X^{(\mathbf{k})}_M Y^{(\mathbf{k})}_M)^2\right\} &= r_{\mathbf{k}M}, \end{aligned}$$

as $\mathbf{k} \longrightarrow \infty, \sigma^2_{\mathbf{k}MR} \to \sigma^2_{MR}, \sigma^2_{\mathbf{k}MI} \longrightarrow \sigma^2_{MI}$, and $r_{\mathbf{k}M} \longrightarrow r_M$. We will be illustrated by just one such calculation. Instead, the calculation of σ^2_{MR} for the first of the fifteen terms as listed in Table III in Rosenblatt and Van Ness (1965), we have

$$\left(\frac{|b_{\mathbf{N}}|^2}{(2\pi)^{4d}}\right) \sum_{|\underline{\mathbf{h}}_1|,\ldots,|\underline{\mathbf{h}}_4|\preceq Ml_{\mathbf{N}}} \sum_{|\underline{\mathbf{y}}|\preceq\mathbf{N}} [(|\mathbf{N}| - |\overline{\mathbf{y}}|)/|\mathbf{N}|] \cos(\mathbf{h}_1.\boldsymbol{\lambda}_1 + \mathbf{h}_2.\boldsymbol{\lambda}_2) \cos(\mathbf{h}_3.\boldsymbol{\lambda}_1 + \mathbf{h}_4.\boldsymbol{\lambda}_2)$$

$$.w(b_{\mathbf{N}} \odot \mathbf{h}_1, b_{\mathbf{N}} \odot \mathbf{h}_2) w(b_{\mathbf{N}} \odot \mathbf{h}_3, b_{\mathbf{N}} \odot \mathbf{h}_4) r(\mathbf{h}_1) r(\mathbf{y} - \mathbf{h}_2) r(\mathbf{h}_4 - \mathbf{h}_3).$$

This behave like

$$f(\mathbf{0})\frac{|b_{\mathbf{N}}|^2}{(2\pi)^{3d}}\sum_{|\underline{\mathbf{h}}_1|,\ldots,|\underline{\mathbf{h}}_4|\preceq Ml_{\mathbf{N}}}\cos(\mathbf{h}_1.\boldsymbol{\lambda}_1+\mathbf{h}_2.\boldsymbol{\lambda}_2)\cos\left[\mathbf{h}_3.(\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2)+\mathbf{h}_4.\boldsymbol{\lambda}_2\right]$$
$$w(\mathbf{0},b_{\mathbf{N}}\odot\mathbf{h}_2)w(-b_{\mathbf{N}}\odot\mathbf{h}_3,0)r(\mathbf{h}_1)r(\mathbf{h}_4),$$

using the modified continuity conditions (see Van Ness (1966)), Using trigonometric identities we have (i.e. $\mathbf{M}=[-M,M]^d$)

$$\begin{aligned}
& \frac{f(\mathbf{0})}{(2\pi)^{3d}}\sum_{|\underline{\mathbf{h}}_1|,\ldots,|\underline{\mathbf{h}}_4|\preceq Ml_{\mathbf{N}}}[\cos\mathbf{h}_1.\boldsymbol{\lambda}_1\cos\mathbf{h}_2.\boldsymbol{\lambda}_2-\sin\mathbf{h}_1.\boldsymbol{\lambda}_1\sin\mathbf{h}_2.\boldsymbol{\lambda}_2)] \\
& [\cos\mathbf{h}_3.(\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2)\cos\mathbf{h}_4.\boldsymbol{\lambda}_2-\sin\mathbf{h}_3.(\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2)\sin\mathbf{h}_4.\boldsymbol{\lambda}_2] \\
& |b_{\mathbf{N}}|^2\,w(\mathbf{0},b_{\mathbf{N}}\odot\mathbf{h}_2)w(-b_{\mathbf{N}}\odot\mathbf{h}_3,0)r(\mathbf{h}_1)r(\mathbf{h}_4) \\
\longrightarrow\ & \frac{f(\mathbf{0})}{(2\pi)^{3d}}\left(\int_{\mathbf{M}}w(\mathbf{0},\mathbf{h})d\mathbf{h}\right)^2\sum_{|\underline{\mathbf{h}}_1|,|\underline{\mathbf{h}}_2|\preceq\infty}[(\cos\mathbf{h}_1.\boldsymbol{\lambda}_1\cos\mathbf{h}_2.\boldsymbol{\lambda}_2)\,.r(\mathbf{h}_1)r(\mathbf{h}_2)\delta_{\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2}\delta_{\boldsymbol{\lambda}_2} \\
=\ & \frac{w_1}{(2\pi)^d}f(\mathbf{0})f(\boldsymbol{\lambda}_1)f(\boldsymbol{\lambda}_2)\delta_{\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2}\delta_{\boldsymbol{\lambda}_2}.
\end{aligned}$$

Proof. Theorem 2.4

Let

$$V_{\mathbf{N}}(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)=|\mathbf{N}|^{\frac{1}{2}}\,|b_{\mathbf{N}}|\left[f_{3\mathbf{N}}(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)-E\left\{f_{3\mathbf{N}}(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)\right\}\right],$$

then

$$\begin{aligned}
\operatorname{Re}V_{\mathbf{N}}(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2) &= [|b_{\mathbf{N}}|/(2\pi)^{2d}\,|\mathbf{N}|^{1/2}]\sum_{\mathbf{1}\preceq\mathbf{h}_1,\mathbf{h}_2,\mathbf{h}_3\preceq\mathbf{N}}\cos[(\mathbf{h}_2-\mathbf{h}_1).\boldsymbol{\lambda}_1+(\mathbf{h}_3-\mathbf{h}_1).\boldsymbol{\lambda}_2] \\
& \quad w(b_{\mathbf{N}}\odot(\mathbf{h}_2-\mathbf{h}_1),b_{\mathbf{N}}\odot(\mathbf{h}_3-\mathbf{h}_1))[X(\mathbf{h}_1)\,X(\mathbf{h}_2)\,X(\mathbf{h}_3)-r_3(\mathbf{h}_2-\mathbf{h}_1,\mathbf{h}_3-\mathbf{h}_1)],
\end{aligned} \tag{2.7.23}$$

and $\operatorname{Im}V_{\mathbf{N}}$ is as above except with a sine instead of cosine. Denote by $[I_1,I_2,I_3]$ the parallelepiped of indices $\{(h_i^{(1)},h_i^{(2)},h_i^{(3)})\backslash h_i^{(1)}\in I_1,h_i^{(2)}\in I_2,h_i^{(3)}\in I_3\}$ where I_1,I_2 and I_3 are intervals. Next choose vectors of sequences $\{\alpha_{\mathbf{N}}\},\{\beta_{\mathbf{N}}\}$ and $\{\gamma_{\mathbf{N}}\}$ of positive integers so that

1. $\alpha_{\mathbf{N}}\odot[\beta_{\mathbf{N}}+\gamma_{\mathbf{N}}]\sim\mathbf{N}$
2. $\alpha_{N_i},\beta_{N_i},\gamma_{N_i}\nearrow\infty,i=1,...,d$
3. $\gamma_{N_i}=o(\beta_{N_i})$

then it will be shown that we can replace the sum $\sum_{\mathbf{1}\preceq\mathbf{h}_1,\mathbf{h}_2,\mathbf{h}_3\preceq\mathbf{N}}$ in (2.7.23) by

$$\sum_{\mathbf{j}=\mathbf{1}}^{\alpha_{\mathbf{N}}}\sum_{\prod_{i=1}^{d}\left[B_{j_i}^{(N_i)},B_{j_i}^{(N_i)},B_{j_i}^{(N_i)}\right]}, \tag{2.7.24}$$

where $B_{j_i}^{(N_i)}=[(j_i-1)(b_{N_i}+\gamma_{N_i})+1, j_i(b_{N_i}+\gamma_{N_i})-\gamma_{N_i}], j_i=1,...,\alpha_{N_i}, i=1,...,d$ and still get the same asymptotic distribution. Having done this, the sum (2.7.24) will be shown to asymptotically normally distributed. Further, the domain summation- hypercube such that the main diagonal of a cube with sides parallel to the x, y and z axes and of length N_i-1, runs from the point $(1,1,1)$ to (N_i,N_i,N_i). Then the sum (2.7.24) is over α_{N_i} smaller cubes whose main diagonals lie on the above diagonal and whose sides are the length $b_{N_i}-1$ and are parallel to chose of the large cube. These smaller cubes are separated by a distance $\sim\gamma_{N_i}$.

We begin the first step by noting that by the properties of w and the summability of the cumulants

$$\begin{aligned} cov[V_{\mathbf{N}}(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2),V_{\mathbf{N}}(\boldsymbol{\lambda}_3,\boldsymbol{\lambda}_4)] &= [|b_{\mathbf{N}}|^2/(2\pi)^{4d}]\sum_{|\underline{\mathbf{h}}_1|,...,|\underline{\mathbf{h}}_4|,|\underline{\mathbf{y}}|\preceq\mathbf{N}}|\mathbf{N}|^{-1}C_{\mathbf{N}}(\mathbf{h}_1,...,\mathbf{h}_4,\mathbf{y}) \\ & e^{-i(\sum_{i=1}^{2}\mathbf{h}_i.\boldsymbol{\lambda}_i-\sum_{i=3}^{4}\mathbf{h}_i.\boldsymbol{\lambda}_i)}.w(b_{\mathbf{N}}\odot\mathbf{h}_1,b_{\mathbf{N}}\odot\mathbf{h}_2)w(b_{\mathbf{N}}\odot\mathbf{h}_3,b_{\mathbf{N}}\odot\mathbf{h}_4) \\ & .\{m_2(\mathbf{0},\mathbf{h}_1)m_2(\mathbf{h}_2,\mathbf{y})m_2(\mathbf{y}+\mathbf{h}_3,\mathbf{y}+\mathbf{h}_4)\}_{15}+O(|b_{\mathbf{N}}|), \end{aligned}$$

where $C_{\mathbf{N}}(\mathbf{h}_1,...,\mathbf{h}_4,\mathbf{y})=\prod_{i=1}^{d}C_{N_i}(h_i^{(1)},...,h_i^{(4)},y)$, and $C_{N_i}(h_i^{(1)},...,h_i^{(4)},y)$ is defined similarly that in Rosenblatt and Van Ness (1965). Also, $0\le C_{N_i}/N_i\le 1, i=1,...,d$ and $|\mathbf{N}|^{-1}C_{\mathbf{N}}(.)\longrightarrow 1$ as $\mathbf{N}\longrightarrow\infty$. The fifteen terms which sum to give the expression $\{m_2(\mathbf{0},\mathbf{h}_1)m_2(\mathbf{h}_2,\mathbf{y})m_2(\mathbf{y}+\mathbf{h}_3,\mathbf{y}+\mathbf{h}_4)\}_{15}$ is given similarly in table III in Rosenblatt and Van Ness (1965).

Lemma 2.6 *If the hypothesis of Theorem 2.4 hold*

i) $\gamma_{\mathbf{N}}\odot b_{\mathbf{N}}\longrightarrow\infty$ *as* $\mathbf{N}\longrightarrow\infty$,

ii) $\frac{\alpha_{\mathbf{N}}^2\odot\gamma_{\mathbf{N}}}{\mathbf{N}}\longrightarrow 0$ *as* $\mathbf{N}\longrightarrow\infty$,

Then

$$\begin{aligned} & \sigma^2[|b_{\mathbf{N}}|^2/(2\pi)^{2d}|\mathbf{N}|^{1/2}]\Big(\sum_{|\underline{\mathbf{h}}_1|,...,|\underline{\mathbf{h}}_3|\preceq\mathbf{N}}-\sum_{\mathbf{j}=\mathbf{1}}^{\alpha_{\mathbf{N}}}\sum_{\prod_{i=1}^{d}\left[B_{j_i}^{(N_i)},B_{j_i}^{(N_i)},B_{j_i}^{(N_i)}\right]}\Big)e^{-i((\mathbf{h}_2-\mathbf{h}_1)\boldsymbol{\mu}_1+(\mathbf{h}_3-\mathbf{h}_1)\boldsymbol{\mu}_2)} \\ & .w(b_{\mathbf{N}}\odot(\mathbf{h}_2-\mathbf{h}_1),b_{\mathbf{N}}\odot(\mathbf{h}_3-\mathbf{h}_1))[X(\mathbf{h}_1)X(\mathbf{h}_2)X(\mathbf{h}_3)-r_3(\mathbf{h}_2-\mathbf{h}_1,\mathbf{h}_3-\mathbf{h}_1)] \\ \longrightarrow\ & 0 \text{ as } \mathbf{N}\longrightarrow\infty. \end{aligned}$$

Proof. To proof this Lemma we use Lemma 1 in Van Ness (1966).

By lemma 2.6, it remains to be shown that the sums of the form (2.7.24) tend to a complex normal distribution in distribution. To do this define

$$\begin{aligned} U_j^{(\mathbf{N})} &= \left(|b_N| / (2\pi)^{2d}\right) \sum_{\prod_{i=1}^{d}\left[B_{j_i}^{(N_i)}, B_{j_i}^{(N_i)}, B_{j_i}^{(N_i)}\right]} \{\kappa_1 \cos[(\mathbf{h}_2 - \mathbf{h}_1)\boldsymbol{\mu}_1 + (\mathbf{h}_3 - \mathbf{h}_1)\boldsymbol{\mu}_2] \\ &+ \kappa_2 \sin[(\mathbf{h}_2 - \mathbf{h}_1)\boldsymbol{\mu}_1 + (\mathbf{h}_3 - \mathbf{h}_1)\boldsymbol{\mu}_2]\} \\ &.w((\mathbf{h}_2 - \mathbf{h}_1) \odot b_{\mathbf{N}}, (\mathbf{h}_3 - \mathbf{h}_1) \odot b_{\mathbf{N}}).[X(\mathbf{h}_1) X(\mathbf{h}_2) X(\mathbf{h}_3) - r_3(\mathbf{h}_2 - \mathbf{h}_1, \mathbf{h}_3 - \mathbf{h}_1)], \end{aligned} \tag{2.7.25}$$

where κ_1 and κ_2 are any two real parameters. By previous results we known that since $b_{\mathbf{N}} \odot \beta_{\mathbf{N}} \longrightarrow \infty$,

$$\lim_{\mathbf{N} \longrightarrow \infty} var(U_{\mathbf{j}}^{(\mathbf{N})} / |\beta_{\mathbf{N}}|^{1/2}) = \sigma_{\boldsymbol{\lambda}}^2,$$

for $\sigma_{\boldsymbol{\lambda}}^2 = \kappa_1 \sigma_R^2 + \kappa_2 \sigma_I^2$ where σ_R^2 and σ_I^2 are defined as the variances of the real and imaginary parts. Then we show that

$$\sum_{\mathbf{r}=\mathbf{1}}^{\alpha_{\mathbf{N}}} U_{\mathbf{r}}^{(\mathbf{N})} / |\mathbf{N}|^{1/2} \sigma_{\boldsymbol{\lambda}} \longrightarrow \sum_{\mathbf{r}=\mathbf{1}}^{\alpha_{\mathbf{N}}} U_{\mathbf{r}}^{(\mathbf{N})} / (|\alpha_{\mathbf{N}}| |\beta_{\mathbf{N}}|)^{1/2} \sigma_{\boldsymbol{\lambda}}.$$

Set

$$G_{\mathbf{r},\mathbf{N}}(x) = P\{U_{\mathbf{r}}^{(\mathbf{N})} / (|\alpha_{\mathbf{N}}| |\beta_{\mathbf{N}}|)^{1/2} \sigma_{\boldsymbol{\lambda}} \leq x\},$$

we see that the distribution we are interested in tends to the convolution

$$G_{\mathbf{1},\mathbf{N}} * \dots * G_{\alpha_{\mathbf{N}},\mathbf{N}}(x)$$

which tend to $\mathcal{N}(0,1)$ (see Rosenblatt (1985)).

Proof. Theorem 2.5

Under the condition of Theorem 2.5, we can shown that

$$\begin{aligned} & (2\pi)^{3d}[f_{\mathbf{4}}(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) - E\{f_{4\mathbf{N}}(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3)\}] \\ = & \sum_{|\underline{\mathbf{h}}_1|, |\underline{\mathbf{h}}_2|, |\underline{\mathbf{h}}_3| \preceq l_{\mathbf{N}}} C_4(\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3)\{w_{\mathbf{N}}(\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3) - 1\} e^{-i\sum_{j=1}^{3} \mathbf{h}_j . \boldsymbol{\lambda}_j} \\ & + \sum_{|\underline{\mathbf{h}}_1|, |\underline{\mathbf{h}}_2|, |\underline{\mathbf{h}}_3| \succ l_{\mathbf{N}}} C_4(\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3) e^{-i\sum_{j=1}^{3} \mathbf{h}_j . \boldsymbol{\lambda}_j} + O\left(|\mathbf{N}^*|^{-1} |b_{\mathbf{N}}|^{-2}\right) \\ \longrightarrow & \ 0 \text{ as } \mathbf{N} \longrightarrow \infty. \end{aligned}$$

From lemmas 2.4 and 2.6, as well as (2.5.9) and (2.5.12). Moreover, by computing $\upsilon_B(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3, \alpha, \beta)$, we can show that $\upsilon_B(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) > 0$ if $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3)$ have submanifolds and that $\upsilon_B(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) = 0$ if $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3)$ have no submanifolds.

Proof. Theorem 2.6

From (2.5.3) we have

$$f_{4\mathbf{N}}(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_3)=h_{\mathbf{N}}(\underline{\boldsymbol{\lambda}})-h_{\mathbf{N}}(\boldsymbol{\lambda}_1)g_{\mathbf{N}}(\boldsymbol{\lambda}_3,\boldsymbol{\lambda}_2)-h_{\mathbf{N}}(\boldsymbol{\lambda}_2)g_{\mathbf{N}}(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_3)-h_{\mathbf{N}}(\boldsymbol{\lambda}_3)g_{\mathbf{N}}(\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_1),$$

where $\underline{\boldsymbol{\lambda}}=(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_3)$

$$\begin{aligned} h_{\mathbf{N}}(\underline{\boldsymbol{\lambda}}) &= \frac{1}{(2\pi)^{3d}}\sum_{\mathbf{h}_1,\mathbf{h}_2,\mathbf{h}_3\in\mathbb{Z}^d}\widehat{r}_4(\mathbf{h}_1,\mathbf{h}_2,\mathbf{h}_3)w_{\mathbf{N}}(\mathbf{h}_1,\mathbf{h}_2,\mathbf{h}_3)e^{-i\sum_{j=1}^3\mathbf{h}_j.\boldsymbol{\lambda}_j},\\ h_{\mathbf{N}}(\boldsymbol{\lambda}_i) &= \frac{1}{(2\pi)^{3d}}\sum_{\mathbf{h}_i\in\mathbb{Z}^d}\widehat{r}_2(\mathbf{h}_i)w_{\mathbf{N}}(\mathbf{h}_1,\mathbf{h}_2,\mathbf{h}_3)e^{-i\sum_{j=1}^3\mathbf{h}_j.\boldsymbol{\lambda}_j},\ i=1,2,3,\end{aligned}$$

$$g_{\mathbf{N}}(\boldsymbol{\lambda}_i,\boldsymbol{\lambda}_j)=\frac{1}{(2\pi)^{3d}}\sum_{\mathbf{h}_i,\mathbf{h}_j\in\mathbb{Z}^d}\widehat{r}_2(\mathbf{h}_i-\mathbf{h}_j)w_{\mathbf{N}}(\mathbf{h}_1,\mathbf{h}_2,\mathbf{h}_3)e^{-i\sum_{j=1}^3\mathbf{h}_j.\boldsymbol{\lambda}_j},\ i,j=1,2,3,$$

we will further let

$$\begin{aligned} \widetilde{h}_{\mathbf{N}}(\underline{\boldsymbol{\lambda}}) &= h_{\mathbf{N}}(\underline{\boldsymbol{\lambda}})-\overline{h}_{\mathbf{N}}(\underline{\boldsymbol{\lambda}})=h_{\mathbf{N}}(\underline{\boldsymbol{\lambda}})-E\left\{h_{\mathbf{N}}(\underline{\boldsymbol{\lambda}})\right\}\ \text{(respectively for } h_{\mathbf{N}}(\boldsymbol{\lambda}_i)),\\ \widetilde{g}_{\mathbf{N}}(\boldsymbol{\lambda}_i,\boldsymbol{\lambda}_j) &= g_{\mathbf{N}}(\boldsymbol{\lambda}_i,\boldsymbol{\lambda}_j)-\overline{g}_{\mathbf{N}}(\boldsymbol{\lambda}_i,\boldsymbol{\lambda}_j)=g_{\mathbf{N}}(\boldsymbol{\lambda}_i,\boldsymbol{\lambda}_j)-E\left\{g_{\mathbf{N}}(\boldsymbol{\lambda}_i,\boldsymbol{\lambda}_j)\right\},i,j=1,2,3.\end{aligned}$$

Then

$$\begin{aligned} f_{4\mathbf{N}}(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_3) &= \widetilde{h}_{\mathbf{N}}(\underline{\boldsymbol{\lambda}})-\widetilde{h}_{\mathbf{N}}(\boldsymbol{\lambda}_1)\overline{g}_{\mathbf{N}}(\boldsymbol{\lambda}_3,\boldsymbol{\lambda}_2)-\widetilde{h}_{\mathbf{N}}(\boldsymbol{\lambda}_2)\overline{g}_{\mathbf{N}}(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_3)-\widetilde{h}_{\mathbf{N}}(\boldsymbol{\lambda}_3)\overline{g}_{\mathbf{N}}(\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_1) \quad (2.7.26)\\ &\quad -\overline{h}_{\mathbf{N}}(\boldsymbol{\lambda}_1)\widetilde{g}_{\mathbf{N}}(\boldsymbol{\lambda}_3,\boldsymbol{\lambda}_2)-\overline{h}_{\mathbf{N}}(\boldsymbol{\lambda}_2)\widetilde{g}_{\mathbf{N}}(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_3)-\overline{h}_{\mathbf{N}}(\boldsymbol{\lambda}_3)\widetilde{g}_{\mathbf{N}}(\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_1)-D1+D2,\end{aligned}$$

with

$$\begin{aligned} D1 &= \widetilde{h}_{\mathbf{N}}(\boldsymbol{\lambda}_1)\widetilde{g}_{\mathbf{N}}(\boldsymbol{\lambda}_3,\boldsymbol{\lambda}_2)+\widetilde{h}_{\mathbf{N}}(\boldsymbol{\lambda}_2)\widetilde{g}_{\mathbf{N}}(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_3)+\widetilde{h}_{\mathbf{N}}(\boldsymbol{\lambda}_3)\widetilde{g}_{\mathbf{N}}(\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_1),\\ D2 &= \widetilde{h}_{\mathbf{N}}(\underline{\boldsymbol{\lambda}})-\overline{h}_{\mathbf{N}}(\boldsymbol{\lambda}_1)\overline{g}_{\mathbf{N}}(\boldsymbol{\lambda}_3,\boldsymbol{\lambda}_2)-\overline{h}_{\mathbf{N}}(\boldsymbol{\lambda}_2)\overline{g}_{\mathbf{N}}(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_3)-\overline{h}_{\mathbf{N}}(\boldsymbol{\lambda}_3)\overline{g}_{\mathbf{N}}(\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_1).\end{aligned}$$

We can prove that in general (see Lii and Rosenblatt (1990)), for $p\geq 2$

$$\overline{h}_{\mathbf{N}}(\underline{\boldsymbol{\lambda}})=\begin{cases} O(|b_{\mathbf{N}}|^{-p/2+1}) \text{ if } p \text{ even},\\ O(|b_{\mathbf{N}}|^{-(p-3)/2}) \text{ if } p \text{ odd},\end{cases} \tag{2.7.27}$$

$$var\left(h_{\mathbf{N}}(\underline{\boldsymbol{\lambda}})\right)=O\left(\left(|\mathbf{N}|\,|b_{\mathbf{N}}|^{p-1}\right)^{-1}\right),$$

respectively for $\overline{h}_{\mathbf{N}}(\boldsymbol{\lambda}_i)$, and for $p\geq 3$, $i,j=1,2,3,$

$$\overline{g}_{\mathbf{N}}(\boldsymbol{\lambda}_i,\boldsymbol{\lambda}_j)=\begin{cases} O(|b_{\mathbf{N}}|^{-p/2+1}) \text{ if } p \text{ even},\\ O(|b_{\mathbf{N}}|^{(-p+1)/2}) \text{ if } p \text{ odd},\end{cases} \tag{2.7.28}$$

$$var\left\{g_{\mathbf{N}}(\boldsymbol{\lambda}_i,\boldsymbol{\lambda}_j)\right\} = O\left((|\mathbf{N}|\,|b_{\mathbf{N}}|^p)^{-1}\right).$$

Therefore, in (2.7.26) we have

$$\begin{aligned} var\left\{\widetilde{h}_{\mathbf{N}}(\underline{\boldsymbol{\lambda}})\right\} &= O\left(\left(|\mathbf{N}|\,|b_{\mathbf{N}}|^3\right)^{-1}\right), \\ var\left\{\widetilde{h}_{\mathbf{N}}(\boldsymbol{\lambda}_i)\right\} &= O\left((|\mathbf{N}|\,|b_{\mathbf{N}}|)^{-1}\right), \\ \overline{h}_{\mathbf{N}}(\boldsymbol{\lambda}_i) &= O(1), \end{aligned} \qquad (2.7.29)$$

$$\begin{aligned} var\left\{\widetilde{g}_{\mathbf{N}}(\boldsymbol{\lambda}_i,\boldsymbol{\lambda}_j)\right\} &= O\left(\left(|\mathbf{N}|\,|b_{\mathbf{N}}|^3\right)^{-1}\right), \\ \overline{g}_{\mathbf{N}}(\boldsymbol{\lambda}_i,\boldsymbol{\lambda}_j) &= O\left(|b_{\mathbf{N}}|^{-1}\right). \end{aligned}$$

Hence we see that in (2.7.26) the magnitude of the first seven terms are $\left(|\mathbf{N}|\,|b_{\mathbf{N}}|^3\right)^{-1/2}$ each while the magnitude of $D1$ is bounded in probability by

$$\left[(|\mathbf{N}|\,|b_{\mathbf{N}}|)^{-1}\left(|\mathbf{N}|\,|b_{\mathbf{N}}|^3\right)^{-1}\right]^{1/2} = \left(|\mathbf{N}|\,|b_{\mathbf{N}}|^2\right)^{-1} = o\left(\left(|\mathbf{N}|\,|b_{\mathbf{N}}|^3\right)^{-1}\right).$$

The nonrandom part $D2$ is approximately $E\left\{f_{4\mathbf{N}}(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_3)\right\}$ because

$$\begin{aligned} \left|E\left\{f_{4\mathbf{N}}(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_3)\right\} - D2\right| &\le \Big|E\left\{h_{\mathbf{N}}(\boldsymbol{\lambda}_1)g_{\mathbf{N}}(\boldsymbol{\lambda}_3,\boldsymbol{\lambda}_2)\right\} - \overline{h}_{\mathbf{N}}(\boldsymbol{\lambda}_1)\overline{g}_{\mathbf{N}}(\boldsymbol{\lambda}_3,\boldsymbol{\lambda}_2) + E\left\{h_{\mathbf{N}}(\boldsymbol{\lambda}_2)g_{\mathbf{N}}(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_3)\right\} \\ &\quad - \overline{h}_{\mathbf{N}}(\boldsymbol{\lambda}_2)\overline{g}_{\mathbf{N}}(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_3) + E\left\{h_{\mathbf{N}}(\boldsymbol{\lambda}_3)g_{\mathbf{N}}(\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_1)\right\} - \overline{h}_{\mathbf{N}}(\boldsymbol{\lambda}_3)\overline{g}_{\mathbf{N}}(\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_1)\Big| \\ &\le \left|cov\left\{h_{\mathbf{N}}(\boldsymbol{\lambda}_1), g_{\mathbf{N}}(\boldsymbol{\lambda}_3,\boldsymbol{\lambda}_2)\right\}\right| + \left|cov\left\{h_{\mathbf{N}}(\boldsymbol{\lambda}_2), g_{\mathbf{N}}(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_3)\right\}\right| \\ &\quad + \left|cov\left\{h_{\mathbf{N}}(\boldsymbol{\lambda}_3), g_{\mathbf{N}}(\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_1)\right\}\right| \\ &= O\left(|\mathbf{N}|\,|b_{\mathbf{N}}|^2\right)^{-1} \\ &= o\left(\left(|\mathbf{N}|\,|b_{\mathbf{N}}|^3\right)^{-1}\right). \end{aligned}$$

We have asymptotically,

$$\begin{aligned} f_{4\mathbf{N}}(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_3) - E\left\{f_{4\mathbf{N}}(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_3)\right\} &\cong \widetilde{h}_{\mathbf{N}}(\underline{\boldsymbol{\lambda}}) - \widetilde{h}_{\mathbf{N}}(\boldsymbol{\lambda}_1)\overline{g}_{\mathbf{N}}(\boldsymbol{\lambda}_3,\boldsymbol{\lambda}_2) - \widetilde{h}_{\mathbf{N}}(\boldsymbol{\lambda}_2)\overline{g}_{\mathbf{N}}(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_3) \\ &\quad - \widetilde{h}_{\mathbf{N}}(\boldsymbol{\lambda}_3)\overline{g}_{\mathbf{N}}(\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_1) - \overline{h}_{\mathbf{N}}(\boldsymbol{\lambda}_1)\widetilde{g}_{\mathbf{N}}(\boldsymbol{\lambda}_3,\boldsymbol{\lambda}_2) \\ &\quad - \overline{h}_{\mathbf{N}}(\boldsymbol{\lambda}_2)\widetilde{g}_{\mathbf{N}}(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_3) - \overline{h}_{\mathbf{N}}(\boldsymbol{\lambda}_3)\widetilde{g}_{\mathbf{N}}(\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_1), \end{aligned}$$

we will show from Theorem 6 p 156 in Rosenblatt (1985), that any fixed finite linear combinations of terms of the form $\widetilde{h}_{\mathbf{N}}(\underline{\boldsymbol{\lambda}})$, $\widetilde{h}_{\mathbf{N}}(\boldsymbol{\lambda}_.)$ and $\widetilde{g}_{\mathbf{N}}(\boldsymbol{\lambda}_., \boldsymbol{\lambda}_.)$ with different weight functions, frequencies, and real and imaginary parts is asymptotically normal with proper normalization, and the exact form of the covariance is obtained in the same manner as Theorem 2 in Li and Rosenblatt (1990).

Proof. Theorem 2.7

Consider that the weight function $w(\mathbf{x})$ is a linear combination of product of functions of one field, from (2.2.2) we have

$$f_{p\mathbf{N}}(\boldsymbol{\lambda}_1, ..., \boldsymbol{\lambda}_{p-1}) = \frac{1}{(2\pi)^{d(p-1)}} \sum_{|\underline{\mathbf{h}}_j| \preceq l_{\mathbf{N}}} \left[\sum_{\nu} (-1)^{m(\nu)-1} [m(\nu)-1]! \prod_{j=1}^{m(\nu)} \widehat{r} \left(\prod_{\mathbf{h}_u \in \nu_j} X(\mathbf{t} + \mathbf{h}_u) \right) \right] \times \prod_{j=1}^{p-1} w_j(b_{\mathbf{N}} \odot \mathbf{h}_j) e^{-i \sum_{j=1}^{p-1} \mathbf{h}_j . \boldsymbol{\lambda}_j}, \quad (2.7.30)$$

where

$$\widehat{r} \left(\prod_{\mathbf{h}_u \in \nu_j} X(\mathbf{t} + \mathbf{h}_u) \right) = |\mathbf{N} - 2l_{\mathbf{N}}|^{-1} \sum_{\mathbf{t}=l_{\mathbf{N}}+\mathbf{1}}^{\mathbf{N}-l_{\mathbf{N}}} \left(\prod_{\mathbf{h}_u \in \nu_j} X(\mathbf{t} + \mathbf{h}_u) \right), |\underline{\mathbf{h}}_u| \preceq l_{\mathbf{N}}.$$

$$\begin{aligned} f_{p\mathbf{N}}(\boldsymbol{\lambda}_1, ..., \boldsymbol{\lambda}_{p-1}) &= \sum_{\nu} (-1)^{m(\nu)-1} [m(\nu)-1]! \\ &\times \left\{ \sum_{|\underline{\mathbf{h}}_j| \preceq l_{\mathbf{N}}} \prod_{j=1}^{m(\nu)} \left[\frac{1}{(2\pi)^{dm(\nu_j)}} \widehat{r} \left(\prod_{\mathbf{h}_u \in \nu_j} X(\mathbf{t} + \mathbf{h}_u) \right) \times \prod_u \left(w_u(b_{\mathbf{N}} \odot \mathbf{h}_u) e^{-i\mathbf{h}_u . \boldsymbol{\lambda}_u} \right) \right] \right\} \\ &= \sum_{\nu} (-1)^{m(\nu)-1} [m(\nu)-1]! \sum_{j=1}^{m(\nu)} g_{\nu_j}(\boldsymbol{\lambda}), \end{aligned} \quad (2.7.31)$$

where $w_j \equiv 1, \boldsymbol{\lambda}_0 = \mathbf{h}_0 = \mathbf{0}, (2\pi)^{dm(\nu_j)} = (2\pi)^{dk}, m(\nu_j)$ is the number of nonzero elements in ν_j, and

$$g_{\nu_j}(\boldsymbol{\lambda}) = \frac{1}{(2\pi)^{dm(\nu_j)}} \sum_{|\underline{\mathbf{h}}_u| \preceq l_{\mathbf{N}}} \widehat{r} \left(\prod_{\mathbf{h}_u \in \nu_j} X(\mathbf{t} + \mathbf{h}_u) \right) \prod_u \left[w_u(b_{\mathbf{N}} \odot \mathbf{h}_u) e^{-i\mathbf{h}_u . \boldsymbol{\lambda}_u} \right],$$

equation (2.7.31) is the generalization of (4.4.2). Just as in (4.4.2), we write

$$f_{p\mathbf{N}}(\boldsymbol{\lambda}_1, ..., \boldsymbol{\lambda}_{p-1}) = \sum_{\nu} (-1)^{m(\nu)-1} [m(\nu)-1]! \sum_{j=1}^{m(\nu)} \left[\widetilde{g}_{\nu_j}(\boldsymbol{\lambda}) + \overline{g}_{\nu_j}(\boldsymbol{\lambda}) \right], \quad (2.7.32)$$

where $\widetilde{g}_{\nu_j}(\boldsymbol{\lambda}) = g_{\nu_j}(\boldsymbol{\lambda}) - E\{g_{\nu_j}(\boldsymbol{\lambda})\}$ and $\overline{g}_{\nu_j}(\boldsymbol{\lambda}) = E\{g_{\nu_j}(\boldsymbol{\lambda})\}$.

In the expansion of the product in (2.7.32), consider a generic term that has $r_1\widetilde{g}$'s and $r_2\overline{g}$'s with $r_1 + r_2 = m(\nu)$. Without loss of generality we consider the partition $\nu = \{\nu_1, ..., \nu_{r_1}; \nu_{r_1+1}, ..., \nu_{r_1+r_2}\}$ such that $m(\nu_i) = m_i, i = 1, ..., r_1$ correspond to $r_1\widetilde{g}$'s, $m(\nu_{r_1+i}) = k_i, i = 1, ..., r_2$ correspond to

$r_2 \overline{g}$'s with $\sum m_i + \sum k_i = p$. If 0 is in one of the ν_i's, $i = 1, ..., r_1$, say ν_1, then the magnitude of the variance of this term is, from (2.7.27) and (2.7.28)

$$\begin{aligned} & \frac{1}{|\mathbf{N}| \, |b_{\mathbf{N}}|^{m_1 - 1}} \prod_{j=2}^{r_1} \left(\frac{1}{|\mathbf{N}| \, |b_{\mathbf{N}}|^{m_j + 1}} \right) \prod_{j=1}^{r_2} \left(\frac{1}{|b_{\mathbf{N}}|^{[k_j/2]}} \right) \\ = \; & |\mathbf{N}|^{-r_1} \, |b_{\mathbf{N}}|^{-1 + \left[m_1 + ... + m_{r_1} - 1 + r_1 - 1 \right]} \, |b_{\mathbf{N}}|^{-2 \left[[k_1/2] + ... + [k_{r_2}/2] \right]} \\ \leq \; & |\mathbf{N}|^{-r_1} \, |b_{\mathbf{N}}|^{-p+1-r_1}, \end{aligned}$$

where $[x]$ is the largest integer less than or equal to x. We know that from (2.7.29) that the order of magnitude of the variance when ν has only one term is $\left(|\mathbf{N}| \, |b_{\mathbf{N}}|^{p-1} \right)^{-1}$. Hence

$$\frac{|\mathbf{N}|^{-r_1} \, |b_{\mathbf{N}}|^{-p+1-r_1}}{|\mathbf{N}|^{-1} \, |b_{\mathbf{N}}|^{-p+1}} = |\mathbf{N}|^{1-r_1} \, |b_{\mathbf{N}}|^{-r_1} \longrightarrow 0.$$

If $r_1 \geq 2$, similarly if 0 is in one of the ν_{r_1+i}'s, $i = 1, ..., r_2$, say ν_{r_1+1}, then the magnitude of the variance of this term is

$$\begin{aligned} & \prod_{j=1}^{r_1} \left(\frac{1}{|\mathbf{N}| \, |b_{\mathbf{N}}|^{m_j+1}} \right) \left(\frac{1}{|b_{\mathbf{N}}|^{[q1/2]-1}} \right)^2 \prod_{j=2}^{r_2} \left(\frac{1}{|b_{\mathbf{N}}|^{[q_2/2]}} \right)^2 \\ \leq \; & |\mathbf{N}|^{-r_1} \, |b_{\mathbf{N}}|^{-p-r_1+2}, \end{aligned}$$

and if $r_1 > 1$

$$\frac{|\mathbf{N}|^{-r_1} \, |b_{\mathbf{N}}|^{-p-r_1+2}}{|\mathbf{N}|^{-1} \, |b_{\mathbf{N}}|^{-p+1}} = |\mathbf{N}|^{1-r_1} \, |b_{\mathbf{N}}|^{-r_1+1} \longrightarrow 0.$$

Hence we only need to consider those terms when $r_1 = 1$. Therefore, in terms of the random part, (2.7.32) can be written as

$$f_{p\mathbf{N}}(\boldsymbol{\lambda}_1, ..., \boldsymbol{\lambda}_{p-1}) = \sum_{\nu} (-1)^{m(\nu)-1} \, [m(\nu) - 1]! \left(\sum_{j=1}^{m(\nu)} \widetilde{g}_{\nu_j}(\boldsymbol{\lambda}) \prod_{\substack{l \neq j \\ l=1}}^{m(\nu)} \overline{g}_{\nu_l}(\boldsymbol{\lambda}) \right).$$

An argument similar to that given in Theorem 6 p 156 in Rosenblatt (1985), we define

$$Y_{\mathbf{t}}^{(\mathbf{N})} = \sum_{u_j, p_j} \sum_{j=1}^{m(\nu)} \delta_{p_j} \, |b_{\mathbf{N}}|^{p_j^*/2} \, \widetilde{g}_{\nu_j}(\boldsymbol{\lambda}^{(u_j)}, w^{(p_j)}),$$

where p_j^* is the exponent of $b_{\mathbf{N}}$ in variance of S_2 and S_3 according to whether 0 is in ν_j or not, and

$$\begin{aligned}\overline{g}_{\nu_j}(\boldsymbol{\lambda}^{(u_j)}, \boldsymbol{\omega}^{(p_j)}) &= \frac{1}{(2\pi)^{dm(\nu_j)}} \sum_{|\underline{\mathbf{h}}_u| \preceq l_{\mathbf{N}}} \left[\prod_{\mathbf{h}_u \in \nu_j} X\left(\mathbf{t}_j + \mathbf{h}_u^{(j)}\right) - E \prod_{\mathbf{h}_u \in \nu_j} X\left(\mathbf{t}_j + \mathbf{h}_u^{(j)}\right) \right] \\ &\quad \times \prod_u w_u^{(p_j)}\left(b_{\mathbf{N}} \odot \mathbf{h}_u^{(j)}\right) \left[\alpha_{u_j} \cos\left(\sum_{\mathbf{h}_u \in \nu_j} \mathbf{h}_u^{(j)}.\boldsymbol{\lambda}_u^{(j)} \right) + \beta_{u_j} \sin\left(\sum_{\mathbf{h}_u \in \nu_j} \mathbf{h}_u^{(j)}.\boldsymbol{\lambda}_u^{(j)} \right) \right],\end{aligned}$$

which asymptotically normal with proper normalization.

Chapter 3

Non-Gaussian estimation

3.1 Introduction

The methods of parameter estimation which are the Gaussian estimates are usually based on either the covariances as Yule-Walker equation or the spectrum (see Rosenblatt (1985)). The idea of non Gaussian estimation for random field by using not only the spectrum but the bispectrum as well is readily extendible from times series analysis (see Terdik (2000)). In this chapter, we consider a functional of the spectrum and the bispectrum for random fields depending on an unknown parameter θ, and we give explicit expression for the asymptotic variance of this estimator who calculated for both the case when the spectra are estimated by the peridogram and by the smoothed periodogram. The consistency and asymptotic normality are proved.

3.2 Estimating a parameter for non-Gaussian random fields

Let $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ be a weakly stationary random field with a zero mean and finite $p-th$ order moments $(p \geq 2)$ on $\mathbb{Z}^d$. we shall assume the following conditions.

Condition 3.1 *Besides the stationarity of p^{th} order, we shall assume that cumulant function of $p - th$ order of the random field depends on a real unknown parameter and*

$$\sum_{\mathbf{h}_1,\dots,\mathbf{h}_{p-1}}^{\infty} (1 + |\mathbf{h}_j|)\, |C_P(\mathbf{h}_1, \dots, \mathbf{h}_{p-1}, \theta)| < \infty, j = 1, 2, \dots, p-1.$$

Thus the spectral densities

$$S_p(\boldsymbol{\omega}_1,...,\boldsymbol{\omega}_{p-1},\theta)=\frac{1}{(2\pi)^{d(p-1)}}\sum_{\mathbf{h}_1\in\mathbb{Z}^d}...\sum_{\mathbf{h}_{p-1}\in\mathbb{Z}^d}C_p(\mathbf{h}_1,...,\mathbf{h}_{p-1},\theta)e^{-i\sum_{j=1}^{p-1}\mathbf{h}_j\boldsymbol{\omega}_j},$$

exist up to p^{th} order, where $\boldsymbol{\omega}=(\omega_1,...,\omega_d)\in\boldsymbol{\pi}=[-\pi,\pi[\times...\times[-\pi,\pi[$, $d-$times.

Condition 3.2 *The unknown parameter θ belongs to a compact set $\Theta\subset\mathbb{R}^d$. Suppose also that the spectrum $S_2(\boldsymbol{\omega},\theta)$ and the bispectrum $S_3(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\theta)$ are twice continuously differentiable with respect to $\theta\in\Theta$. These derivatives are continuous and bounded above and suppose further that they are bounded away from zero in modulus with respect to the frequencies of the sets Λ_1^d and Λ_2^d to be defined below.*

From a finite realization $\{X(\mathbf{t}),\mathbf{t}=\mathbf{1},...,\mathbf{N}\}$, we can write

$$\begin{aligned}I_{2\mathbf{N}}(\boldsymbol{\omega})&=(2\pi)^{-d}|\mathbf{N}|^{-1}d_{\mathbf{N}}(\boldsymbol{\omega})\overline{d_{\mathbf{N}}(\boldsymbol{\omega})},\\ I_{3\mathbf{N}}(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2)&=(2\pi)^{-2d}|\mathbf{N}|^{-1}d_{\mathbf{N}}(\boldsymbol{\omega}_1)d_{\mathbf{N}}(\boldsymbol{\omega}_2)\overline{d_{\mathbf{N}}(\boldsymbol{\omega}_1+\boldsymbol{\omega}_2)},\end{aligned}$$

at the standard Fourier frequencies, where

$$d_{\mathbf{N}}(\boldsymbol{\omega})=\sum_{\mathbf{t}=\mathbf{1}}^{\mathbf{N}}X_{\mathbf{t}}e^{-i\mathbf{t}.\boldsymbol{\omega}},\ \mathbf{t}.\boldsymbol{\omega}=\sum_{i=1}^{d}t_i\omega_i,$$

is the (finite) Fourier transform of the data.

We are going to apply some well known methods for the estimation of the spectral densities. We shall deal with the discrete Fourier frequencies $\mu_{\mathbf{k}}=(\frac{k_1}{N_1},...,\frac{k_d}{N_d})$,i.e. $k_i=1,...,N_i, i=1,...,d$.

Consider the following smoothed estimate for the spectral density

$$S_{2\mathbf{N}}(\boldsymbol{\omega})=|\mathbf{N}|^{-1}\sum_{\mathbf{k}}W_{1\mathbf{N}}(\boldsymbol{\omega}-\mu_{\mathbf{k}})I_{2\mathbf{N}}(\mu_{\mathbf{k}}),\boldsymbol{\omega}\in\boldsymbol{\pi},\tag{3.2.1}$$

where the weights $W_{1\mathbf{N}}(\boldsymbol{\omega})$ are defined by a real valued, even weight function $W_1(\boldsymbol{\omega})$ of finite support with $\int_{\mathbb{R}^d}W_1(\boldsymbol{\omega})d\boldsymbol{\omega}=1$, and $\int_{\mathbb{R}^d}W_1^2(\boldsymbol{\omega})d\boldsymbol{\omega}=\|W_1\|^2<\infty$, and $W_{1\mathbf{N}}(\boldsymbol{\omega})=W_1(\frac{\boldsymbol{\omega}}{b_{1\mathbf{N}}})|b_{1\mathbf{N}}|^{-1}, b_{1\mathbf{N}}\longrightarrow\mathbf{0},\mathbf{N}\odot b_{1\mathbf{N}}\longrightarrow\infty$ as $\mathbf{N}\longrightarrow\infty$. It is easy to show that the first order moments of $S_{2\mathbf{N}}(\boldsymbol{\omega})$ is (see Brillinger (1965))

$$E\{S_{2\mathbf{N}}(\boldsymbol{\omega})\}=S_2(\boldsymbol{\omega})+O(|b_{1\mathbf{N}}|)+O(|\mathbf{N}|^{-1}|b_{1\mathbf{N}}|^{-1}),\tag{3.2.2}$$

uniformly in $\boldsymbol{\omega}$, if $\boldsymbol{\omega} \neq \mathbf{0} \bmod (2\pi)$, and the covariance

$$Cov\left\{S_{2\mathbf{N}}(\boldsymbol{\omega}_1), S_{2\mathbf{N}}(\boldsymbol{\omega}_2)\right\} = |\mathbf{N}|^{-1}\left[S_4(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, -\boldsymbol{\omega}_2) + \frac{\|W_1\|^2}{|b_{1\mathbf{N}}|}(S_2^2(\boldsymbol{\omega}_1) + O(|b_{1\mathbf{N}}|))(\delta_{\boldsymbol{\omega}_1-\boldsymbol{\omega}_2} + \delta_{\boldsymbol{\omega}_1+\boldsymbol{\omega}_2})\right] + O(|\mathbf{N}|^{-2}|b_{1\mathbf{N}}|^{-2}), \tag{3.2.3}$$

uniformly in $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2$, where

$$\delta_{\boldsymbol{\omega}} = \begin{cases} 1 \text{ if } \omega_i \equiv 0 \bmod (2\pi), \\ 0 \quad \text{otherwise.} \end{cases}$$

The method of smoothing the biperiodogram is analogue of the previous one. A consistent estimate of the bispectrum is

$$S_{3\mathbf{N}}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2) = |\mathbf{N}|^{-2} \sum_{\mathbf{k}} \sum_{\mathbf{l}} W_{2\mathbf{N}}(\boldsymbol{\omega}_1 - \mu_{\mathbf{k}}, \boldsymbol{\omega}_2 - \mu_{\mathbf{l}}) I_{3\mathbf{N}}(\mu_{\mathbf{k}}, \mu_{\mathbf{l}}), \tag{3.2.4}$$

where $(\mu_{\mathbf{k}}, \mu_{\mathbf{l}}) = (\frac{\mathbf{k}}{\mathbf{N}}, \frac{\mathbf{l}}{\mathbf{N}}), \mathbf{k} = (k_1, ..., k_d)$, and $\mathbf{l} = (l_1, ..., l_d)$ i.e. $k_i, l_i = 1, ..., N_i$, are the Fourier frequencies, $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in \boldsymbol{\pi}$, the weights $W_{2\mathbf{N}}(\boldsymbol{\omega})$ are defined by a non-negative symmetric weight function $W_2(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2)$ of finite support with $\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W_2(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2) d\boldsymbol{\omega}_1 d\boldsymbol{\omega}_2 = 1$, $\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W_2^2(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2) = \|W_2\|^2 < \infty$, and $W_{2\mathbf{N}}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2) = W_2(\frac{\boldsymbol{\omega}_1}{b_{2\mathbf{N}}}, \frac{\boldsymbol{\omega}_2}{b_{2\mathbf{N}}})\,|b_{2\mathbf{N}}|^{-2}$, $b_{2\mathbf{N}} \longrightarrow \mathbf{0}, \mathbf{N} \odot b_{2\mathbf{N}}^2 \longrightarrow \infty$ as $\mathbf{N} \longrightarrow \infty, b_{1\mathbf{N}} \preceq b_{2\mathbf{N}}$ and there exists the limit $\lim_{N_i \longrightarrow \infty} \frac{b_{1N_i}}{b_{2N_i}} = \rho_i, i = 1, ..., d$.

The following expansion shows that the smoothed estimator is asymptotically unbiased

$$E\left\{S_{3\mathbf{N}}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2)\right\} = S_3(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2) + O(|b_{2\mathbf{N}}|) + O(|\mathbf{N}|^{-1}|b_{2\mathbf{N}}|^{-1}). \tag{3.2.5}$$

Put $\boldsymbol{\omega}_3 = -\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2, \boldsymbol{\lambda}_3 = -\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2$, the cross-covariance between the smoothed periodogram and the biperiodogram is

$$\begin{aligned} Cov\left\{S_{2\mathbf{N}}(\boldsymbol{\lambda}), S_{3\mathbf{N}}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2)\right\} &= |\mathbf{N}|^{-1}\Big\{S_5(\boldsymbol{\lambda}, -\boldsymbol{\omega}_1, -\boldsymbol{\omega}_2, -\boldsymbol{\lambda}) \\ &\quad + \frac{W_{12}(\boldsymbol{\rho})}{|b_{2\mathbf{N}}|} \sum_{k=1}^{3} S_2(\boldsymbol{\omega}_k)\overline{S}_3(\boldsymbol{\omega}_{k+1}, \boldsymbol{\omega}_{k+2})(\delta_{\boldsymbol{\lambda}+\boldsymbol{\omega}_k} + \delta_{\boldsymbol{\lambda}-\boldsymbol{\omega}_k}) \\ &\quad + \frac{W_{20}}{|b_{2\mathbf{N}}|} \sum_{k=1}^{3} [S_2(\boldsymbol{\omega}_k) S_3(\boldsymbol{\lambda}, \mathbf{0}) + O(|b_{2\mathbf{N}}|)]\delta_{\boldsymbol{\omega}_{k+1}}\Big\} + O\left((|\mathbf{N}|\,|b_{1\mathbf{N}}|)^{-2}\right), \end{aligned} \tag{3.2.6}$$

where the constants W_{12} and W_{20} are defined by

$$\begin{aligned} W_{12}(\boldsymbol{\rho}) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W_1(\boldsymbol{\omega}_1) W_2(\boldsymbol{\rho} \odot \boldsymbol{\omega}_1, \boldsymbol{\omega}_2) d\boldsymbol{\omega}_1 d\boldsymbol{\omega}_2, \\ W_{20} &= \int_{\mathbb{R}^d} W_2(\boldsymbol{\omega}_1, -\boldsymbol{\omega}_1) d\boldsymbol{\omega}_1, \end{aligned}$$

and $W_{12}(\boldsymbol{\rho})=0$ if $\boldsymbol{\rho}=\mathbf{0}$. The covariance according to the smoothed biperiodogram is

$$
\begin{aligned}
& Cov\left\{S_{3\mathbf{N}}(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2),S_{3\mathbf{N}}(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2)\right\} \qquad (3.2.7)\\
= & \left|\mathbf{N}\right|^{-1}\{S_6(\boldsymbol{\lambda}_1,\boldsymbol{\lambda_2}{-}\boldsymbol{\omega}_1,-\boldsymbol{\omega}_2,\boldsymbol{\lambda}_3)\\
& +\frac{W_{23}}{|b_{2\mathbf{N}}|}\left[\sum_{m,n=1}^{3}\left(S\left({}_2(\boldsymbol{\lambda}_m)S_4(\boldsymbol{\lambda}_{m+1},\boldsymbol{\lambda}_{m+2},-\boldsymbol{\omega}_{n+1}\right)+O(|b_{2\mathbf{N}}|)\right)\delta_{\boldsymbol{\lambda}_m-\boldsymbol{\omega}_n}\right.\\
& \left.+\sum_{m,n=1}^{3}(S_3(\boldsymbol{\lambda}_m,\boldsymbol{\lambda}_{m+1})S_3(\boldsymbol{\lambda}_{m+2},-\boldsymbol{\omega}_{n+1})+O(|b_{2\mathbf{N}}|))\delta_{\boldsymbol{\lambda}_m+\boldsymbol{\lambda}_{m+1}-\boldsymbol{\omega}_n}\right]\\
& +\frac{W_{20}}{|b_{2\mathbf{N}}|}\left[U(\boldsymbol{\omega},\boldsymbol{\lambda}){+}U(\boldsymbol{\lambda},\boldsymbol{\omega})\right]+\frac{W_{20}^2}{|b_{2\mathbf{N}}|^2}\left[S_2(\mathbf{0})\left(\sum_{m=1}^{3}(S_2(\boldsymbol{\lambda}_m)\delta_{\boldsymbol{\lambda}_{m+1}}\right)\right.\\
& \left.\left(\sum_{m=1}^{3}S_2(\boldsymbol{\omega}_m)\delta_{\boldsymbol{\omega}_{m+1}}\right)+O(|b_{2\mathbf{N}}|)\sum_{m,n=1}^{3}\delta_{\boldsymbol{\lambda}_m}\delta_{\boldsymbol{\omega}_n}\right]\\
& +\frac{\|W_2\|^2}{|B_{2\mathbf{N}}|^2}\left[S_2(\boldsymbol{\omega}_1)S_2(\boldsymbol{\omega}_2)S_2(\boldsymbol{\omega}_1+\boldsymbol{\omega}_2){+}O(|b_{2\mathbf{N}}|)\right]\\
& \times\sum_{m=1}^{3}\delta_{\boldsymbol{\lambda}_1-\boldsymbol{\omega}_m}(\delta_{\boldsymbol{\lambda}_2-\boldsymbol{\omega}_{m+1}}+\delta_{\boldsymbol{\lambda}_2-\boldsymbol{\omega}_{m+2}})\}+O(|\mathbf{N}|^{-2}|b_{2\mathbf{N}}|^{-2}),
\end{aligned}
$$

where $U(\boldsymbol{\omega},\boldsymbol{\lambda})=S_4(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\boldsymbol{\omega}_3)\sum_{m=1}^{3}(S_2(\boldsymbol{\lambda}_m)+O(|b_{2\mathbf{N}}|))\delta_{\boldsymbol{\lambda}_{m+1}}$, and

$$
W_{23}=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}W_2(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)W_2(\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2,\boldsymbol{\omega})d\boldsymbol{\lambda}_1 d\boldsymbol{\lambda}_2 d\boldsymbol{\omega},
$$

Let us suppose that the spectrum and the bispectrum of the random field $X(\mathbf{t})$ depend on a parameter θ which is not a multiplicative[1] one. Put

$$
\begin{aligned}
\mathcal{F}_{2\mathbf{N}}(\mu_{1\mathbf{k}},\theta) &= \left(\frac{S_2(\mu_{1\mathbf{k}},\theta)-S_{2\mathbf{N}}(\mu_{1\mathbf{k}})}{S_2(\mu_{1\mathbf{k}},\theta)}\right)^2,\\
\mathcal{F}_{3\mathbf{N}}(\mu_{2\mathbf{m}},\mu_{2\mathbf{l}},\theta) &= \frac{|S_3(\mu_{2\mathbf{m}},a_{2\mathbf{l}},\theta)-S_{3\mathbf{N}}(\mu_{2\mathbf{m}},\mu_{2\mathbf{l}})|^2}{S_2(\mu_{2\mathbf{m}},\theta)S_2(\mu_{2\mathbf{l}},\theta)S_2(\mu_{2\mathbf{m}}+\mu_{2\mathbf{l}},\theta)},
\end{aligned}
$$

and define

$$
Q_{\mathbf{N}}(\theta)=\frac{p_1\,|b_{1\mathbf{N}}|}{\beta_1^d}\sum_{\mu_{1\mathbf{k}}\in\Lambda_1^d}\mathcal{F}_{2\mathbf{N}}(\mu_{1\mathbf{k}},\theta)+\frac{q_1\,|b_{2\mathbf{N}}|^2}{\beta_2^d}\sum_{(\mu_{2\mathbf{m}},\mu_{2\mathbf{l}})\in\Lambda_2^d}\mathcal{F}_{3\mathbf{N}}(\mu_{2\mathbf{m}},\mu_{2\mathbf{l}},\theta), \qquad (3.2.8)
$$

[1] The parameter θ is multiplicative if $S_2(\mathbf{w},c\theta)=cS_2(\mathbf{w},\theta)$ and $S_3(\mathbf{w}_1,\mathbf{w}_2,c\theta)=cS_3(\mathbf{w}_1,\mathbf{w}_2,\theta)$ for any positive real number c.

where $\Lambda_1 \subset [0,1/2]$ is a finite union of closed intervals, Λ_2 be some finite union compact domains lying inside the open triangle Δ with vertices $(0,0),(\pi,0),(\frac{2\pi}{3},\frac{2\pi}{3})$ and the frequencies $\mu_{1\mathbf{k}}$ are equally spaced in $\boldsymbol{\pi}$ by bandwidth $b_{1\mathbf{N}}$ as well as frequencies $\mu_{2\mathbf{k}}$ by bandwidth $b_{2\mathbf{N}}$, $p_1 \in (0,1), q_1 = 1-p_1$, and constants β_1^d, β_2^d denote the d-dimensional product Lebesgue measure of Λ_1^d and Λ_2^d, respectively. A multiplicative parameter can not be estimated by minimization of (3.2.8) , we consider the estimate $\theta_{\mathbf{N}}$ for the unknown parameters θ obtained from minimization of the function (3.2.8) .

Remark 3.1 *The role of* β_1 *and* β_2 *is that both sums in* (3.2.8) *are averaged since the numbers of terms are about to* $\frac{\beta_1^d}{|b_{1\mathbf{N}}|}$ *and* $\frac{\beta_2^d}{|b_{2\mathbf{N}}|^2}$.

Denote Λ_1' and Λ_2' the sets with origins Λ_1 and Λ_2, respectively, and with property that they are invariant according to transformations

$$\begin{aligned} \mathcal{T}_1(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2) &= (\boldsymbol{\omega}_2,\boldsymbol{\omega}_1), \qquad & \mathcal{T}_2(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2) &=(\boldsymbol{\omega}_1,1-\boldsymbol{\omega}_2-\boldsymbol{\omega}_1), \\ \mathcal{T}_3(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2) &= (1-\boldsymbol{\omega}_1-\boldsymbol{\omega}_2,\boldsymbol{\omega}_2), & \mathcal{T}_4(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2) &=(1-\boldsymbol{\omega}_1,1-\boldsymbol{\omega}_2). \end{aligned}$$

Actually $\Lambda_1' = \Lambda_1 \cup \{\boldsymbol{\omega} = \mathbf{1}-\boldsymbol{\lambda} \,|\boldsymbol{\lambda} \in \Lambda_1\}$. For technical reasons we shall consider an equivalent form of (3.2.8) as

$$Q_{\mathbf{N}}(\theta) = \frac{p\,|b_{1\mathbf{N}}|}{2^d} \sum_{\mu_{1\mathbf{k}} \in \boldsymbol{\Lambda}_1'} \mathcal{F}_{2\mathbf{N}}(\mu_{1\mathbf{k}},\theta) + \frac{q\,|b_{2\mathbf{N}}|^2}{12^d} \sum_{(\mu_{2\mathbf{m}},\mu_{2\mathbf{l}}) \in \boldsymbol{\Lambda}_2'} \mathcal{F}_{3\mathbf{N}}(\mu_{2\mathbf{m}},\mu_{2\mathbf{l}},\theta), \tag{3.2.9}$$

where $\boldsymbol{\Lambda}_1' = \Lambda_1' \times ... \times \Lambda_1'$, $\boldsymbol{\Lambda}_2' = \Lambda_2' \times ... \times \Lambda_2'$, $d-$times, $p = \frac{p_1}{\beta_1^d}$ and $q = \frac{q_1}{\beta_2^d}$.

Remark 3.2 *There will be some advantage of changing the domains of summations in* (3.2.8) *in to the symmetric ones because the results of summations over set* Λ_2' *of the expression of the complex valued bispectrum will then be real.*

Remark 3.3 *For a Gaussian random fields this method based on the first term of* $Q_{\mathbf{N}}(\theta)$ *for estimating parameters. When a process is non-Gaussian, we suggested applying both the second and the third order periodogram in* (3.2.8) *(see Brillinger (1975)).*

Let θ_0 be the true value of the parameter $\theta \in \Theta$ and put

$$Q(\theta) = p\int_{\Lambda_1^d} \mathcal{F}_2(\boldsymbol{\omega},\theta_0)d\boldsymbol{\omega} + q\int\int_{\Lambda_2^d} \mathcal{F}_3(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\theta_0)d\boldsymbol{\omega}_1 d\boldsymbol{\omega}_1, \tag{3.2.10}$$

where

$$\mathcal{F}_2(\boldsymbol{\omega},\theta_0) = \left(\frac{S_2(\boldsymbol{\omega},\theta)-S_2(\boldsymbol{\omega},\theta_0)}{S_2(\boldsymbol{\omega},\theta)}\right)^2,$$
$$\mathcal{F}_3(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\theta_0) = \frac{|S_3(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\theta)-S_3(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\theta_0)|^2}{S_2(\boldsymbol{\omega}_1,\theta)S_2(\boldsymbol{\omega}_2,\theta)S_2(\boldsymbol{\omega}_1+\boldsymbol{\omega}_2,\theta)}.$$

3.3 Consistency and asymptotic variance

In this section we shall give conditions under which $Q_{\mathbf{N}}(\theta) \xrightarrow{p} Q(\theta)$ and $\theta_{\mathbf{N}} \xrightarrow{p} \theta_0$ as $\mathbf{N} \longrightarrow \infty$.

Lemma 3.1 *Suppose that conditions 3.1–3.2 are satisfied and* $|S_3(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\theta)| < \infty$*. Let* $C_{\mathbf{N}} = (C_{N_1},...,C_{N_d}), V_{\mathbf{N}} = (V_{N_1},...,V_{N_d})$ *are two vectors of sequences of positive integers where* $N_i = 1,2,...,$ $C_{N_i} \longrightarrow 0$ *and* V_{N_i} *is increasing. Then for any* $\epsilon > 0$

$$\sup_{\mathbf{p}=\mathbf{0},...,V_{\mathbf{N}}} |S_{2\mathbf{N}}(C_{\mathbf{N}}\odot\mathbf{p}) - S_2(C_{\mathbf{N}}\odot\mathbf{p})| = o_p(|V_{\mathbf{N}}|^{\epsilon}|b_{2\mathbf{N}}|^{-1/2}|\mathbf{N}|^{-1/2}),$$
$$\sup_{\mathbf{q},\mathbf{p}=\mathbf{0},...,V_{\mathbf{N}}} |S_{3\mathbf{N}}(C_{\mathbf{N}}\odot\mathbf{q}, C_{\mathbf{N}}\odot\mathbf{p}) - S_3(C_{\mathbf{N}}\odot\mathbf{q}, C_{\mathbf{N}}\odot\mathbf{p})| = o_p(|V_{\mathbf{N}}|^{\epsilon}|b_{2\mathbf{N}}|^{-1}|\mathbf{N}|^{-1/2}).$$

Proof. The proof of this Lemma is similar as Lemma 98 in Terdik (1998).

Our theorem concerning the consistency is the following

Theorem 3.1 *Suppose that* $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^d}$ *satisfies conditions 3.1–3.2 with* $p \geq 2$, $|S_3(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\theta)| < \infty$ *and that both* $\mathcal{F}_2(\boldsymbol{\omega},\theta_0)$ *and* $\mathcal{F}_3(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\theta_0)$ *have finite total variations on* Λ_1^d *and* Λ_2^d*, respectively. Suppose moreover that* $Q(\theta)$ *has a unique minimum at* θ_0 *and is continuous in* θ*. Then*

$$Q_{\mathbf{N}}(\theta) \xrightarrow{P} Q(\theta) \text{ and } \theta_{\mathbf{N}} \xrightarrow{P} \theta_0 \text{ as } \mathbf{N} \longrightarrow \infty,$$

where $Q_{\mathbf{N}}(\theta)$ *and* $Q(\theta)$ *are define by* (3.2.8) *and* (3.2.10)*, respectively. Moreover,* $\theta_{\mathbf{N}}$ *is an asymptotically unbiased estimate of* θ_0.

Proof. We have

$$\begin{aligned}
Q_{\mathbf{N}}(\theta)-Q(\theta) &= \frac{p\left|b_{1\mathbf{N}}\right|}{2^d}\sum_{\mu_{1\mathbf{k}}\in\mathbf{\Lambda}_1'}\left[\mathcal{F}_{2\mathbf{N}}(\mu_{1\mathbf{k}},\theta)-\mathcal{F}_2(\mu_{1\mathbf{k}},\theta_0)\right]+\\
&+\frac{p\left|b_{1\mathbf{N}}\right|}{2^d}\sum_{\mu_{1\mathbf{k}}\in\mathbf{\Lambda}_1'}\mathcal{F}_2(\mu_{1\mathbf{k}},\theta_0)-p\int_{\Lambda_1^d}\mathcal{F}_2(\boldsymbol{\omega},\theta_0)d\boldsymbol{\omega}\\
&+\frac{q\left|b_{2\mathbf{N}}\right|^2}{12^d}\sum_{(\mu_{2\mathbf{m}},\mu_{2\mathbf{l}})\in\mathbf{\Lambda}_2'}\left[\mathcal{F}_{3\mathbf{N}}(\mu_{2\mathbf{m}},\mu_{2\mathbf{l}},\theta)-\mathcal{F}_3(\mu_{2\mathbf{m}},\mu_{2\mathbf{l}},\theta_0)\right]\\
&+\frac{q\left|b_{2\mathbf{N}}\right|^2}{12^d}\sum_{(\mu_{2\mathbf{m}},\mu_{2\mathbf{l}})\in\mathbf{\Lambda}_2'}\mathcal{F}_3(\mu_{2\mathbf{m}},\mu_{2\mathbf{l}},\theta_0)-q\int\int_{\Lambda_2^d}\mathcal{F}_3(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\theta_0)d\boldsymbol{\omega}_1d\boldsymbol{\omega}_1\\
&= Q_{\mathbf{N}}^{(1)}+Q_{\mathbf{N}}^{(2)}+Q_{\mathbf{N}}^{(3)}+Q_{\mathbf{N}}^{(4)}.
\end{aligned}$$

From lemma 3.1 $Q_{\mathbf{N}}^{(2)}$ is $O(|b_{1\mathbf{N}}|)$ and $Q_{\mathbf{N}}^{(4)}$ is $O(|b_{2\mathbf{N}}|^2)$. Consider

$$Q_{\mathbf{N}}^{(1)}=\frac{p\left|b_{1\mathbf{N}}\right|}{2^d}\sum_{\mu_{1\mathbf{k}}\in\mathbf{\Lambda}_1'}\frac{\left[S_2(\mu_{1\mathbf{k}},\theta_0)-S_{2\mathbf{N}}(\mu_{1\mathbf{k}})\right]\left[2S_2(\mu_{1\mathbf{k}},\theta)-S_{2\mathbf{N}}(\mu_{1\mathbf{k}})-S_2(\mu_{1\mathbf{k}},\theta_0)\right]}{S_2^2(\mu_{1\mathbf{k}},\theta)}.$$

By lemma 3.1

$$\sup_{\mathbf{k}}\left|S_2(\mu_{1\mathbf{k}},\theta_0)-S_{2\mathbf{N}}(\mu_{1\mathbf{k}})\right|=o_p\left(\left|b_{1\mathbf{N}}\right|^{-\epsilon}\left|b_{1\mathbf{N}}\right|^{-1/2}\left|\mathbf{N}\right|^{-1/2}\right),$$

for any $\epsilon>0$, and under the condition of the theorem $Q_{\mathbf{N}}^{(1)}\overset{P}{\longrightarrow}0$ as $\mathbf{N}\longrightarrow\infty$. The same argument shows that $Q_{\mathbf{N}}^{(3)}\overset{P}{\longrightarrow}0$ as $\mathbf{N}\longrightarrow\infty$. As the final step, we use lemma 1 from Brillinger (1975), which contains general conditions for the consistency of an estimator based on some functional.

Under the regularity conditions above $\theta_{\mathbf{N}}$ tends in probability to the true value θ_0, and for $N_i, i=1,...,d$ sufficiently large:

$$\left.\frac{\partial}{\partial\theta}Q_{\mathbf{N}}(\theta)\right|_{\theta=\theta_{\mathbf{N}}}=\left.\frac{\partial}{\partial\theta}Q_{\mathbf{N}}(\theta)\right|_{\theta=\theta_{\mathbf{0}}}+\left.\frac{\partial^2}{\partial\theta^2}Q_{\mathbf{N}}(\theta)\right|_{\theta=\theta_{\mathbf{N}}^*}(\theta_{\mathbf{N}}-\theta_{\mathbf{0}}),$$

where $\|\theta_{\mathbf{N}}^*-\theta_{\mathbf{N}}\|<\|\theta_{\mathbf{N}}-\theta_{\mathbf{0}}\|$. Since $\theta_{\mathbf{N}}$ minimizes $Q_{\mathbf{N}}(\theta)$, it follows that $\left.\frac{\partial}{\partial\theta}Q_{\mathbf{N}}(\theta)\right|_{\theta=\theta_{\mathbf{N}}}=\mathbf{0}$. Thus

$$\theta_{\mathbf{N}}-\theta_{\mathbf{0}}=-\left(\left.\frac{\partial^2}{\partial\theta^2}Q_{\mathbf{N}}(\theta)\right|_{\theta=\theta_{\mathbf{N}}^*}\right)^{-1}\left(\left.\frac{\partial}{\partial\theta}Q_{\mathbf{N}}(\theta)\right|_{\theta=\theta_{\mathbf{0}}}\right). \tag{3.3.1}$$

We obtain

$$\begin{aligned}
\frac{\partial}{\partial\theta}Q_{\mathbf{N}}(\theta) &= \frac{p}{2^{d-1}}\left|b_{1\mathbf{N}}\right|\sum_{\mu_{1\mathbf{k}}\in\mathbf{\Lambda}_1'}\left(\mathcal{F}_2^{1/2}(\mu_{1\mathbf{k}},\theta)-\mathcal{F}_2(\mu_{1\mathbf{k}},\theta)\right)A(\mu_{1\mathbf{k}},\theta) \qquad (3.3.2)\\
&+\frac{q\left|b_{2\mathbf{N}}\right|^2}{12^d}\sum_{(\mu_{2\mathbf{m}},\mu_{2\mathbf{l}})\in\mathbf{\Lambda}_2'}\left(2\mathcal{F}_3^{1/2}(\mu_{2\mathbf{m}},\mu_{2\mathbf{l}},\theta)B(\mu_{2\mathbf{m}},\mu_{2\mathbf{l}},\theta)-\mathcal{F}_3(\mu_{2\mathbf{m}},\mu_{2\mathbf{l}},\theta)C(\mu_{2\mathbf{m}},\mu_{2\mathbf{l}},\theta)\right),
\end{aligned}$$

where

$$\begin{aligned}
A(\mu_{1\mathbf{k}},\theta) &= \frac{\partial}{\partial\theta}\log(S_2(\mu_{1\mathbf{k}},\theta)),\\
B(\mu_{2\mathbf{m}},\mu_{2\mathbf{l}},\theta) &= \frac{\partial}{\partial\theta}S_3(-\mu_{2\mathbf{m}},-\mu_{2\mathbf{l}},\theta),\\
C(\mu_{2\mathbf{m}},\mu_{2\mathbf{l}},\theta) &= \frac{\partial}{\partial\theta}\log(S_2(\mu_{2\mathbf{m}},\theta)S_2(\mu_{2\mathbf{l}},\theta)S_2(\mu_{2\mathbf{m}}+\mu_{2\mathbf{l}},\theta)).
\end{aligned}$$

Further algebra leads to the expression for the second derivative:

$$\begin{aligned}
\frac{\partial^2}{\partial\theta^2}Q_{\mathbf{N}}(\theta) &= \frac{p}{2^{d-1}}\left\{\int_{\mathbf{\Lambda}_1'} A(\boldsymbol{\omega},\theta)A'(\boldsymbol{\omega},\theta)d\boldsymbol{\omega}\right.\\
&+\int_{\mathbf{\Lambda}_1'}\mathcal{F}_{2\mathbf{N}}^{1/2}(\boldsymbol{\omega},\theta)\left[\mathcal{A}(\boldsymbol{\omega},\theta)-3A(\boldsymbol{\omega},\theta)A'(\boldsymbol{\omega},\theta)\right]d\boldsymbol{\omega}\\
&\left.-\int_{\mathbf{\Lambda}_1'}\mathcal{F}_{2\mathbf{N}}(\boldsymbol{\omega},\theta)\left[\mathcal{A}(\boldsymbol{\omega},\theta)-2A(\boldsymbol{\omega},\theta)A'(\boldsymbol{\omega},\theta)\right]d\boldsymbol{\omega}\right\}\\
&+\frac{q}{12^d}\left\{\int\int_{\mathbf{\Lambda}_2'}\mathcal{F}_{3\mathbf{N}}(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\theta)\left[\mathcal{C}(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\theta)-C(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\theta)C'(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\theta)\right]\right.\\
&+2\mathcal{G}_{3\mathbf{N}}(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\theta)\left[\mathcal{B}(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\theta)-2B(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\theta)C'(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\theta)\right])d\boldsymbol{\omega}_1 d\boldsymbol{\omega}_2\\
&\left.+\int\int_{\mathbf{\Lambda}_2'}2\Gamma^{-1}(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\theta)B(-\boldsymbol{\omega}_1,-\boldsymbol{\omega}_2,\theta)B'(-\boldsymbol{\omega}_1,-\boldsymbol{\omega}_2,\theta)d\boldsymbol{\omega}_1 d\boldsymbol{\omega}_2\right\}+O(|B_{\mathbf{N}}|),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{A}(\boldsymbol{\omega},\theta) &= \frac{\partial^2}{\partial\theta^2}\log(S_2(\boldsymbol{\omega},\theta)),\\
\mathcal{B}(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\theta) &= \frac{\partial^2}{\partial\theta^2}S_3(-\boldsymbol{\omega}_1,-\boldsymbol{\omega}_2,\theta),\\
\mathcal{C}(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\theta) &= \frac{\partial^2}{\partial\theta^2}\log(S_2(\boldsymbol{\omega}_1,\theta)S_2(\boldsymbol{\omega}_2,\theta)S_2(\boldsymbol{\omega}_1+\boldsymbol{\omega}_2,\theta)),\\
\mathcal{G}_{3\mathbf{N}}(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\theta) &= \frac{S_3(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\theta)-S_{3\mathbf{N}}(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2)}{S_2(\boldsymbol{\omega}_1,\theta)S_2(\boldsymbol{\omega}_2,\theta)S_2(\boldsymbol{\omega}_1+\boldsymbol{\omega}_2,\theta)},\\
\Gamma(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\theta) &= \prod_{k=1}^{3}S_2(\boldsymbol{\omega}_k,\theta),\boldsymbol{\omega}_1+\boldsymbol{\omega}_2+\boldsymbol{\omega}_3=\mathbf{0}.
\end{aligned}$$

Now we denote $(\frac{\partial^2}{\partial\theta^2}Q_{\mathbf{N}}(\theta)\Big|_{\theta=\theta^*_{\mathbf{N}}})$ of (3.3.1) by $\Sigma_0^{1/2}(\theta_0)$, and we obtain the asymptotic variance for the estimator $\theta_{\mathbf{N}}$.

Lemma 3.2 *Under the assumption of Theorem 3.1*

$$\begin{aligned}\lim_{\mathbf{N}\to\infty} P(\frac{\partial^2}{\partial\theta^2} Q_{\mathbf{N}}(\theta)\Big|_{\theta=\theta_{\mathbf{N}}}) &= \Sigma_0^{1/2}(\theta_0) \qquad (3.3.3)\\ &= \frac{p}{2^{d-1}} \int_{\mathbf{\Lambda}_1'} A(\boldsymbol{\omega},\theta)A'(\boldsymbol{\omega},\theta)d\boldsymbol{\omega}\\ &\quad + \frac{2q}{12^d}\int\int_{\mathbf{\Lambda}_2'} \Gamma^{-1}(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\theta_0)B(-\boldsymbol{\omega}_1,-\boldsymbol{\omega}_2,\theta_0)B'(-\boldsymbol{\omega}_1,-\boldsymbol{\omega}_2,\theta_0)d\boldsymbol{\omega}_1 d\boldsymbol{\omega}_2.\end{aligned}$$

Let us turn to $(\frac{\partial}{\partial\theta}Q_{\mathbf{N}}(\theta)\big|_{\theta=\theta_{\mathbf{0}}})$ of the vector defined by $\theta_{\mathbf{N}}-\theta_0$. It influence the limiting behavior of the estimator $\theta_{\mathbf{N}}$

$$\begin{aligned}Q_{1\mathbf{N}}(\theta_0) &= \frac{p\,|b_{1\mathbf{N}}|}{2^{d-1}} \sum_{\mu_{1\mathbf{k}}\in\mathbf{\Lambda}_1'} \mathcal{F}_{2\mathbf{N}}^{1/2}(\mu_{1\mathbf{k}},\theta)A(\mu_{1\mathbf{k}},\theta_0)\\ &\quad + \frac{2q\,|b_{2\mathbf{N}}|^2}{12^d} \sum_{(\mu_{2\mathbf{m}},\mu_{2\mathbf{l}})\in\mathbf{\Lambda}_2'} \mathcal{G}_{3\mathbf{N}}(\mu_{2\mathbf{m}},\mu_{2\mathbf{l}},\theta_0)B(\mu_{2\mathbf{m}},\mu_{2\mathbf{l}},\theta_0).\end{aligned}$$

Denote the asymptotic variance of $Q_{1\mathbf{N}}(\theta_0)$ by

$$\Sigma_1(\theta_0) = \lim_{\mathbf{N}\to\infty} |\mathbf{N}|\, var\left\{Q_{1\mathbf{N}}(\theta_0)\right\}. \qquad (3.3.4)$$

Therefore the variance of $Q_{1\mathbf{N}}(\theta_0)$ will be given as

$$\Sigma_1(\theta_0) = \Sigma_2(\theta_0) + \Sigma_3(\theta_0) + 2\Sigma_{23}(\theta_0).$$

The asymptotic variance of the second order term $\Sigma_2(\theta_0)$ has been given by

$$\begin{aligned}\Sigma_2(\theta_0) &= \frac{p^2}{2^{2(d-1)}} \Bigg[\int\int_{\mathbf{\Lambda}_1'\times\mathbf{\Lambda}_1'} \frac{S_4(\boldsymbol{\omega},\boldsymbol{\lambda},-\boldsymbol{\lambda},\theta_0)}{S_2(\boldsymbol{\omega},\theta_0)S_2(\boldsymbol{\lambda},\theta_0)} A(\boldsymbol{\omega},\theta_0)A'(\boldsymbol{\lambda},\theta_0)d\boldsymbol{\omega}d\boldsymbol{\lambda}\\ &\quad + 2\,\|W_1\|^2 \int_{\mathbf{\Lambda}_1'} A(\boldsymbol{\omega},\theta_0)A'(\boldsymbol{\omega},\theta_0)d\boldsymbol{\omega}\Bigg].\end{aligned}$$

From the application of the asymptotic covariance (3.2.7), the variance of the third order term is

$$
\begin{aligned}
\Sigma_3(\theta_0) &= \frac{4q^2}{12^{2d}} \Bigg\{ \int\int\int\int_{\mathbf{\Lambda}_2' \times \mathbf{\Lambda}_2'} \Gamma^{-1}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \theta_0)\Gamma^{-1}(-\boldsymbol{\lambda}_1, -\boldsymbol{\lambda}_2, \theta_0) S_6(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, -\boldsymbol{\lambda}_1, -\boldsymbol{\lambda}_2, \boldsymbol{\omega}_3, \theta_0) \\
& B(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \theta_0) B'(-\boldsymbol{\lambda}_1, -\boldsymbol{\lambda}_2, \theta_0)\, d\boldsymbol{\omega}_1 d\boldsymbol{\omega}_2 d\boldsymbol{\lambda}_1 d\boldsymbol{\lambda}_2 \\
& + 9W_{23} \int\int\int\int_{\mathbf{\Lambda}_2' \times \mathbf{\Lambda}_2'} \Gamma^{-1}(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \theta_0)\Gamma^{-1}(\boldsymbol{\lambda}_3, -\boldsymbol{\omega}_2, \theta_0) S_3(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \theta_0) S_3(\boldsymbol{\lambda}_3, -\boldsymbol{\omega}_2, \theta_0) \\
& B(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \theta_0) B'(\boldsymbol{\lambda}_3, -\boldsymbol{\omega}_2, \theta_0) \\
& + \Gamma^{-1}(\boldsymbol{\omega}_2 + \boldsymbol{\lambda}_3, -\boldsymbol{\omega}_2, \theta_0)\Gamma^{-1}(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \theta_0) S_4(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, -\boldsymbol{\omega}_2, \theta_0) S_2(\boldsymbol{\lambda}_3, \theta_0) \\
& \times B(\boldsymbol{\omega}_2 + \boldsymbol{\lambda}_3, -\boldsymbol{\omega}_2, \theta_0) B'(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \theta_0) d\boldsymbol{\lambda}_1 d\boldsymbol{\lambda}_2 d\boldsymbol{\omega}_2 \\
& + 6W_{20} \int\int\int_{\mathbf{\Lambda}_{21}' \times \mathbf{\Lambda}_2'} \Gamma^{-1}(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \theta_0)\Gamma^{-1}(\mathbf{0}, -\boldsymbol{\omega}_2, \theta_0) S_4(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, 0, \theta_0) S_2(\boldsymbol{\omega}_2, \theta_0) \\
& \times B(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \theta_0) B'(\mathbf{0}, -\boldsymbol{\omega}_2, \theta_0) d\boldsymbol{\lambda}_1 d\boldsymbol{\lambda}_2 d\boldsymbol{\omega}_2 \\
& + 9W_{20}^2 S_2^{-1}(\mathbf{0}, \theta_0) \int_{\mathbf{\Lambda}_1'} S_2^{-2}(\boldsymbol{\omega}_1, \theta_0) B(-\boldsymbol{\omega}_1, \mathbf{0}, \theta_0) B'(-\boldsymbol{\omega}_1, \mathbf{0}, \theta_0) d\boldsymbol{\omega}_1 \\
& + 6\left\|W_2\right\|^2 \int\int_{\mathbf{\Lambda}_2'} \Gamma^{-1}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \theta_0) B(-\boldsymbol{\omega}_1, -\boldsymbol{\omega}_2, \theta_0) B'(-\boldsymbol{\omega}_1, -\boldsymbol{\omega}_2, \theta_0) d\boldsymbol{\omega}_1 d\boldsymbol{\omega}_2 \Bigg\},
\end{aligned}
$$

where set $\mathbf{\Lambda}_{21}'$ is the orthogonal projection of $\mathbf{\Lambda}_2'$ onto $[0, 2\pi]^d$. The covariance between the second and third order terms is

$$
\begin{aligned}
\Sigma_{23}(\theta_0) &= \frac{pq}{2^{d-1}12^d} \Bigg\{ \int\int\int_{\mathbf{\Lambda}_2' \times \mathbf{\Lambda}_1'} 2\Gamma^{-1}(-\boldsymbol{\omega}_1, -\boldsymbol{\omega}_2, \theta_0) S_2^{-1}(\boldsymbol{\omega}, \theta_0) S_5(\boldsymbol{\omega}, -\boldsymbol{\omega}, -\boldsymbol{\omega}_1, -\boldsymbol{\omega}_2, \theta_0) \\
& B(-\boldsymbol{\omega}_1, -\boldsymbol{\omega}_2, \theta_0) A'(\boldsymbol{\omega}, \theta_0) d\boldsymbol{\omega}_1 d\boldsymbol{\omega}_2 d\boldsymbol{\omega} \\
& + W_{12}(\boldsymbol{\rho}) \int\int_{\mathbf{\Lambda}_{21}' \times \mathbf{\Lambda}_1'} \Gamma^{-1}(-\boldsymbol{\omega}_1, -\boldsymbol{\omega}_2, \theta_0) S_3(-\boldsymbol{\omega}_1, -\boldsymbol{\omega}_2, \theta_0) \\
& B(-\boldsymbol{\omega}_1, -\boldsymbol{\omega}_2, \theta_0) A(\boldsymbol{\omega}_1, \theta_0) d\boldsymbol{\omega}_1 d\boldsymbol{\omega}_2 \\
& + \frac{W_{20}}{6} \int\int_{\mathbf{\Lambda}_2' \times \mathbf{\Lambda}_1'} \Gamma^{-1}(-\boldsymbol{\omega}_1, \mathbf{0}, \theta_0) S_3(\boldsymbol{\omega}_1, \mathbf{0}, \theta_0) B(-\boldsymbol{\omega}_1, \mathbf{0}, \theta_0) A'(\boldsymbol{\omega}_2, \theta_0) d\boldsymbol{\omega}_1 d\boldsymbol{\omega}_2 \Bigg\}.
\end{aligned}
$$

Consider the following functional (depending on the periodogram of the second and the third orders of the smoothed periodograms) taken at Fourier frequencies $\mu_{\mathbf{k}}, \mu_{\mathbf{l}}, \mathbf{k}, \mathbf{l} = \mathbf{1}, ..., \mathbf{N}$

$$
\begin{aligned}
R_{\mathbf{N}}(\theta_0) &= \frac{p}{2^{d-1}\left|\mathbf{N}\right|} \sum_{\mu_{\mathbf{k}} \in \mathbf{\Lambda}_1'} \left[\frac{S_2(\mu_{\mathbf{k}}, \theta_0) - I_{2\mathbf{N}}(\mu_{\mathbf{k}})}{S_2(\mu_{\mathbf{k}}, \theta_0)} \right] A(\mu_{\mathbf{k}}, \theta_0) \\
& + \frac{2q}{12^d\left|\mathbf{N}\right|^2} \sum_{(\mu_{\mathbf{k}}, \mu_{\mathbf{l}}) \in \mathbf{\Lambda}_2'} \left[\frac{S_3(\mu_{\mathbf{k}}, \mu_{\mathbf{l}}, \theta_0) - I_{3\mathbf{N}}(\mu_{\mathbf{k}}, \mu_{\mathbf{l}})}{S_3(\mu_{\mathbf{k}}, \mu_{\mathbf{l}}, \theta_0)} \right] B(\mu_{\mathbf{k}}, \mu_{\mathbf{l}}, \theta_0).
\end{aligned}
$$

Lemma 3.3 *Under the assumption of Theorem* 3.1 *the asymptotic variance*

$$\Sigma_{R_1}(\theta_0) = \lim_{\mathbf{N}\to\infty} |\mathbf{N}|\, var\left\{Q_{1\mathbf{N}}(\theta_0)\right\}, \tag{3.3.5}$$

of $R_{\mathbf{N}}(\theta_0)$ *is given as the sum*

$$\Sigma_{R_1}(\theta_0) = \Sigma_{R_2}(\theta_0) + \Sigma_{R_3}(\theta_0) + 2\Sigma_{R_{23}}(\theta_0).$$

Remark 3.4 *The asymptotic variance* $\Sigma_{R_1}(\theta_0)$ *of* $R_{\mathbf{N}}(\theta_0)$ *is the same as* $\Sigma_1(\theta_0)$*. Then* $\Sigma_{R_2}(\theta_0), \Sigma_{R_3}(\theta_0)$ *and* $\Sigma_{R_{23}}(\theta_0)$ *are the same as* $\Sigma_2(\theta_0), \Sigma_3(\theta_0)$ *and* $\Sigma_{23}(\theta_0)$*, respectively, where all constants depending on the weight functions* $W's$ *are changed to* 1*.*

There is an interesting case concerning the statistic $R_{\mathbf{N}}$ if one shows down the convergence by bandwidth $b_{\mathbf{N}} = b_{1\mathbf{N}} = b_{2\mathbf{N}}$ and considers the vector

$$\begin{aligned} R^*_{\mathbf{N}}(\theta_0) &= \frac{p\,|b_{\mathbf{N}}|^{1/2}}{2^{d-1}} \sum_{\mu_{\mathbf{k}}\in\mathbf{\Lambda}'_1} \left[\frac{S_2(\mu_{\mathbf{k}},\theta_0) - I_{2\mathbf{N}}(\mu_{\mathbf{k}})}{S_2(\mu_{\mathbf{k}},\theta_0)}\right] A(\mu_{\mathbf{k}},\theta_0) \\ &\quad + \frac{2q\,|b_{\mathbf{N}}|^{1/2}}{12^d\,|\mathbf{N}|} \sum_{(\mu_{\mathbf{k}},\mu_{\mathbf{l}})\in\mathbf{\Lambda}'_2} \left[\frac{S_3(\mu_{\mathbf{k}},\mu_{\mathbf{l}},\theta_0) - I_{3\mathbf{N}}(\mu_{\mathbf{k}},\mu_{\mathbf{l}})}{S_3(\mu_{\mathbf{k}},\mu_{\mathbf{l}},\theta_0)}\right] B(\mu_{\mathbf{k}},\mu_{\mathbf{l}},\theta_0). \end{aligned}$$

Then the asymptotic variance $\Sigma_{R^*_1}(\theta_0)$ of $R^*_{\mathbf{N}}(\theta_0)$ is the same as $\Sigma_0(\theta_0)$.

3.4 Asymptotic normality

Theorem 3.2 *Suppose that conditions of Theorem* 3.1 *are satisfied with* $p \geq 1$*. Then the estimator* $\theta_{\mathbf{N}}$ *defined by* (3.2.8) *is asymptotically Gaussian:*

$$\sqrt{|\mathbf{N}|}(\theta_{\mathbf{N}} - \theta_{\mathbf{0}}) \xrightarrow{D} \mathcal{N}(O, \Sigma_1(\theta_0)\Sigma_0^{-1}(\theta_0)) \text{ as } \mathbf{N} \longrightarrow \infty,$$

where $\Sigma_0(\theta_0)$ *is defined by* (3.3.3) *and* $\Sigma_1(\theta_0)$ *by* (3.3.4)*.*

Proof. To prove Theorem 3.2, we use Theorem 3.1, lemmas 3.2, 3.3 and Slutsky's argument.

Part II

Wavelet Analysis

Chapter 4

Wavelets and random fields

4.1 Introduction

Wavelets are mathematical functions that allow us divide data into different frequency components and then study each component with a resolution appropriate for its overall scale. They have advantages over traditional Fourier methods in analyzing physical situations where the signal contains discontinuities and sharp spikes. Wavelets were developed independently in the fields of mathematics, quantum physics, electrical engineering, and seismic geology. Interchanges between these fields during the last forty years have led to many new wavelet applications such as image compression, turbulence, human vision, radar, and earthquake prediction.

In recent years, wavelet methods are advocated as an alternative to Fourier methods for the analysis of both deterministic and nondeterministic signals. It is generally believed that the wavelet methods are more appropriate for the analysis of nonlinear and nonstationary signals. But, so far the methods that have been proposed are restricted to the analysis of continuous random fields. In this chapter we develop an approach to deal with the discrete random field and wavelet transforms, and then study the probabilistic structures.

4.2 Multiresolution analysis in $\mathbb{R}^d$

Multiresolution analysis in $\mathbb{R}^d$ provides an efficient framework for the decomposition of random fields. Recently, a considerable attention was given to the properties of the wavelet transform and of the wavelet orthonormal representations of random fields (see Antoine et al. (2004)).

Definition 4.1 *A d-dimensional multiresolution analysis (MRA) is an increasing sequence of subspaces* $\{V_j\} \subset L^2(\mathbb{R})$ *defined for* $j \in \mathbb{Z}$ *with*

$$... \subset V_{-1} \subset V_0 \subset V_1 \subset ...$$

together with a scaling function $\phi \in L^2(\mathbb{R})$ *such that*

(i) $\cup_{j\in\mathbb{Z}} V_j$ *is dense in* $L^2(\mathbb{R}), \cap_{j\in\mathbb{Z}} V_j = \{0\}$.

(ii) $X(\mathbf{t}) \in V_j$ *if and only if* $X(2^j\mathbf{t}) \in V_0$.

(iii) $X(\mathbf{t}) \in V_j$ *if and only if* $X(\mathbf{t} - 2^j\mathbf{k}) \in V_j$.

Definition 4.2 *For any* $\mathbf{x} = (x_1, ..., x_d) \in \mathbb{R}^d$, *a function* $\Phi(\mathbf{x}) \in V_0$ *which satisfies*

$$\Phi(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} h_{\mathbf{k}} 2^{d/2} \Phi(2\mathbf{x} - \mathbf{k}),$$

where

$$\Phi(\mathbf{x}) = \prod_{i=1}^{d} \phi(x_i), \tag{4.2.1}$$

and

$$\phi(x_i) = \sum_{k_i\in\mathbb{Z}} h_{k_i} \sqrt{2} \phi(2x_i - k_i),$$

is called a scaling function (or refinable function). If $\{\Phi(\mathbf{x} - \mathbf{n})\}_{\mathbf{n}\in\mathbb{Z}^d}$ *is an orthonormal system, then* Φ *is called an orthonormal scaling function, and the wavelet function is given by*

$$\Psi_u(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} g_{\mathbf{k}}^{(u)} 2^{d/2} \Phi(2\mathbf{x} - \mathbf{k}), u = 1, ..., 2^d - 1,$$

where

$$g_{\mathbf{k}}^{(u)} = \begin{cases} g_{k_i} \prod_{i=1}^{d} h_{k_i} & \text{when} \quad u \in \{1, ..., d\}, \\ \prod_{i\in A_u} g_{k_i} \prod_{i\notin A_u} h_{k_i} & \text{when } u \in \{d+1, ..., 2^d - 1\}, \end{cases}$$

where $(A_u)_{u\in\{d+1,...,2^d-1\}}$ *forms the set of all non void subsets of* $\{1, ..., d\}$ *of cardinality greater or equal to 2.*

Remark 4.1 *We can proceed analogously to construct wavelets using products of one-dimensional functions as*

$$\Psi_u(\mathbf{x}) = \begin{cases} \psi(x_i)\prod_{i=1}^{d}\phi(x_i) & \textit{when} \quad u \in \{1, ..., d\}, \\ \prod_{i\in A_u}\psi(x_i)\prod_{i\notin A_u}\phi(x_i) & \textit{when } u \in \{d+1, ..., 2^d - 1\}, \end{cases} \tag{4.2.2}$$

where

$$\psi(x_i) = \sum_{k_i\in\mathbb{Z}} g_{k_i}\sqrt{2}\phi(2x_i - k_i).$$

Usual choice for a two-dimensional scaling function or wavelet is a product of two one-dimensional functions as the following example

Example 4.1 *The scaling function has the form*

$$\Phi(x, y) = \sum h_{kl} 2\Phi(2x - k, 2y - l),$$

where

$$\Phi(x, y) = \phi(x)\phi(y), \tag{4.2.3}$$

and $h_{kl} = h_k h_l$. Since $\phi(x)$ and $\phi(y)$ both satisfy the scaling equation

$$\phi(x) = \sum h_k\sqrt{2}\phi(2x - k).$$

Thus two dimensional scaling equation is product of two one dimensional scaling equations.

However, unlike one-dimensional case, we have three rather than one basic wavelet. They are:

$$\begin{aligned} \Psi^{(h)}(x, y) &= \phi(x)\psi(y), \\ \Psi^{(v)}(x, y) &= \psi(x)\phi(y), \\ \Psi^{(d)}(x, y) &= \psi(x)\psi(y). \end{aligned} \tag{4.2.4}$$

The generalization of the one-dimensional wavelet equation leads to the following relations:

$$\begin{aligned} \Psi^{(h)}(x, y) &= \sum g_{kl}^{(h)} 2\Phi(2x - k, 2y - l), \\ \Psi^{(v)}(x, y) &= \sum g_{kl}^{(v)} 2\Phi(2x - k, 2y - l), \\ \Psi^{(d)}(x, y) &= \sum g_{kl}^{(d)} 2\Phi(2x - k, 2y - l), \end{aligned}$$

where $g_{kl}^{(h)} = h_k g_l, g_{kl}^{(v)} = g_k h_l, g_{kl}^{(d)} = g_k g_l$.

Remark 4.2 *In the wavelet literature, the reader may encounter an indexing of the multiresolution subspaces, which is the reverse of that in definition* 4.1

$$... \subset V_1 \subset V_0 \subset V_{-1} \subset ... \tag{4.2.5}$$

This convention (both have advantages and inconveniences), sometimes called "Daubechies" convention, as opposed to "Mallat's" convention in (4.2.5)*, is almost equally often used. however, the family* $\{\Phi_{j,\mathbf{k}}(\mathbf{t}) = 2^{jd/2}\phi(2^j t_1 - k_1, ..., 2^j t_d - k_d), \mathbf{k} \in \mathbb{Z}^d\}$ *is a basis of* V_j *according to Mallat's indexing, while* $\{\Phi_{j,\mathbf{k}}(\mathbf{t}) = 2^{-jd/2}\phi(2^{-j} t_1 - k_1, ..., 2^{-j} t_d - k_d), \mathbf{k} \in \mathbb{Z}^d\}$ *is a basis of* V_j *according to Daubechies's indexing.*

The approximation of a function $X(\mathbf{t})$ on to a subspace V_j is given in terms of scaling functions as

$$\widehat{X}_j(\mathbf{t}) = \sum_{\mathbf{k} \in \mathbb{Z}^d} \alpha_{j,\mathbf{k}} \Phi_{j,\mathbf{k}}(\mathbf{t}), \tag{4.2.6}$$

where $\alpha_{j,\mathbf{k}}$ is the scaling coefficient at resolution j and translation $\mathbf{k}$ and

$$\Phi_{j,\mathbf{k}}(\mathbf{t}) = 2^{jd/2}\Phi(2^j t_1 - k_1, ..., 2^j t_d - k_d). \tag{4.2.7}$$

Therefore, a function $X(\mathbf{t}) \in L^2(\mathbb{R}^d)$ can either be represented by a set of orthogonal scaling functions as

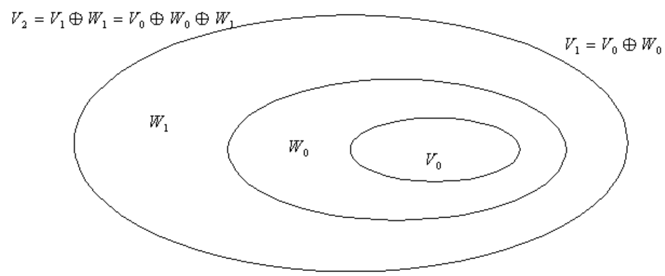
$$X(\mathbf{t}) = \lim \sum_{\mathbf{k} \in \mathbb{Z}^d} \alpha_{j,\mathbf{k}} \Phi_{j,\mathbf{k}}(\mathbf{t}).$$

Definition 4.3 *For each* $j \in \mathbb{Z}$*, the wavelet subspace* W_j *is defined by*

$$W_j = \overline{span}\{\Psi_{j,\mathbf{k}}(\mathbf{x})\}_{\mathbf{k} \in \mathbb{Z}^d}.$$

Since $\{V_j, j \in \mathbb{Z}\}$ are the nest subspaces, we can represent the subspaces V_{j-1} as a direct sum of coarsely approximated subspaces V_j and its orthogonal complement subspaces W_j as

$$V_j = V_{j-1} \oplus W_{j-1}.$$



The relationship between scaling and wavelet function spaces

This shows that the projection of a function $X(\mathbf{t})$ on to subspaces W_j gives the detailed information lost in approximating the function over the subspaces V_j. An orthonormal basis can now be constructed for W_j subspaces. A collection of all such basis functions for subspaces $\left\{W_j, j \in \mathbb{Z}^d\right\}$, form a new orthogonal bases for $L^2(\mathbb{R}^d)$.

Thus the same approximation in terms of basis function of W_j is given by

$$\widehat{X}_j(\mathbf{t}) = \sum_{u=1}^{2^d-1} \sum_{j=-\infty}^{l} \sum_{\mathbf{k}\in\mathbb{Z}^d} \beta_{j,\mathbf{k},u} \Psi_{j,\mathbf{k},u}(\mathbf{t}),$$

where $\beta_{j,\mathbf{k},u}$ is the wavelet coefficient at resolution j and translation $\mathbf{k}$ and

$$\Psi_{j,\mathbf{k},u}(\mathbf{t}) = 2^{jd/2}\Psi_u(2^j t_1 - k_1, ..., 2^j t_d - k_d), \text{ for any } u \in \{1, ..., 2^d - 1\}, \tag{4.2.8}$$

Any square integrable function $X(\mathbf{t}) \in L^2(\mathbb{R}^d)$, it can be written in terms of scaling and wavelet functions as

$$X(\mathbf{t}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} \alpha_{j,\mathbf{k}} \Phi_{j,\mathbf{k}}(\mathbf{t}) + \sum_{u=1}^{2^d-1} \sum_{j=-\infty}^{l} \sum_{\mathbf{k}\in\mathbb{Z}^d} \beta_{j,\mathbf{k},u} \Psi_{j,\mathbf{k},u}(\mathbf{t}), \tag{4.2.9}$$

and the coefficients $\{\alpha_{j,\mathbf{k}}\}$ and $\left\{\beta_{j,\mathbf{k},u}\right\}$ are the sequences that describe the signal while the basis functions are fixed.

Remark 4.3 *The basis functions can either be orthonormal or just linearly independent. When these are orthonormal, they fit into a general multiresolution framework. In other words, the multiresolution analysis provides a method for constructing a set of orthonormal function which form a bases for $L^2(\mathbb{R}^d)$ space and satisfy the properties of a wavelet function. In this analysis functions are approximated at different resolutions to give smoothed versions of a functions. The increment* *information lost while approximating a function at different lower resolutions can be studies using wavelet coefficient.*

Example 4.2 *Let $\phi(x)$ an $\psi(x)$ be the scaling and wavelet functions as associated with some MRA, for each $j,k,l,\in\mathbb{Z}$, define*

$$\begin{aligned}\Phi_{j,k,l}(x,y) &= 2^j\Phi(2^jx-k,2^jy-l),\\ \Psi^{(h)}_{j,k,l}(x,y) &= 2^j\Psi^{(h)}(2^jx-k,2^jy-l),\\ \Psi^{(v)}_{j,k,l}(x,y) &= 2^j\Psi^{(v)}(2^jx-k,2^jy-l),\\ \Psi^{(d)}_{j,k,l}(x,y) &= 2^j\Psi^{(d)}(2^jx-k,2^jy-l),\end{aligned}$$

the collection $\{\Phi_{j,k,l}(x,y),j,k,l\in\mathbb{Z}\}\cup\left\{\Psi^{(u)}_{j,k,l}(x,y),j,k,l\in\mathbb{Z},u\in\{v,h,d\}\right\}$ is an orthonorrmal basis on $\mathbb{R}^2$ satisfying (4.2.3) *and* (4.2.4). *However, any function $X(x,y)\in L^2(\mathbb{R}^2)$ can be written in terms of scaling and wavelet as*

$$X(x,y)=\sum_{k,l\in\mathbb{Z}}\alpha_{j,k,l}\Phi_{j,k,l}(x,y)+\sum_{u\in\{v,h,d\}}\sum_{j\geq J=-\infty}^{l}\sum_{k,l\in\mathbb{Z}}\beta^{(u)}_{j,k,l}\Psi^{(u)}_{j,k,l}(x,y).$$

From Daubechies convention we can express any function $\Phi_{j,\mathbf{n}}(\mathbf{t})$ in the subspaces V_j as a linear combination of the basis functions $\left\{\Phi_{j-1,\mathbf{k}}(\mathbf{t});\mathbf{k}\in\mathbb{Z}^d\right\}$ of V_{j-1} as

$$\Phi_{j,\mathbf{n}}(\mathbf{t})=\sum_{\mathbf{k}\in\mathbb{Z}^d}h_{\mathbf{k}-2\mathbf{n}}\Phi_{j-1,\mathbf{k}}(\mathbf{t}), \tag{4.2.10}$$

where $h_{\mathbf{k}-2\mathbf{n}}=\int\Phi_{j,\mathbf{n}}(\mathbf{t})\Phi^*_{j-1,\mathbf{k}}(\mathbf{t})d\mathbf{t}$ and $\sum_{\mathbf{k}\in\mathbb{Z}^d}h_{\mathbf{k}}=1$. For a compact support wavelet basis, $\{h_{\mathbf{k}}\}$ is finite length $|\mathbf{N}|$ (i.e. $h_{\mathbf{k}}$ is nonzero in the interval $\mathbf{0}\preceq\mathbf{k}\preceq\mathbf{N}-\mathbf{1}$ and zero outside the interval). Since W_j is also a subspace of V_{j-1}, we can express any function $\Psi_{j,\mathbf{n},u}(\mathbf{t})$ in W_j subspaces as a linear combination of the basis functions $\left\{\Phi_{j-1,\mathbf{k}}(\mathbf{t});\mathbf{k}\in\mathbb{Z}^d\right\}$ of V_{j-1} as

$$\Psi_{j,\mathbf{n},u}(\mathbf{t})=\sum_{\mathbf{k}\in\mathbb{Z}^d}g^{(u)}_{\mathbf{k}-2\mathbf{n}}\Phi_{j-1,\mathbf{k}}(\mathbf{t}),u=1,...,2^d-1, \tag{4.2.11}$$

where $g^{(u)}_{\mathbf{k}-2\mathbf{n}}=\int\Psi_{j,\mathbf{n},u}(\mathbf{t})\Phi^*_{j-1,\mathbf{k}}(\mathbf{t})d\mathbf{t},u=1,...,2^d-1$. Many choices of $h_{\mathbf{k}}$ and $g^{(u)}_{\mathbf{k}}$ exist which satisfy (4.2.10) and (4.2.11). One such choice is by choosing coefficients $\left\{g^{(u)}_{\mathbf{k}},u=1,...,2^d-1\right\}$ such that $g_{k_i}=(-1)^{k_i}h_{1-k_i}$. The relation between coefficients that gives the information about smoothed (scaling coefficient) and detailed version (wavelet coefficient) of a function at different resolution can be obtained by permultilying the conjugates of (4.2.10) and (4.2.11) with $X(\mathbf{t})$ and integrating with respect to $\mathbf{t}$. Therefore, we have

$$\begin{aligned} \alpha_{j,\mathbf{n}} &= \sum_{\mathbf{k}\in\mathbb{Z}^d} h^*_{\mathbf{k}-2\mathbf{n}}\alpha_{j-1,\mathbf{k}}, \\ \beta_{j,\mathbf{n},u} &= \sum_{\mathbf{k}\in\mathbb{Z}^d} g^{(u)*}_{\mathbf{k}-2\mathbf{n}}\alpha_{j-1,\mathbf{k}}, \text{ for } u = 1, ..., 2^d - 1, \end{aligned} \tag{4.2.12}$$

thus

$$\alpha_{j,\mathbf{n}} = \sum_{\mathbf{k}\in\mathbb{Z}^d} h_{\mathbf{n}-2\mathbf{k}}\alpha_{j+1,\mathbf{k}} + \sum_{u=1}^{2^d-1}\sum_{\mathbf{k}\in\mathbb{Z}^d} g^{(u)}_{\mathbf{n}-2\mathbf{k}}\beta_{j+1,\mathbf{k},u}.$$

Equation (4.2.12) indicate that all the scaling and wavelet coefficient at resolution $(j, j+1, j+2, ...)$ can be obtained from a set of coefficient $\{h_{\mathbf{k-2n}}\}$ which describe the wavelet basis and the initial set of $\left\{\alpha_{j-1,\mathbf{k}}; \mathbf{k}\in\mathbb{Z}^d\right\}$.

4.3 Random fields

Random fields have found numerous applications in diverse areas such as image processing (see for example Jain (1981)), oceanography (see Sylvester (1974)), geology (see Harbaugh and Preston (1968)), forestry (see Matern (1960)), turbulence (see Mandelbort (1975)), and geomorphology (see Mandelbort (1975)). In previous section, a function $X(\mathbf{t})$ is assumed to be an element of $L^2(\mathbb{R}^d)$. In the case of random fields, all may not have sample paths on $L^2(\mathbb{R}^d)$. however, if $X(\mathbf{t},\boldsymbol{\epsilon})$ is a measurable function defined on $\mathbb{R}^d\times\Lambda$ (Λ is the sample space) and satisfies $\int E\left\{X^2(\mathbf{t},\boldsymbol{\epsilon})\right\}d\mathbf{t} < \infty$, then $X(\mathbf{t},\boldsymbol{\epsilon})\in L^2(\mathbb{R}^d)$ with probability 1 in Λ.

If $X(\mathbf{t})$ is a continuous parameter random fields and $X(\mathbf{t})\in L^2(\mathbb{R}^d)$, then we have the wavelet and scaling representations as

$$X(\mathbf{t}) = \sum_{u=1}^{2^d-1}\sum_{j=-\infty}^{l}\sum_{\mathbf{k}\in\mathbb{Z}^d}\beta_{j,\mathbf{k},u}\Psi_{j,\mathbf{k},u}(\mathbf{t}),$$

and

$$X(\mathbf{t}) = \lim \sum_{\mathbf{k}\in\mathbb{Z}^d}\alpha_{j,\mathbf{k}}\Phi_{j,\mathbf{k}}(\mathbf{t}),$$

where $\beta_{j,\mathbf{k},u}$ and $\alpha_{j,\mathbf{k}}$ are the new random fields defined as

$$\begin{aligned} \alpha_{j,\mathbf{k}} &= \int_{\mathbb{R}^d} X(\mathbf{t})\Phi_{j,\mathbf{k}}(\mathbf{t})d\mathbf{t}, \\ \beta_{j,\mathbf{k},u} &= \int_{\mathbb{R}^d} X(\mathbf{t})\Psi_{j,\mathbf{k},u}(\mathbf{t})d\mathbf{t}. \end{aligned}$$

In many practical situations it is required to analyses wavelet transforms of discrete parameter random fields, for example, in digital image processing. In the next, we develop an alternative procedure in which a continuous random fields is first generated by interpolation of the discrete process under study that preserves stationarity, linearity and moments of the discrete process.

Let $(X(\mathbf{n}))_{\mathbf{n}\in\mathbb{Z}^d}$ be a zero mean discrete parameter random fields having finite power spectrum $f(\boldsymbol{\lambda})$, where $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_d) \in \boldsymbol{\pi} = [-\pi, \pi[\times ... \times [-\pi, \pi[$, $d-$times. We now construct a new continuous parameter random fields, $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{R}^d}$ as

$$X(\mathbf{t}) = \sum_{\mathbf{n}\in\mathbb{Z}^d} X(\mathbf{n})\gamma(\mathbf{t}-\mathbf{n}), \tag{4.3.1}$$

where $\gamma(.)$ belongs to a family of scaling functions.

Lemma 4.1 *Let $(X(\mathbf{n}))_{\mathbf{n}\in\mathbb{Z}^d}$ be a zero mean stationary discrete random fields with spectral density $f^d(\boldsymbol{\lambda})$. Then the covariance $C^c(\mathbf{t},\mathbf{s})$ for random fields satisfies* (4.3.1) *and having finite power spectrum $f^c(\boldsymbol{\lambda})$ is given by*

$$C^c(\mathbf{t},\mathbf{s}) = \sum_{\mathbf{l}\in\mathbb{Z}^d} C^d(\mathbf{l}) \sum_{\mathbf{n}\in\mathbb{Z}^d} \gamma(\mathbf{t}-\mathbf{n})\gamma^*(\mathbf{s}-\mathbf{n}-\mathbf{l}),$$

where $C^d(\mathbf{l}) = E\{X(\mathbf{n})X(\mathbf{l}+\mathbf{n})\}$, and

$$f^c(\boldsymbol{\lambda}) = (2\pi)^d |\Gamma(-\boldsymbol{\lambda})|^2 f^d(\boldsymbol{\lambda}), \tag{4.3.2}$$

where $\Gamma(\boldsymbol{\lambda})$ is the Fourier transform of $\gamma(\mathbf{t})$.

Proof. Straightforward and hence omitted.

Lemma 4.2 *Let $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{R}^d}$ be a stationary continuous random fields satisfies* (4.3.1) *and having finite power bispectrum $f^c(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)$. Then*

$$\begin{aligned} E\{X(\mathbf{t})X(\mathbf{t}+\mathbf{s})X(\mathbf{t}+\mathbf{r})\} &= \sum_{\mathbf{n}\in\mathbb{Z}^d}\sum_{\mathbf{m}\in\mathbb{Z}^d} E\{X(\mathbf{l})X(\mathbf{l}+\mathbf{m})X(\mathbf{l}+\mathbf{n})\} \frac{1}{\sqrt{(2\pi)^d}} \\ &\int_{\boldsymbol{\pi}}\int_{\boldsymbol{\pi}} \Gamma(\boldsymbol{\lambda}_1)\Gamma(\boldsymbol{\lambda}_2)\Gamma(-\boldsymbol{\lambda}_1-\boldsymbol{\lambda}_2) \times e^{i[(\mathbf{n}-\mathbf{r})\boldsymbol{\lambda}_1+(\mathbf{n}-\mathbf{r}+\mathbf{s}-\mathbf{m})\boldsymbol{\lambda}_2} d\boldsymbol{\lambda}_1 d\boldsymbol{\lambda}_2, \end{aligned}$$

and

$$f^c(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) = \left(\sqrt{(2\pi)^d}\right)^3 \Gamma(-\boldsymbol{\lambda}_1)\,\Gamma(-\boldsymbol{\lambda}_2)\,\Gamma(\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2)\, f^d(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2), \tag{4.3.3}$$

where $f^d(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)$ *is bispectral density of the discrete random fields.*

Proposition 4.1 *From* (4.3.2) *and* (4.3.3)*, we have*

- $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{R}^d}$ *is second order stationary whenever* $(X(\mathbf{n}))_{\mathbf{n}\in\mathbb{Z}^d}$ *is second order stationary.*
- $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{R}^d}$ *is third order stationary whenever* $(X(\mathbf{n}))_{\mathbf{n}\in\mathbb{Z}^d}$ *is third order stationary.*
- *If* $(X(\mathbf{n}))_{\mathbf{n}\in\mathbb{Z}^d}$ *is Gaussian* $f^d(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) = \mathbf{0}$ *and hence* $f^c(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) = \mathbf{0}$*. (Gaussianity of* $X(\mathbf{n})_{\mathbf{n}\in\mathbb{Z}^d}$ *implies Gaussianity of* $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{R}^d}$*).*
- *If* $(X(\mathbf{n}))_{\mathbf{n}\in\mathbb{Z}^d}$ *is linear random fields, then* $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{R}^d}$ *is linear random fields.*

Example 4.3 *let* $\gamma(\mathbf{t}) = \prod_{i=1}^d \frac{\sin(\pi.t_i)}{(\pi.t_i)}$*, we have*

$$X(\mathbf{t}) = \sum_{\mathbf{n}\in\mathbb{Z}^d} X(\mathbf{n}) \prod_{i=1}^d \frac{\sin[\pi.(t_i - n_i)]}{[\pi.(t_i - n_i)]}, \tag{4.3.4}$$

For this choice of $\gamma(.)$*, we observe the following*

- $\Gamma(\boldsymbol{\lambda}) = (2\pi)^{-d/2}$.
- $C^c(\boldsymbol{\tau}) = \sum_{\mathbf{l}\in\mathbb{Z}^d} C^d(\mathbf{l})[\prod_{i=1}^d \frac{\sin[\pi.(\tau_i - l_i)]}{[\pi.(\tau_i - l_i)]}]$ *and hence ,* $f^c(\boldsymbol{\lambda}) = f^d(\boldsymbol{\lambda})$.
- $E\{X(\mathbf{t})X(\mathbf{t}+\mathbf{s})X(\mathbf{t}+\mathbf{r})\} = \sum_{\mathbf{m}\in\mathbb{Z}^d}\sum_{\mathbf{n}\in\mathbb{Z}^d} E\{X(\mathbf{l})X(\mathbf{l}+\mathbf{m})X(\mathbf{l}+\mathbf{n})\}$ $\left[\prod_{i=1}^d \frac{\sin\pi.(r_i-n_i)}{\pi.(r_i-n_i)}\right]\left[\prod_{i=1}^d \frac{\sin\pi.(r_i-n_i+m_i-s_i)}{\pi.(r_i-n_i+m_i-s_i)}\right]$ *and hence* $f^c(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) = f^d(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)$.

The above relation show that the covariance functions of the continuous and discrete processes are related and the spectra (power spectrum and bispectrum) of continuous and discrete random fields are identical in the range.

Remark 4.4 *1) Since for only* $\gamma(\mathbf{t}) = \prod_{i=1}^d \frac{\sin[\pi.(t_i-n_i)]}{[\pi.(t_i-n_i)]}, \Gamma(\boldsymbol{\lambda}) = (2\pi)^{-d/2}$ *, it is evident from* (4.3.2) *and* (4.3.3) *that for any other choice of scaling function* $\gamma(\mathbf{t})$*, the spectral density function and the bispectral density function of continuous and discrete random fields are not identical, although they preserve the two important properties of linearity and stationarity.*

2) Any choice of scaling function other than $\gamma(\mathbf{t}) = \prod_{i=1}^d \frac{\sin[\pi.(t_i-n_i)]}{[\pi.(t_i-n_i)]}$ *result in a continuous random fields which does not have the same properties as that of the discrete random field under study. This indicates that the use of discrete random field itself as an initial set of scaling coefficients to compute wavelet transform may not be an optimum procedure.*

Wavelet representation of discrete random fields

Let $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{R}^d}$ be a continuous random fields constructed from a discrete random filed having zero mean and finite power, and let $\left\{\Psi_{j,\mathbf{k},u}(\mathbf{t}), u=1,...,2^d-1, \mathbf{k}\in\mathbb{Z}^d, j\in\mathbb{Z}\right\}$ and $\left\{\Phi_{j,\mathbf{k}}(\mathbf{t}); \mathbf{k}\in\mathbb{Z}^d\right\}$ be multiresolution wavelet and scaling orthonormal bases. Further, let us assume $X(\mathbf{t})\in V_{\mathbf{0}}$, then from (4.2.6), we have

$$X(\mathbf{t}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} \alpha_{0,\mathbf{k}}\Phi_{0,\mathbf{k}}(\mathbf{t}),$$

where $\alpha_{0,\mathbf{k}}$ denote the scaling coefficient at zeroth resolution, given by

$$\alpha_{0,\mathbf{k}} = \int_{\mathbb{R}^d} X(\mathbf{t})\Phi_{0,\mathbf{k}}(\mathbf{t})d\mathbf{t}.$$

Substituting for $X(\mathbf{t})$ in terms of $X(\mathbf{n})$ by using (4.3.4), and rearranging, we obtain

$$\alpha_{0,\mathbf{k}} = \sum_{\mathbf{n}\in\mathbb{Z}^d} X(\mathbf{n})b_{\mathbf{k}-\mathbf{n}}, \tag{4.3.5}$$

where the sequence $(b_{\mathbf{m}})$ is computed as

$$b_{\mathbf{m}} = (2\pi)^{d/2}\int_{\boldsymbol{\pi}} \widetilde{\Phi}(-\boldsymbol{\lambda})e^{i\mathbf{m}.\boldsymbol{\lambda}}d\boldsymbol{\lambda}, \tag{4.3.6}$$

with $\widetilde{\Phi}(\boldsymbol{\lambda})$ indicating the Fourier transform of $\Phi(\mathbf{t})$.

The relation given by (4.3.5) shows how amenable it is to theoretically analyse a discrete random field when these processes can be represented by expressions in closed form. The random field $(b_{\mathbf{m}})$ can be precomputed for any particular wavelet transform and used in the analysis of discrete random field. For orthonormal multiresolution scaling functions, $\widetilde{\Phi}(\boldsymbol{\lambda})$ is given by

$$\widetilde{\Phi}_J(\boldsymbol{\lambda}) = \frac{1}{(2\pi)^{\frac{d}{2}}}\prod_{j=1}^{J}\frac{1}{2^{\frac{d}{2}}}\sum_{\mathbf{m}\in\mathbb{Z}^d} h_{\mathbf{m}}e^{-i2^{-j}\mathbf{m}.\boldsymbol{\lambda}}, \tag{4.3.7}$$

when $j=\infty, \widetilde{\Phi}_J(\boldsymbol{\lambda}) = \widetilde{\boldsymbol{\Phi}}(\boldsymbol{\lambda})$. However, for large values of J, $\widetilde{\Phi}_J(\boldsymbol{\lambda}) \approx \widetilde{\boldsymbol{\Phi}}(\boldsymbol{\lambda})$.

4.4 Covariances structure

4.4.1 Second order covariances

The second order properties of wavelet coefficients for continuous and discrete parameter random processes have been studied in Dijkerman and Mazumdar (1994), Mary (1993), Tewfik (1992) and

Subba Rao and Indukumar (1996). However, in this section, we obtain explicit expressions for the second order covariances of discrete random fields.

The following lemma is an extension of the result obtained by Subba Rao and Indukumar (1996) in one dimensional framework.

Lemma 4.3 *Let* $(X(\mathbf{n}))_{\mathbf{n}\in\mathbb{Z}^d}$ *be a stationary zero mean discrete random fields belongs to* $V_\mathbf{0}$ *space and let* $C(\mathbf{l}) =: Cov\left\{X\left(\mathbf{n}\right)X\left(\mathbf{n}+\mathbf{l}\right)\right\}$ *the covariance of the discrete random field. Then the covariance of scaling coefficient at Zeroth resolution* $C_0^s(\mathbf{l}) =: E\left\{\alpha_{0,\mathbf{k}}\alpha_{0,\mathbf{k}+\mathbf{l}}^*\right\}$ *is*

$$C_0^s(\mathbf{l}) = \sum_{\mathbf{m}\in\mathbb{Z}^d}\sum_{\mathbf{n}\in\mathbb{Z}^d} b_\mathbf{m} b_\mathbf{n}^* C(\mathbf{l}+\mathbf{m}-\mathbf{n}), \tag{4.4.1}$$

where $b_\mathbf{m}$ *defined by* (4.3.6).

Proof. Straightforward and hence omitted

Example 4.4 *Consider the spatial* $AR(1,1)$ *process*

$$X\left(\mathbf{t}\right) = a_1 X\left(\mathbf{t}-\mathbf{e}_1\right) + a_2 X\left(\mathbf{t}-\mathbf{e}_2\right) - a_1 a_2 X\left(\mathbf{t}-\mathbf{1}\right) + e\left(\mathbf{t}\right), \tag{4.4.2}$$

where $(e(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^2}$ *is a Gaussian white noise with zero mean and variance* $\sigma^2 = 1$ *and Max* $\{|a_1|, |a_2|\} < 1$. *Then the Model* (4.4.2) *has a regular second order stationary solution if* $0 < (1-a_1^2)(1-a_2^2)$. *Under these conditions,*

$$C(h_1,h_2) = \frac{(a_1+a_2-a_1a_2)^{|h_1|+|h_2|}}{(1-a_1^2)(1-a_2^2)},$$

and covariance of scaling coefficients at zeroth resolution is given by

$$C_0^s(h_1,h_2) = \sum_{m_1,m_2\in\mathbb{Z}}\sum_{n_1,n_2\in\mathbb{Z}} b_{m_1,m_2} b_{n_1,n_2}^* \frac{(a_1+a_2-a_1a_2)^{|h_1+m_1-n_1|+|h_2+m_2-n_2|}}{(1-a_1^2)(1-a_2^2)}.$$

Equation (4.4.1) measures the linear relationship at the zeroth resolution, and since the coefficients at different resolutions are related, we can evaluate covariances at lower resolutions as follows. We know from (4.2.12) that the scaling coefficients and the wavelet coefficient at zeroth resolution and the next lower resolution are related by the following lemma

Lemma 4.4 *Let* $\alpha_{1,\mathbf{k}}$ *and* $\beta_{1,\mathbf{k},u}$ *are the scaling and the wavelet coefficients at the first resolution defined by*

$$\begin{aligned} \alpha_{1,\mathbf{k}} &= \sum_{\mathbf{k}\in\mathbb{Z}^d} h_\mathbf{m}\alpha_{0,\mathbf{m}+2\mathbf{k}}, \qquad (4.4.3) \\ \beta_{1,\mathbf{k},u} &= \sum_{\mathbf{k}\in\mathbb{Z}^d} g_\mathbf{m}^{(u)}\alpha_{0,\mathbf{m}+2\mathbf{k}} \text{ for } u = 1,...,2^d-1. \end{aligned}$$

Then the covariance of the scaling and the wavelet coefficient satisfying (4.4.3) *is*

$$\begin{aligned} C_1^s(\mathbf{l}) &= \sum_{\mathbf{m}\in\mathbb{Z}^d}\sum_{\mathbf{n}\in\mathbb{Z}^d} h_{\mathbf{m}} h_{\mathbf{n}}^* C_0^s(\mathbf{n}-\mathbf{m}+2\mathbf{l}), \\ C_{1,u}^w(\mathbf{l}) &= \sum_{\mathbf{m}\in\mathbb{Z}^d}\sum_{\mathbf{n}\in\mathbb{Z}^d} g_{\mathbf{m}}^{(u)} g_{\mathbf{n}}^{(u)*} C_0^s(\mathbf{n}-\mathbf{m}+2\mathbf{l}) \text{ for } u=1,...,2^d-1. \end{aligned} \tag{4.4.4}$$

Proof. Straightforward and hence omitted

Corollary 4.1 *Let* $\alpha_{r,\mathbf{k}}$ *and* $\beta_{r,\mathbf{k},u}$ *are the scaling and the wavelet coefficients at the rth resolution given by the recursion formula given by* (4.4.3)*. Then the covariance function at lower resolutions of the scaling and the wavelet coefficients are*

$$\begin{aligned} C_r^s(\mathbf{l}) &= \sum_{\mathbf{m}\in\mathbb{Z}^d}\sum_{\mathbf{n}\in\mathbb{Z}^d} h_{\mathbf{m}} h_{\mathbf{n}}^* C_{r-1}^s(\mathbf{n}-\mathbf{m}+2\mathbf{l}), \\ C_{r,u}^w(\mathbf{l}) &= \sum_{\mathbf{m}\in\mathbb{Z}^d}\sum_{\mathbf{n}\in\mathbb{Z}^d} g_{\mathbf{m}}^{(u)} g_{\mathbf{n}}^{(u)*} C_{r-1}^s(\mathbf{n}-\mathbf{m}+2\mathbf{l}) \text{ for } u=1,...,2^d-1. \end{aligned} \tag{4.4.5}$$

Remark 4.5 *For any rth resolution, we can write* (4.4.5) *as*

$$\begin{aligned} C_r^s(\mathbf{l}) &= \left|\sum_{\mathbf{m}\in\mathbb{Z}^d} h_{\mathbf{m}-2\mathbf{k}}\right|^{2r} C_0^s(\mathbf{l}), \\ C_{r,u}^w(\mathbf{l}) &= \left|\sum_{\mathbf{m}\in\mathbb{Z}^d} g_{\mathbf{m}-2\mathbf{k}}^{(u)}\right|^{2r} C_0^s(\mathbf{l}) \text{ for } u=1,...,2^d-1. \end{aligned}$$

Example 4.5 *Consider the* $2-D$ *harmonic process as*

$$X(\mathbf{t}) = A\cos(\boldsymbol{\omega}.\mathbf{t}+\theta), \text{ for } \mathbf{t}\in\mathbb{Z}^2, \tag{4.4.6}$$

where $\boldsymbol{\omega}\in[-\pi,\pi[^2$ *and the Phase* $\theta\in[-\pi,\pi[$*. Then*

$$C(\mathbf{l}) = \frac{A^2}{2}\cos(\boldsymbol{\omega}.\mathbf{l}), \text{ where } \mathbf{l}\in\mathbb{Z}^2,$$

and covariance of scaling coefficients at rth resolution is given by

$$C_r^s(\mathbf{l}) = \frac{A^2}{2}\left|\sum_{\mathbf{m}\in\mathbb{Z}^2} b_{\mathbf{m}} e^{i(\boldsymbol{\omega}.\mathbf{m})}\right|^2 \left|\sum_{\mathbf{m}\in\mathbb{Z}^2} h_{\mathbf{m}} e^{i(\boldsymbol{\omega}.\mathbf{m})}\right|^{2r} \cos[2^r(\boldsymbol{\omega}.\mathbf{l})],$$

and for the wavelet functions as

$$C^{w}_{r,u}(\mathbf{l}) = \frac{A^2}{2}\left|\sum_{\mathbf{m}\in\mathbb{Z}^2} b_{\mathbf{m}} e^{i(\boldsymbol{\omega}.\mathbf{m})}\right|^2 \left|\sum_{\mathbf{m}\in\mathbb{Z}^2} h_{\mathbf{m}} e^{i(\boldsymbol{\omega}.\mathbf{m})}\right|^{2(r-1)} \cdot \left|\sum_{\mathbf{m}\in\mathbb{Z}^2} g^{(u)}_{\mathbf{m}} e^{i(\boldsymbol{\omega}.\mathbf{m})}\right|^2 \cos[2^r(\boldsymbol{\omega}.\mathbf{l})], u = \{h, v, d\},$$

where

$$g^{(h)}_{\mathbf{m}} = h_{m_1} g_{m_2}, g^{(v)}_{\mathbf{m}} = g_{m_1} h_{m_2}, g^{(d)}_{\mathbf{m}} = g_{m_1} g_{m_2}.$$

4.4.2 Third order covariances

It is clear from (4.4.1) that the scaling and wavelet covariance for any two different stationary spatial processes having the same covariance cannot be distinguished. For example $SARMA(\mathbf{p}, \mathbf{q})$ processes and $SBL_d(\mathbf{p}, \mathbf{q}, \mathbf{P}, \mathbf{Q})$ have the same covariance structure (chap 1). This shows that linear and nonlinear processes cannot be identified using second order statistics. This makes it necessary to study higher order statistics in wavelet analysis.

In this section, we develop expression for the third order covariances between wavelet and scaling coefficient for discrete random fields.

Lemma 4.5 *Let $(X(\mathbf{n}))_{\mathbf{n}\in\mathbb{Z}^d}$ be a zero mean discrete random fields belongs to $V_{\mathbf{0}}$ space then third order covariance of scaling coefficient at Zeroth resolution which related to third order covariance of the discrete random field as*

$$E\{\alpha_{0,\mathbf{k}}\alpha_{0,\mathbf{k+p}}\alpha_{0,\mathbf{k+q}}\} = \sum_{\mathbf{l}\in\mathbb{Z}^d}\sum_{\mathbf{m}\in\mathbb{Z}^d}\sum_{\mathbf{n}\in\mathbb{Z}^d} E\{X(\mathbf{l})X(\mathbf{l}+\mathbf{m})X(\mathbf{l}+\mathbf{n})\} b_{\mathbf{k-l}} b_{\mathbf{k+p-l-m}} b_{\mathbf{k+q-l-n}}, \quad (4.4.7)$$

where

$$\sum_{\mathbf{l}\in\mathbb{Z}^d} b_{\mathbf{k-l}} b_{\mathbf{k+p-l-m}} b_{\mathbf{k+q-l-n}} = \frac{1}{\sqrt{(2\pi)^d}} \int_{\boldsymbol{\pi}}\int_{\boldsymbol{\pi}} \widetilde{\Phi}(-\boldsymbol{\lambda}_1)\,\widetilde{\Phi}(-\boldsymbol{\lambda}_2)\,\widetilde{\Phi}(\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2)\, e^{i[(\mathbf{n-q}).\boldsymbol{\lambda}_1 + (\mathbf{p-m-q+n}).\boldsymbol{\lambda}_2]} d\boldsymbol{\lambda}_1 d\boldsymbol{\lambda}_2. \quad (4.4.8)$$

Proof. Straightforward and hence omitted

Remark 4.6 *Note that* (4.4.8) *is independent of* $\mathbf{k}$ *and* $\mathbf{l}$, *indicating that* $E\{\alpha_{0,\mathbf{k}}\alpha_{0,\mathbf{k+p}}\alpha_{0,\mathbf{k+q}}\}$ *independent of* $\mathbf{k}$. *This implies that the random fields* $(\alpha_{0,\mathbf{k}})$ *is a third order stationary whenever* $(X(\mathbf{n}))_{\mathbf{n}\in\mathbb{Z}^d}$ *is a third order stationary random fields.*

Corollary 4.2 *Let $\alpha_{0,\mathbf{k}}$ the scaling coefficient at Zero resolution satisfying* (4.3.5)*. Then the third order covariance of scaling and wavelet coefficients at rth resolution is given by*

$$\begin{aligned} E\left\{\alpha_{r,\mathbf{k}}\alpha_{r,\mathbf{k+j}}\alpha_{r,\mathbf{k+l}}\right\} &= \sum_{\mathbf{m}\in\mathbb{Z}^d}\sum_{\mathbf{n}\in\mathbb{Z}^d}\sum_{\mathbf{p}\in\mathbb{Z}^d} h_{\mathbf{m}}h_{\mathbf{n}}h_{\mathbf{p}}E\left\{\alpha_{r-1,\mathbf{m+2k}}\alpha_{r-1,\mathbf{n+2j}}\alpha_{r-1,\mathbf{p+2l}}\right\}, \quad (4.4.9)\\ E\left\{\beta_{r,\mathbf{k},u}\beta_{r,\mathbf{k+j},u}\beta_{r,\mathbf{k+l},u}\right\} &= \sum_{\mathbf{m}\in\mathbb{Z}^d}\sum_{\mathbf{n}\in\mathbb{Z}^d}\sum_{\mathbf{p}\in\mathbb{Z}^d} g^{(u)}_{\mathbf{m}}g^{(u)}_{\mathbf{n}}g^{(u)}_{\mathbf{p}}E\left\{\alpha_{r-1,\mathbf{m+2k}}\alpha_{r-1,\mathbf{n+2j}}\alpha_{r-1,\mathbf{p+2l}}\right\},\\ \mathit{for}\ u &= 1,...,2^d-1. \end{aligned}$$

Remark 4.7 *From the relation between scaling and wavelet coefficient, we can write* (4.4.9) *as*

$$\begin{aligned} E\left\{\alpha_{r,\mathbf{k}}\alpha_{r,\mathbf{k+j}}\alpha_{r,\mathbf{k+l}}\right\} &= \left|\sum_{\mathbf{m}\in\mathbb{Z}^d} h_{\mathbf{m-2k}}\right|^r \left|\sum_{\mathbf{n}\in\mathbb{Z}^d} h_{\mathbf{n-2j}}\right|^r \left|\sum_{\mathbf{p}\in\mathbb{Z}^d} h_{\mathbf{p-2l}}\right|^r E\left\{\alpha_{0,\mathbf{k}}\alpha_{0,\mathbf{k+j}}\alpha_{0,\mathbf{k+l}}\right\},\\ E\left\{\beta_{r,\mathbf{k},u}\beta_{r,\mathbf{k+j},u}\beta_{r,\mathbf{k+l},u}\right\} &= \left|\sum_{\mathbf{m}\in\mathbb{Z}^d} g^{(u)}_{\mathbf{m-2k}}\right|^r \left|\sum_{\mathbf{n}\in\mathbb{Z}^d} g^{(u)}_{\mathbf{n-2j}}\right|^r \left|\sum_{\mathbf{p}\in\mathbb{Z}^d} g^{(u)}_{\mathbf{p-2l}}\right|^r E\left\{\alpha_{0,\mathbf{k}}\alpha_{0,\mathbf{k+j}}\alpha_{0,\mathbf{k+l}}\right\},\\ \mathit{for}\ u &= 1,...,2^d-1. \end{aligned}$$

4.4.3 Dependence structure in terms of cumulants

The dependence structure of Gaussian random fields is entirely characterized by the covariance. When the normality assumption no longer holds, higher order cumulants are necessary. The covariance and spectral properties of the wavelet transform and discrete wavelet coefficient for random fields have been extensively studied in the past (see Masry (1998)). In this section, we obtain a new expression for cumulant of the scaling coefficient, and the dependence structure between wavelet coefficients is closely related to the dependence of scaling coefficients. Hence, to explain how to obtain the joint cumulants of the scaling coefficients from the joint cumulants of random fields, we have

Proposition 4.2 *Let $(\alpha_{0,\mathbf{k}})_{\mathbf{t}\in\mathbb{Z}^d}$ be the scaling coefficient at Zero resolution satisfies* (4.3.5)*, and suppose that joint cumulants of $(X(\mathbf{n}))_{\mathbf{n}\in\mathbb{Z}^d}$ of order s exist. Then*

$$Cum\left(\alpha_{0,\mathbf{k}_1},...,\alpha_{0,\mathbf{k}_s}\right) = \sum_{\mathbf{n}_1\in\mathbb{Z}^d}\cdots\sum_{\mathbf{n}_s\in\mathbb{Z}^d}\prod_{i=1}^{s} b_{\mathbf{k}_i-\mathbf{n}_i}Cum(X(\mathbf{n}_1),...,X(\mathbf{n}_s)).$$

Corollary 4.3 *Let* $(\alpha_{1,\mathbf{k}})$ *and* $(\beta_{1,\mathbf{k},u})$ *are the scaling and wavelet coefficients at the first resolution satisfies* (4.2.12). *Then*

$$\begin{aligned} Cum(\underline{\alpha_{1,\mathbf{k}}}) &= \sum_{\mathbf{m}_1\in\mathbb{Z}^d}\cdots\sum_{\mathbf{m}_s\in\mathbb{Z}^d}\prod_{i=1}^{s} h_{\mathbf{m}_i} Cum(\underline{\alpha_{0,\mathbf{m}+2\mathbf{k}}}), \\ Cum(\underline{\beta_{1,\mathbf{k},u}}) &= \sum_{\mathbf{m}_1\in\mathbb{Z}^d}\cdots\sum_{\mathbf{m}_s\in\mathbb{Z}^d}\prod_{i=1}^{s} g^{(u)}_{\mathbf{m}_i} Cum(\underline{\alpha_{0,\mathbf{m}+2\mathbf{k}}}) \text{ for } u = 1,...,2^d-1, \end{aligned}$$

where

$$\begin{aligned} Cum(\underline{\alpha_{1,\mathbf{k}}}) &= Cum(\alpha_{1,\mathbf{k}_1},...,\alpha_{1,\mathbf{k}_s}), \\ Cum(\underline{\beta_{1,\mathbf{k},u}}) &= Cum\left(\beta_{1,\mathbf{k}_1,u},,...,\beta_{1,\mathbf{k}_s,u}\right). \end{aligned}$$

Corollary 4.4 *Let* $(\alpha_{r,\mathbf{k}})$ *and* $(\beta_{r,\mathbf{k},u})$ *are the scaling and wavelet coefficients at the* r*th resolution satisfies* (4.2.12). *Then*

$$\begin{aligned} Cum_r(\underline{\alpha_{r,\mathbf{k}}}) &= \sum_{\mathbf{m}_1\in\mathbb{Z}^d}\cdots\sum_{\mathbf{m}_s\in\mathbb{Z}^d}\prod_{i=1}^{s} h_{\mathbf{m}_i} Cum_{r-1}(\underline{\alpha_{r-1,\mathbf{m}+2\mathbf{k}}}), \\ Cum_r(\underline{\beta_{r,\mathbf{m},u}}) &= \sum_{\mathbf{m}_1\in\mathbb{Z}^d}\cdots\sum_{\mathbf{m}_s\in\mathbb{Z}^d}\prod_{i=1}^{s} g^{(u)}_{\mathbf{m}_i} Cum_{r-1}(\underline{\alpha_{r-1,\mathbf{m}+2\mathbf{k}}}) \text{ for } u = 1,...,2^d-1. \end{aligned}$$

4.5 The discrete wavelet transform

Let $(X(\mathbf{n}))_{\mathbf{n}\in\mathbb{Z}^d}$ be a zero mean discrete stationary random fields, we define the discrete wavelet transform with respect to Ψ as

$$d_{j,\mathbf{k}}(\mathbf{n}) = 2^{-jd/2}\sum_{\mathbf{n}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} X(\mathbf{n})\,\Psi(2^{-j}\mathbf{n}-\mathbf{k}), \tag{4.5.1}$$

Condition 4.1 *Let* $C(\mathbf{l})$ *be a covariance function of a random fields* $(X(\mathbf{n}))_{\mathbf{n}\in\mathbb{Z}^d}$ *satisfies*

$$\sum_{\mathbf{l}\in\mathbb{Z}^d}[1+|\mathbf{l}|]\,|C(\mathbf{l})| < \infty,$$

We have $E\{d_{j,\mathbf{k}}(\mathbf{n})\} = 0$ and

$$\begin{aligned} var\{d_{j,\mathbf{k}}(\mathbf{n})\} &= 2^{dj}\sum_{\mathbf{n}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}}\sum_{\mathbf{m}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}} C(\mathbf{n}-\mathbf{m})\,\Psi(2^{-j}\mathbf{n}-\mathbf{k})\Psi(2^{-j}\mathbf{m}-\mathbf{k}) \\ &= 2^{dj}\sum_{\mathbf{l}=-\mathbf{N}+\mathbf{1}}^{\mathbf{N}-\mathbf{1}} C(\mathbf{l})\,\widetilde{\Psi}(|\underline{\mathbf{l}}|), \end{aligned}$$

where

$$\widetilde{\Psi}(|\underline{\mathbf{l}}|) = \sum_{\mathbf{n}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}-|\mathbf{l}|} \Psi(2^{-j}\mathbf{n}-\mathbf{k})\Psi(2^{-j}\mathbf{n}+|\underline{\mathbf{l}}|-\mathbf{k}).$$

If Condition 4.1 holds, then

$$var\left\{d_{j,\mathbf{k}}(\mathbf{n})\right\} \longrightarrow \eta_{j,\mathbf{k}}(\mathbf{n}) \text{ as } \mathbf{N} \longrightarrow \infty,$$

where

$$\eta_{j,\mathbf{k}}(\mathbf{n}) = 2^{dj} \sum_{\mathbf{l}\in\mathbb{Z}^d} C\left(\mathbf{l}\right) \widetilde{\Psi}_{\infty}\left(|\underline{\mathbf{l}}|\right), \tag{4.5.2}$$

is the wavelet spectrum of $(X\left(\mathbf{n}\right))_{\mathbf{n}\in\mathbb{Z}^d}$ and

$$\widetilde{\Psi}_{\infty}\left(|\underline{\mathbf{l}}|\right) = \sum_{\mathbf{n}=\mathbf{0}}^{\infty} \Psi(2^{-j}\mathbf{n}-\mathbf{k})\Psi(2^{-j}\mathbf{n}+|\underline{\mathbf{l}}|-\mathbf{k}),$$

is called the wavelet autocorrelation function, at $(j, \mathbf{k})$.

Theorem 4.1 *Let $(X\left(\mathbf{n}\right))_{\mathbf{n}\in\mathbb{Z}^d}$ be a zero mean discrete stationary random fields, with covariance function satisfies Condition 4.1. Then $\eta_{j,\mathbf{k}}(\mathbf{n})$ is bounded and non-negative.*

Proof. The proof follows from similar arguments as Theorem 1 in Chiann and Morettin (1998).

Theorem 4.2 *Let $(X\left(\mathbf{n}\right))_{\mathbf{n}\in\mathbb{Z}^d}$ be a zero mean discrete stationary random fields, with covariance function $C\left(\mathbf{l}\right)$ satisfies Condition 4.1, and let*

$$\eta_{(j_1,j_2),(\mathbf{k}_1,\mathbf{k}_2)}(\mathbf{n}) = 2^{d(j_1+j_2)/2} \sum_{\mathbf{l}\in\mathbb{Z}^d} \sum_{\mathbf{n}=\mathbf{0}}^{\infty} \gamma\left(\mathbf{l}\right) \Psi(2^{-j_1}\mathbf{n}-\mathbf{k}_1+|\underline{\mathbf{l}}|\, I_{\{\mathbf{u}\succ\mathbf{0}\}})\Psi(2^{-j_2}\mathbf{n}-\mathbf{k}_2+|\underline{\mathbf{l}}|\, I_{\{\mathbf{u}\preceq\mathbf{0}\}}),$$

for $(j_1, j_2) \in \mathbb{Z}^2, (\mathbf{k}_1, \mathbf{k}_2) \in \mathbb{Z}^{2d}$, the covariance of the wavelet transform with respect to Ψ. Then,

i) $E\left\{d_{j_1,\mathbf{k}_1}(\mathbf{n})d_{j_2,\mathbf{k}_2}(\mathbf{n})\right\} \longrightarrow \eta_{(j_1,j_2),(\mathbf{k}_1,\mathbf{k}_2)}(\mathbf{n})$ *as* $\mathbf{N} \longrightarrow \infty$,

ii) *If* $j_1 = j_2, \mathbf{k}_1 = \mathbf{k}_2$, *then* $\eta_{(j_1,j_2),(\mathbf{k}_1,\mathbf{k}_2)}(\mathbf{n}) = \eta_{j,\mathbf{k}}(\mathbf{n})$,

iii) $E\left\{d_{j_1,\mathbf{k}_1}(\mathbf{n})d_{j_2,\mathbf{k}_2}(\mathbf{n})\right\} = O\left(1\right)$ *as* $\mathbf{N} \longrightarrow \infty$.

Proof. The proof is similar as that in Theorem 2 in Chiann and Morettin (1998).

Theorem 4.3 *Let* $(X(\mathbf{n}))_{\mathbf{n}\in\mathbb{Z}^d}$ *is a second order stationary random fields with zero mean and the covariance function* $C(\mathbf{l}) \neq 0$ *for* $|\underline{\mathbf{l}}| \preceq \mathbf{L}$, $\mathbf{L} \prec\prec \mathbf{N}$. *If* $\Psi(\mathbf{n})$ *has support* $[K_1, K_2]$, *where* $K_1 \geq 0, K_2 \geq 0$, *then*

$$E\left\{d_{j_1,\mathbf{k}_1}(\mathbf{n})d_{j_2,\mathbf{k}_2}(\mathbf{n})\right\} = 0,$$

for $|\mathbf{k}_1 - \mathbf{k}_2| \succ K_2 - K_1 + (2^j\mathbf{L})$, $j = 1, ..., M$ *and* $k = 0, 1, ..., (2^{M-j} - 1)$.

Proof. The proof is similar as that in Theorem 3 in Chiann and Morettin (1998).

Definition 4.4 *A real-valued second-order random field* $(X(\mathbf{n}))_{\mathbf{n}\in\mathbb{Z}^d}$ *is said to be weakly homogeneous if*

i) $m(\mathbf{n}) = E\{X(\mathbf{n})\}$ for all $\mathbf{n} \in \mathbb{Z}^d$.

ii) $C(\mathbf{n}+\mathbf{u}, \mathbf{m}+\mathbf{u}) = C(\mathbf{n}, \mathbf{m})$ for all $\mathbf{u} \in \mathbb{Z}^d$.

We assume that $C(\mathbf{l})$ is continuous and has the spectral representation

$$C(\mathbf{l}) = \int_{\boldsymbol{\pi}} e^{i\mathbf{l}.\boldsymbol{\lambda}} dF(\boldsymbol{\lambda}),$$

where $dF(\boldsymbol{\lambda})$ is a finite measure on $\boldsymbol{\pi}$.

Theorem 4.4 *Assume that* $\Psi(\mathbf{n}) \in L(\mathbb{R}^d)$. *The random fields* $(d_{j,\mathbf{k}}(\mathbf{n}))_{\mathbf{n}\in\mathbb{Z}^d}$, *are jointly weakly homogeneous with zero means and covariance/ cross-covariance functions*

$$\begin{aligned} C_{d_{j,\mathbf{k}}}(\mathbf{l}) &= E\left\{d_{j,\mathbf{k}}(\mathbf{n})d^*_{j,\mathbf{k}}(\mathbf{n}+\mathbf{l})\right\}, \\ C_{d_{j_1,\mathbf{k}_1},d_{j_2,\mathbf{k}_2}}(\mathbf{l}) &= E\left\{d_{j_1,\mathbf{k}_1}(\mathbf{n})d^*_{j_2,\mathbf{k}_2}(\mathbf{n}+\mathbf{l})\right\}, \end{aligned}$$

having the spectral representations

$$\begin{aligned} C_{d_{j,\mathbf{k}}}(\mathbf{l}) &= 2^{dj} \sum_{\mathbf{l}=-\mathbf{N}-\mathbf{1}}^{\mathbf{N}-\mathbf{1}} \int_{\boldsymbol{\pi}} \widetilde{\boldsymbol{\Psi}}(|\underline{\mathbf{l}}|) e^{i\mathbf{l}.\boldsymbol{\lambda}} dF_x(\boldsymbol{\lambda}), \\ C_{d_{j_1,\mathbf{k}_1},d_{j_2,\mathbf{k}_2}}(\mathbf{l}) &= 2^{d(j_1+j_2)/2} \sum_{\mathbf{l}=-\mathbf{N}-\mathbf{1}}^{\mathbf{N}-\mathbf{1}} \int_{\boldsymbol{\pi}} \widetilde{\boldsymbol{\Psi}}(\mathbf{0}) e^{i\mathbf{l}.\boldsymbol{\lambda}} dF_X(\boldsymbol{\lambda}), \end{aligned}$$

where $\widetilde{\boldsymbol{\Psi}}(.)$ is the Fourier transform of $\widetilde{\Psi}(.)$ and

$$\begin{aligned} \widetilde{\Psi}(|\underline{\mathbf{l}}|) &= \sum_{\mathbf{n}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}-|\underline{\mathbf{l}}|} \Psi(2^j\mathbf{n}-\mathbf{k})\Psi(2^j\mathbf{n}+|\underline{\mathbf{l}}|-\mathbf{k}), \\ \widetilde{\Psi}(\mathbf{0}) &= \sum_{\mathbf{n}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}-|\mathbf{l}|} \Psi(2^{j_1}\mathbf{n}-\mathbf{k}_1)\Psi(2^{j_2}\mathbf{n}-\mathbf{k}_2). \end{aligned}$$

Proof. The proof follows from Theorem 2.1 in Masry (1998).

Remark 4.8 *By the above Theorem, the wavelet transform* $\left\{d_{j,\mathbf{k}}(\mathbf{n}), \mathbf{n} \in \mathbb{Z}^d\right\}$ *is a weakly homogenous random field with spectral measure*

$$dF_d(\boldsymbol{\lambda}) = 2^{dj} \sum_{\mathbf{l}=-\mathbf{N}-\mathbf{1}}^{\mathbf{N}-\mathbf{1}} \widetilde{\boldsymbol{\Psi}}(|\underline{\mathbf{l}}|) dF_X(\boldsymbol{\lambda}),$$

In particular, if the input random field $(X(\mathbf{n}))_{\mathbf{n}\in\mathbb{Z}^d}$ *possesses a spectral density* $f_X(\boldsymbol{\lambda})$*, then so does the wavelet transform and*

$$f_d(\boldsymbol{\lambda}) = 2^{dj} \sum_{\mathbf{l}=-\mathbf{N}-\mathbf{1}}^{\mathbf{N}-\mathbf{1}} \widetilde{\boldsymbol{\Psi}}(|\underline{\mathbf{l}}|) f_X(\boldsymbol{\lambda}).$$

Theorem 4.5 *Let* $(X(\mathbf{n}))_{\mathbf{n}\in\mathbb{Z}^d}$ *be a zero mean stationary discrete random fields belongs to* $V_{\mathbf{0}}$ *space with covariance function* $C(\mathbf{l})$ *and* $\alpha_{0,\mathbf{k}}$ *the scaling coefficient at Zero resolution satisfying* (4.3.5)*. Then the discrete random field* $(\alpha_{0,\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$ *is weakly homogenous with zero mean and covariance function* $C_0^s(\mathbf{l})$ *having the spectral representation as*

$$C_0^s(\mathbf{l}) = \sum_{\mathbf{m}\in\mathbb{Z}^d} \sum_{\mathbf{n}\in\mathbb{Z}^d} \int_{\boldsymbol{\pi}} \left|\widetilde{\Phi}(\boldsymbol{\lambda})\right|^2 e^{i(\mathbf{l}+\mathbf{m}-\mathbf{n}).\boldsymbol{\lambda}} dF_X(\boldsymbol{\lambda}).$$

Proof. The proof follows from Theorem 2.3 in Masry (1998).

Chapter 5

Wavelet spectral and bispectral density estimation

5.1 Introduction

Wavelet density estimation of times series has been well developed theoretically and has found many applications in vast areas of applied sciences. However, in signal processing, the spectral density is an appropriate tool for the description of second-order statistics. It is well known that it characterizes completely stationary signals which have Gaussian distributions. If the signal under study is non-Gaussian, or of it is the results of nonlinear dynamics, knowledge of the mean value and the spectral density is not sufficient to fully characterize the signal (see for example Nikias and Petropulu (1993)). Unlike spectral density, the bispectral density has received special attention in the literature (see Swami et al. (1997)).

Some of the techniques are extendible to random fields with varying degree of success, and there remain to be solved many inherent problems that are not present in the times series case. In this chapter, we consider the theoretical aspects of wavelet spectral and bispectral density for random fields.

5.2 Nonlinear wavelet spectral density estimation

In this section, we obtain empirical versions of the coefficients of f which are treated with the same methods as Neumann (1996), and we drive the uniform estimates of the cumulants of the empirical wavelet coefficients. These results allow us to conclude the risk equivalence between all monotone estimators based on the empirical coefficients. We shown that the optimality thresholded wavelet attain the minimax rate of convergence in a large scale of Besov smoothness classes. Finally, we discuss a possibility to adapt the smoothing parameters involved in the procedure.

5.2.1 Cumulant of the empirical wavelet coefficients

Let $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^2}$ be a stationary random fields with zero mean and the spectral density

$$f(\boldsymbol{\omega}) = \frac{1}{(2\pi)^2}\sum_{\mathbf{h}\in\mathbb{Z}^2} C(\mathbf{h})e^{-i\mathbf{h}.\boldsymbol{\omega}}, \qquad (5.2.1)$$

and consider an orthonormal-wavelet basis of $L^2(\mathbb{R}^2)$, associated to the following scaling and wavelet functions:

$$\begin{aligned}\widetilde{\Phi}_{j,\mathbf{k}}(\mathbf{t}) &= 2^j\Phi(2^jt_1-k_1,2^jt_2-k_2),\\ \widetilde{\Psi}_{j,\mathbf{k},u}(\mathbf{t}) &= 2^j\Psi_u(2^jt_1-k_1,2^jt_2-k_2), \text{ for any } u\in\{1,2,3\}.\end{aligned}$$

It is easy to see that with $\Lambda_j=\{1,...,2^j\}^2$,

$$\Phi_{j,\mathbf{k}}(\mathbf{t}) = \sum_{\mathbf{n}\in\mathbb{Z}^2}(2\pi)^{-1}\widetilde{\Phi}_{j,\mathbf{k}}((2\pi)^{-1}\mathbf{t}+\mathbf{n}),$$

and

$$\Psi_{j,\mathbf{k},u}(\mathbf{t}) = \sum_{\mathbf{n}\in\mathbb{Z}^2}(2\pi)^{-1}\widetilde{\Psi}_{j,\mathbf{k},u}((2\pi)^{-1}\mathbf{t}+\mathbf{n}),$$

is an orthonormal basis of $\widetilde{L}^2(\pi_2)$ (i.e. $\pi_2=[-\pi,\pi[\times[-\pi,\pi[$). For $f\in\widetilde{L}^2(\pi_2)$ we have the representation

$$f(\boldsymbol{\omega}) = \sum_{\mathbf{k}\in\Lambda_l}\alpha_{l,\mathbf{k}}\Phi_{l,\mathbf{k}}(\boldsymbol{\omega}) + \sum_{u=1}^{3}\sum_{j\geq l}\sum_{\mathbf{k}\in\Lambda_j}\beta_{j,\mathbf{k},u}\Psi_{j,\mathbf{k},u}(\boldsymbol{\omega}), \qquad (5.2.3)$$

where $\alpha_{l,\mathbf{k}}=\int_{\pi_2}f(\boldsymbol{\omega})\Phi_{l,\mathbf{k}}(\boldsymbol{\omega})d\boldsymbol{\omega}$ and $\beta_{j,\mathbf{k},u}=\int_{\pi_2}f(\boldsymbol{\omega})\Psi_{j,\mathbf{k},u}(\boldsymbol{\omega})d\boldsymbol{\omega}$.

From the sample observation $\{X(\mathbf{t}),\mathbf{t}=\mathbf{1},...,\mathbf{N}\}$ the tapered periodogram

$$I_{\mathbf{N}}(\boldsymbol{\omega}) = \frac{1}{(2\pi)^2H_2^{(\mathbf{N})}}\left|d_{\mathbf{N}}(\boldsymbol{\omega})\right|^2, \qquad (5.2.2)$$

where $d_{\mathbf{N}}(\boldsymbol{\omega}) = \sum_{\mathbf{t}=\mathbf{0}}^{\mathbf{N-1}}h(\frac{\mathbf{t}}{\mathbf{N}})X_{\mathbf{t}}e^{-i\mathbf{t}.\boldsymbol{\omega}}$, $H_2^{(\mathbf{N})}=\sum_{\mathbf{t}=\mathbf{1}}^{\mathbf{N}}h^2(\frac{\mathbf{t}}{\mathbf{N}})$, is asymptotically unbiased for $f(\boldsymbol{\omega})$ under quite general assumptions (i.e. for $\mathbf{N}=(N,N)$ we obtain $H_2^{(\mathbf{N})}\sim N^2H$), however it is not a consistent estimator of $f(\boldsymbol{\omega})$ (i.e. $I_{\mathbf{N}}(\boldsymbol{\omega})\equiv 0$ if $H_2^{(\mathbf{N})}=0$). Therefore, there is some hope that one can obtain via smoothing estimators that are consistent under certain smoothness conditions on f.

Under the following assumptions we can derive appropriate estimates for bias and variance of the empirical coefficients defined as

$$\begin{aligned} \widetilde{\alpha}_{l,\mathbf{k}} &= \int_{\pi_2} \Phi_{l,\mathbf{k}}(\boldsymbol{\omega}) I_{\mathbf{N}}(\boldsymbol{\omega})\, d\boldsymbol{\omega}, \\ \widetilde{\beta}_{j,\mathbf{k},u} &= \int_{\pi_2} \Psi_{j,\mathbf{k},u}(\boldsymbol{\omega}) I_{\mathbf{N}}(\boldsymbol{\omega})\, d\boldsymbol{\omega}. \end{aligned} \tag{5.2.4}$$

Condition 5.1 *The taper function h is of bounded variation and satisfies $H = \int h^2(\mathbf{x})d\mathbf{x} > 0$.*

Condition 5.2 $\forall k \geq 2$, *there exists $C > 0$ and $\gamma \geq 0$ such that*

$$\sup_{1 \preceq \mathbf{t}_1 \preceq \infty} \left\{ \sum_{\mathbf{t}_2,\ldots,\mathbf{t}_{p-1}} \left| cum(X(\mathbf{t}_1), \ldots X(\mathbf{t}_{p-1}), X(\mathbf{t}_p)) \right| \right\} \leq C^k (p!)^{\gamma+1}.$$

Condition 5.3 *f is of finite total variation over π_2, $\|f\|_\infty < D$, $\forall D > 0$.*

Condition 5.4 - *$\Phi(\mathbf{t})$ and $\Psi_u(\mathbf{t})$ are in C^r, for any $r > m$.*

- *$\int \Phi(\mathbf{t})d\mathbf{t} = 1$ and $\int \Psi_u(\mathbf{t}) |\mathbf{t}|^k d\mathbf{t} = 1$, for $0 \leq k \leq r$.(i.e. $|\mathbf{t}| = t_1 t_2$).*
- *$C = \max\left(\left\|\widetilde{\Phi}\right\|_{L^1}, \left\|\widetilde{\Psi_u}\right\|_{L^1}\right)$ and $D = \max\left(\left\|\widetilde{\Phi}'\right\|_{L^1}, \left\|\widetilde{\Psi}'_u\right\|_{L^1}\right)$ are finite, and $\max\left(\|\Phi_{j,\mathbf{k}}\|_\infty, \|\Psi_{j,\mathbf{k},u}\|_\infty\right) \leq A 2^{j/2}$.*

These assumptions are widely satisfied. In particular for Daubechies's wavelets with support $2M$, the last assumption is satisfied with $A = 2M \max\left(\left\|\widetilde{\Phi}\right\|_\infty, \left\|\widetilde{\Psi}_u\right\|_\infty\right)$.

Remark 5.1 *Note that the regularity r of the scaling function $\Phi(\mathbf{t})$ and the wavelet $\Psi_u(\mathbf{t})$ has tom be chosen higher than the assumed smoothness m of the spectral density in order to make the optimal rate of convergence of the estimators possible.*

Lemma 5.1 *For $\mathbf{N} = (N, N)$, let $j \geq \tau, \mathbf{k} \in \Lambda_j, u \in \{1,2,3\}, \widetilde{\beta}_{j,\mathbf{k},u}$ defined by (5.2.4) satisfying conditions 5.1 through 5.4. Then*

$$E\left\{\widetilde{\beta}_{j,\mathbf{k},u}\right\} = \beta_{j,\mathbf{k},u} + O(2^{j/2+1} N^{-2} \log N),$$

$$Var\left\{\widetilde{\beta}_{j,\mathbf{k},u}\right\} \leq C N^{-2} 2^{-j}, C > 0.$$

Proposition 5.1 *Let $j \geq \tau$, such that $2^j \leq C N^{2(1-\alpha)}$ and $\alpha > 0, \mathbf{k} \in \Lambda_j, u \in \{1,2,3\}$, $\mathbf{N} = (N, N), \widetilde{\beta}_{j,\mathbf{k},u}$ defined by (5.2.4) satisfying conditions 5.1 through 5.4. Then there exists a constant $C > 0$ such that*

$$\left| Cum(\widetilde{\beta}_{j,\mathbf{k},u}) \right| \leq C^n (p!)^{2+2\gamma} N^{-2} (2^{j/2+1} N^{-2} \log N)^{p-2}.$$

Proof. the proof is similarly as that in Proposition 3.1 in Neumann (1996).

5.2.2 Asymptotic normality of the empirical wavelet coefficients

We say that a function $f \in L_2(\pi_2)$ belongs to the two dimensional Besov ball $\mathbf{B}^m_{p,q}(M)$ if and only if there exists a constant M_* (depending on M), such that the associated wavelet coefficient of f satisfy

$$2^{\tau(1/2-1/p)}\left(\sum_{\mathbf{k}\in\Lambda_\tau}|\alpha_{\tau,\mathbf{k}}|^p\right)^{1/p}+\left(\sum_{u=1}^{3}\sum_{j=\tau}^{\infty}\left(2^{j(m+1/2-1/p)}\left(\sum_{\mathbf{k}\in\Lambda_j}\left|\beta_{j,\mathbf{k},u}\right|^p\right)^{1/p}\right)^q\right)^{1/q}\leq M_*<\infty,$$

Besov balls are able to model different kind of smoothness features in a function. For a particular choice of parameters m, p and q, they contain the Hölder and Sobolev ball (see for example Mallat (2009) and Meyer (1992)). Details and results on wavelets and Besov balls in nonparametric estimation can be found in Härdle et al. (1998).

On the other hand, We have for any ball $\mathcal{F}$ in a Besov space $\mathbf{B}^m_{p,q}$ that

$$\sup_{f\in\mathcal{F}}\{\sum_{j>J}\sum_{\mathbf{k}\in\Lambda_j}\beta^2_{j,\mathbf{k},u}\} = O(2^{-2J(m+1/2-1/(p\wedge 2))}), \tag{5.2.5}$$

if $2^{-J} = O(N^{-4/3})$ then

$$\sup_{f\in\mathcal{F}}\{\sum_{j>J}\sum_{\mathbf{k}\in\Lambda_j}\beta^2_{j,\mathbf{k},u}\} = O(N^{-4m/(2m+1)}),$$

we restrict our considerations in this section to coefficient with indices $(j,\mathbf{k})$ from a set

$$\mathcal{J} = \mathcal{J}(\mathbf{N}) = \{(j,\mathbf{k})\backslash 2^j \leq CN^{2(1-\alpha)}, \mathbf{k}\in\Lambda_j, C<\infty, 0<\alpha\leq 1/3\}.$$

Let $\sigma^2_{j,\mathbf{k},u}$ denote the variance of the coefficients $\widetilde{\beta}_{j,\mathbf{k},u}$, then by lemma 5.1 we obtain that

$$\sup_{j,\mathbf{k}}\{\sigma_{j,\mathbf{k},u}\} = O(N^{-1}), \tag{5.2.6}$$

and by proposition 5.1

$$\left|Cum(\widetilde{\beta}_{j,\mathbf{k},u}/\sigma_{j,\mathbf{k},u}\right| \leq (p!)^{2+2\gamma}C^p(2N^{-\alpha}\log(N))^{p-2}, \tag{5.2.7}$$

holds uniformly in $(j,\mathbf{k}) \in \mathcal{J}^0$ where $\mathcal{J}^0 = \mathcal{J}^0(\mathbf{N}) = \{(j,\mathbf{k}) \in \mathcal{J}/\sigma_{j,\mathbf{k}} \geq C_0N^{-1}\}$ for some fixed $C_0 > 0$.

Theorem 5.1 *Let* $\widetilde{\beta}_{j,\mathbf{k},u}$ *defined by* (5.2.4) *satisfying conditions 5.1 through 5.4. Then*

$$\frac{P(\left|\widetilde{\beta}_{j,\mathbf{k},u}-\beta_{j,\mathbf{k},u}\right|/\sigma_{j,\mathbf{k},u}\geq x)}{1-\Phi(x)} \longrightarrow 1,$$

holds uniformly in $(j,\mathbf{k}) \in \mathcal{J}^0, -\infty < x \leq \varepsilon_\gamma$,*where* $\varepsilon_\gamma = o(\varepsilon^{1/(3+4\gamma)})$ *and* $\varepsilon = 2N^\alpha(\log N)^{-1}$, *and* $\Phi(x)$ *be the cumulative distribution function of the standard normal distribution* $\mathcal{N}(0,1)$.

Proof. Using Proposition 5.1 and Lemma 3.1 and Theorem 4.1 in Neuman (1996).

Let

$$\sigma_{\mathbf{N}} = \max\{\max_{(j,\mathbf{k})\in\mathcal{J}} (\sigma_{j,\mathbf{k},u}), C_0 N^{-1}\},$$

and let $\theta_{j,\mathbf{k},u} \sim N(0, \sigma_{\mathbf{N}}^2 - \sigma_{j,\mathbf{k},u}^2)$ be independent of $\widetilde{\beta}_{j,\mathbf{k},u}$. Then the new random field $\widetilde{\beta}_{j,\mathbf{k},u} + \theta_{j,\mathbf{k},u}$ has the same mean and the same p-order cumulants for $p \geq 3$ as $\widetilde{\beta}_{j,\mathbf{k},u}$, whereas its variance is equal to $\sigma_{\mathbf{N}}^2 \asymp N^{-2}$. Therefore, we can derive in complete analogy to Theorem 5.1 the following result.

Corollary 5.1 *Assume* $\widetilde{\beta}_{j,\mathbf{k},u}$ *defined by* (5.2.4) *satisfying condition of Theorem 5.1. Then*

$$\frac{P(\left|(\widetilde{\beta}_{j,\mathbf{k},u} + \theta_{j,\mathbf{k},u}) - \beta_{j,\mathbf{k},u}\right| / \sigma_{j,\mathbf{k},u} \geq x)}{1 - \Phi(x)} \longrightarrow 1.$$

5.2.3 Derivation of thresholding schemes

Let

$$\delta^{(h)}(\widetilde{\beta}_{j,\mathbf{k},u}, \lambda) = \widetilde{\beta}_{j,\mathbf{k},u} \mathbf{1}_{(|\widetilde{\beta}_{j,\mathbf{k},u}| \geq \lambda)}, \tag{5.2.9}$$

$$\delta^{(s)}(\widetilde{\beta}_{j,\mathbf{k},u}, \lambda) = sgn\left(\widetilde{\beta}_{j,\mathbf{k},u}\right)(\left|\widetilde{\beta}_{j,\mathbf{k},u}\right| - \lambda)_+, \tag{5.2.10}$$

where these two nonlinear procedures on the empirical coefficients are usually called hard and soft thersholding, respectively. We consider as approximating models for our empirical wavelet coefficients

$$\zeta_{j,\mathbf{k},u} = \beta_{j,\mathbf{k},u} + \sigma_{j,\mathbf{k},u}\varepsilon_{j,\mathbf{k},u}, \tag{5.2.11}$$

and

$$\overline{\zeta}_{j,\mathbf{k},u} = \beta_{j,\mathbf{k},u} + (\sigma_{j,\mathbf{k},u} \vee \sigma_{\mathbf{N}})\varepsilon_{j,\mathbf{k},u}, \tag{5.2.12}$$

where $\varepsilon_{j,\mathbf{k},u} \sim \mathcal{N}(0,1)$. Then we have the following basic result for monotone coordinate wise estimators.

Theorem 5.2 *Let* $\delta_{j,\mathbf{k}} = \delta_{j,\mathbf{k},\mathbf{N}}$ *be monotone non decreasing functions with*

$$\delta_{j,\mathbf{k}}(y) \leq |y|, \tag{5.2.13}$$

and assume that Conditions 5.1 through 5.4 holds. Then, for $0 < p' < \infty$,

i) $\sum_{(j,\mathbf{k})\in\mathcal{J}^o} E\left\{\left|\delta_{j,\mathbf{k}}(\widetilde{\beta}_{j,\mathbf{k},u})-\beta_{j,\mathbf{k},u}\right|^{p'}\right\} = (1+o(1))\sum_{(j,\mathbf{k})\in\mathcal{J}^o} E\left\{\left|\delta_{j,\mathbf{k}}(\zeta_{j,\mathbf{k},u})-\beta_{j,\mathbf{k},u}\right|^{p'}\right\}+O(N^{-p'})$.

ii) $\sum_{(j,\mathbf{k})\in\mathcal{J}} E\left\{\left|\delta_{j,\mathbf{k}}(\widetilde{\beta}_{j,\mathbf{k},u})-\beta_{j,\mathbf{k},u}\right|^{p'}\right\} \le (2+o(1))\sum_{(j,\mathbf{k})\in\mathcal{J}} E\left\{\left|\delta_{j,\mathbf{k}}(\overline{\zeta}_{j,\mathbf{k},u})-\beta_{j,\mathbf{k},u}\right|^{p'}\right\}+O(N^{-p'})$.

Proof.

i) Since $\delta_{j,\mathbf{k}}$ is monotone, there exists a constant $\gamma_{j,\mathbf{k}}$ such that

$$\begin{aligned}\delta_{j,\mathbf{k}}(y) &\ge \beta_{j,\mathbf{k},u},\ \text{if } y>\gamma_{j,\mathbf{k}},\\ \delta_{j,\mathbf{k}}(y) &\le \beta_{j,\mathbf{k},u},\ \text{if } y<\gamma_{j,\mathbf{k}},\end{aligned}$$

(we assume $\delta_{j,\mathbf{k}}(\widetilde{\beta}_{j,\mathbf{k},u})\ge\beta_{j,\mathbf{k},u}$) Now we split up

$$\begin{aligned}E\left\{\left|\delta_{j,\mathbf{k}}(\widetilde{\beta}_{j,\mathbf{k},u})-\beta_{j,\mathbf{k},u}\right|^{p'}\right\} &= E\left\{\mathbf{1}_{(\gamma_{j,\mathbf{k}}\le\widetilde{\beta}_{j,\mathbf{k},u}<\beta_{j,\mathbf{k},u}+\sigma_{j,\mathbf{k},u}\varepsilon_\gamma)}\left|\delta_{j,\mathbf{k}}(\widetilde{\beta}_{j,\mathbf{k},u})-\beta_{j,\mathbf{k},u}\right|^{p'}\right\} \qquad (5.2.14)\\ &\quad +E\left\{\mathbf{1}_{(\beta_{j,\mathbf{k},u}-\sigma_{j,\mathbf{k},u}\varepsilon_\gamma\le\widetilde{\beta}_{j,\mathbf{k},u}<\gamma_{j,\mathbf{k}})}\left|\delta_{j,\mathbf{k}}(\widetilde{\beta}_{j,\mathbf{k},u})-\beta_{j,\mathbf{k},u}\right|^{p'}\right\}\\ &\quad +E\left\{\mathbf{1}_{(\left|\widetilde{\beta}_{j,\mathbf{k},u}-\beta_{j,\mathbf{k},u}\right|\ge\sigma_{j,\mathbf{k},u}\varepsilon_\gamma)}\left|\delta_{j,\mathbf{k}}(\widetilde{\beta}_{j,\mathbf{k},u})-\beta_{j,\mathbf{k},u}\right|^{p'}\right\}\\ &= R_1+R_2+R_3.\end{aligned}$$

According to the assertion of Theorem 5.1 there exist $C_{\mathbf{N}}^{(l)}, C_{\mathbf{N}}^{(h)}$ both tending to $\mathbf{1}$ as $\mathbf{N}\longrightarrow\infty$, such that

$$\left|C_{\mathbf{N}}^{(l)}\right|(1-\Phi(x)) \le \left(P(\left|\widetilde{\beta}_{j,\mathbf{k},u}-\beta_{j,\mathbf{k},u}\right|/\sigma_{j,\mathbf{k},u}\ge x\right) \le \left|C_{\mathbf{N}}^{(h)}\right|(1-\Phi(x)),\ \forall x\le\varepsilon_\gamma.$$

Since $\left|\delta_{j,\mathbf{k}}(y)-\beta_{j,\mathbf{k},u}\right|$ is monotone nondecreasing for $y\ge\gamma_{j,\mathbf{k}}$, we obtain by integration by parts that

$$\begin{aligned}R_1 &= -\int\left[\mathbf{1}_{(\gamma_{j,\mathbf{k}}\le x<\beta_{j,\mathbf{k},u}+\sigma_{j,\mathbf{k},u}\varepsilon_\gamma)}\left|\delta_{j,\mathbf{k}}(x)-\beta_{j,\mathbf{k},u}\right|^{p'}\right]dP(\widetilde{\beta}_{j,\mathbf{k},u}\ge x) \qquad (5.2.15)\\ &= P(\widetilde{\beta}_{j,\mathbf{k},u}\ge x)d\left[\mathbf{1}_{(\gamma_{j,\mathbf{k}}\le x<\beta_{j,\mathbf{k},u}+\sigma_{j,\mathbf{k},u}\varepsilon_\gamma)}\left|\delta_{j,\mathbf{k}}(x)-\beta_{j,\mathbf{k},u}\right|^{p'}\right]\\ +P(\widetilde{\beta}_{j,\mathbf{k},u} &\ge \gamma_{j,\mathbf{k}})\left|\delta_{j,\mathbf{k}}(\gamma_{j,\mathbf{k}})-\beta_{j,\mathbf{k},u}\right|^{p'}\\ &\le \left|C_{\mathbf{N}}^{(h)}\right|\left\{\int p(\zeta_{j,\mathbf{k},u}\ge x)d\left[\mathbf{1}_{(\gamma_{j,\mathbf{k}}\le x<\beta_{j,\mathbf{k},u}+\sigma_{j,\mathbf{k},u}\varepsilon_\gamma)}\left|\delta_{j,\mathbf{k}}(x)-\beta_{j,\mathbf{k},u}\right|^{p'}\right]\right.\\ &\quad \left.+P(\zeta_{j,\mathbf{k},u}\ge\gamma_{j,\mathbf{k}})\left|\delta_{j,\mathbf{k}}(x)-\beta_{j,\mathbf{k},u}\right|^{p'}\right\}\\ &= \left|C_{\mathbf{N}}^{(h)}\right|E\left\{\mathbf{1}_{(\gamma_{j,\mathbf{k}}\le\zeta_{j,\mathbf{k},u}<\beta_{j,\mathbf{k},u}+\sigma_{j,\mathbf{k},u}\varepsilon_\gamma)}\left|\delta_{j,\mathbf{k}}(\zeta_{j,\mathbf{k},u})-\beta_{j,\mathbf{k},u}\right|^{p'}\right\},\end{aligned}$$

holds uniformly in $(j, \mathbf{k}) \in \mathcal{J}^o$. Analogously we get

$$R_2 \leq \left|C_{\mathbf{N}}^{(h)}\right| E\left\{\mathbf{1}_{(\gamma_{j,\mathbf{k}} \leq \zeta_{j,\mathbf{k},u} < \beta_{j,\mathbf{k},u} + \sigma_{j,\mathbf{k},u}\varepsilon_\gamma)} \left|\delta_{j,\mathbf{k}}(\zeta_{j,\mathbf{k},u}) - \beta_{j,\mathbf{k},u}\right|^{p'}\right\}.$$

Using Proposition 5.1 we obtain, for arbitrary even p, that

$$\begin{aligned} E\left\{(\widetilde{\beta}_{j,\mathbf{k},u} - \beta_{j,\mathbf{k},u})^p\right\} &= O\left(\sum_{r=1}^{p} \prod_{\substack{i_1+\ldots i_r = p \\ i_j \geq 1}} \left|cum_{i_j}(\widetilde{\beta}_{j,\mathbf{k},u})\right|\right) \\ &= O\left(N^{-p}\right). \end{aligned} \tag{5.2.16}$$

Further we have, with $\varepsilon_\gamma = N^{2\nu}$ for some $\nu > 0$, that

$$P(\left|\widetilde{\beta}_{j,\mathbf{k},u} - \beta_{j,\mathbf{k},u}\right| \geq \sigma_{j,\mathbf{k},u}\varepsilon_\gamma) \leq C(1 - \Phi(\varepsilon_\gamma)) = O\left(N^{-2\mu}\right),$$

for arbitrary $\mu < \infty$, which implies by $\left|\delta_{j,\mathbf{k}}(\widetilde{\beta}_{j,\mathbf{k},u}) - \beta_{j,\mathbf{k},u}\right| \leq \left|\widetilde{\beta}_{j,\mathbf{k},u} - \beta_{j,\mathbf{k},u}\right| + 2\left|\beta_{j,\mathbf{k},u}\right|$ and the Cauchy-Schawrz inequality that

$$R_3 \leq \sqrt{P(\left|\widetilde{\beta}_{j,\mathbf{k},u} - \beta_{j,\mathbf{k},u}\right| \geq \sigma_{j,\mathbf{k},u}\varepsilon_\gamma)} \sqrt{E\left\{\left|\delta_{j,\mathbf{k}}(\widetilde{\beta}_{j,\mathbf{k},u}) - \beta_{j,\mathbf{k},u}\right|^{2p'}\right\}} = O\left(N^{-p'-2}\right), \tag{5.2.17}$$

by (5.2.14) and (5.2.15) through (5.2.17) we conclude that

$$E\left\{\left|\delta_{j,\mathbf{k}}(\widetilde{\beta}_{j,\mathbf{k},u}) - \beta_{j,\mathbf{k},u}\right|^{p'}\right\} \leq (|C_{\mathbf{N}_1}| \vee |C_{\mathbf{N}_2}|)\, E\left\{\left|\delta_{j,\mathbf{k}}(\zeta_{j,\mathbf{k},u}) - \beta_{j,\mathbf{k},u}\right|^{p'}\right\} + O\left(N^{-p'-2}\right).$$

A lower bound can be proved analogously.

ii) Let $\theta_{j,\mathbf{k},u} \sim N(0, \sigma_{\mathbf{N}}^2 - \sigma_{j,\mathbf{k},u}^2)$ be independent of $\widetilde{\beta}_{j,\mathbf{k},u}$. Then

$$\begin{aligned} E\left\{\left|\delta_{j,\mathbf{k}}(\widetilde{\beta}_{j,\mathbf{k},u}) - \beta_{j,\mathbf{k},u}\right|^{p'}\right\} &= E\left\{\mathbf{1}_{(\widetilde{\beta}_{j,\mathbf{k},u} \geq \gamma_{j,\mathbf{k}})} \left|\delta_{j,\mathbf{k}}(\gamma_{j,\mathbf{k}}) - \beta_{j,\mathbf{k},u}\right|^{p'}\right\} \\ &\quad + E\left\{\mathbf{1}_{(\widetilde{\beta}_{j,\mathbf{k},u} < \gamma_{j,\mathbf{k}})} \left|\delta_{j,\mathbf{k}}(\gamma_{j,\mathbf{k}}) - \beta_{j,\mathbf{k},u}\right|^{p'}\right\} \\ &= 2E\left\{\mathbf{1}_{(\widetilde{\beta}_{j,\mathbf{k},u} \geq \gamma_{j,\mathbf{k}}, \theta_{j,\mathbf{k},u} \geq 0)} \left|\delta_{j,\mathbf{k}}(\widetilde{\beta}_{j,\mathbf{k},u}) - \beta_{j,\mathbf{k},u}\right|^{p'}\right\} \\ &\quad + 2E\left\{\mathbf{1}_{(\widetilde{\beta}_{j,\mathbf{k},u} < \gamma_{j,\mathbf{k}}, \theta_{j,\mathbf{k},u} \geq 0)} \left|\delta_{j,\mathbf{k}}(\widetilde{\beta}_{j,\mathbf{k},u}) - \beta_{j,\mathbf{k},u}\right|^{p'}\right\} \\ &\leq 2E\left\{\left|\delta_{j,\mathbf{k}}(\widetilde{\beta}_{j,\mathbf{k},u} + \theta_{j,\mathbf{k},u}) - \beta_{j,\mathbf{k},u}\right|^{p'}\right\}, \end{aligned}$$

which yields (ii) due to Corollary 2.1.

Remark 5.2 *From the above theorem we can obtain risk properties of thresholded wavelet estimators. Since the estimators* (5.2.9) *and* (5.2.10) *obey the assumption* (5.2.13)*, we can immediately derive due to Theorem 5.1 the risk equivalence of our spectral density estimators to analogous estimators in the much simpler models* (5.2.11) *and* (5.2.12)*.*

Let $\delta^{(\cdot)}$ denote either the hard-threshold rule $\delta^{(h)}$ defined by (5.2.9) or the soft-threshold rule $\delta^{(s)}$ given by (5.2.10). Then we can state the following assertion.

Corollary 5.2 *Let $\widetilde{\beta}_{j,\mathbf{k},u}$ defined by* (5.2.4) *and assume Conditions 5.1 through 5.4 holds. Then, for nonrandom thresholds $\lambda_{j,\mathbf{k},u}$*

i) $\sum_{(j,\mathbf{k})\in\mathcal{J}^o} E\left\{\left(\delta^{(\cdot)}(\widetilde{\beta}_{j,\mathbf{k},u},\lambda_{j,\mathbf{k},u})-\beta_{j,\mathbf{k},u}\right)^2\right\} = (1+o(1))\sum_{(j,\mathbf{k})\in\mathcal{J}^o} E\left\{\left|\delta^{(\cdot)}(\zeta_{j,\mathbf{k},u},\lambda_{j,\mathbf{k},u})-\beta_{j,\mathbf{k},u}\right|^2\right\}+ O(N^{-2})$.

ii) $\sum_{(j,\mathbf{k})\in\mathcal{J}} E\left\{\left(\delta^{(\cdot)}(\widetilde{\beta}_{j,\mathbf{k},u},\lambda_{j,\mathbf{k},u})-\beta_{j,\mathbf{k},u}\right)^2\right\} \leq (2+o(1))\sum_{(j,\mathbf{k})\in\mathcal{J}} E\left\{\left|\delta_{j,k}(\overline{\zeta}_{j,\mathbf{k},u},\lambda_{j,\mathbf{k},u})-\beta_{j,\mathbf{k},u}\right|^2\right\}+ O(N^{-2})$.

Let us now assume that the spectral density $f(\boldsymbol{\omega})$ lies in a set of the following type:

$$\mathcal{F}=\mathcal{F}(C)=\left\{ f(\boldsymbol{\omega})=\sum_{\mathbf{k}\in\Lambda_l}\alpha_{l,\mathbf{k}}\Phi_{l,\mathbf{k}}(\boldsymbol{\omega})+\sum_{u=1}^{3}\sum_{j\geq l}\sum_{\mathbf{k}\in\Lambda_j}\beta_{j,\mathbf{k},u}\Psi_{j,\mathbf{k},u}(\boldsymbol{\omega})\,\middle|\, \|\alpha\|_{m,p,q}\leq C\right\}, \tag{5.2.18}$$

where

$$\|\alpha\|_{m,p,q}=\left(\sum_{u=1}^{3}\sum_{j\geq l}\left(2^{js}\left(\sum_{\mathbf{k}\in\Lambda_j}\left|\beta_{j,\mathbf{k},u}\right|^p\right)^{1/p}\right)^q\right)^{1/q}, \tag{5.2.19}$$

with $s=m+1/2-1/p$. It is known that this norm is essentially equivalent to the norm in the two dimensional Besov ball $\mathbf{B}^m_{p,q}(M)$, if the basis functions $\Phi_{l,\mathbf{k}}(\boldsymbol{\omega})$ and $\Psi_{j,\mathbf{k},u}(\boldsymbol{\omega})$ satisfy condition 5.4. Moreover, we see by the relation $\mathbf{B}^m_{p,1}\subseteq\mathbf{W}^m_p\subseteq\mathbf{B}^m_{p,\infty}$, that smoothness classes from the scale of two dimensional Sobolev spaces $\mathbf{W}^m_p$ are also covered by our results.

Let

$$\widehat{f}^o(\boldsymbol{\omega})=\sum_{\mathbf{k}\in\Lambda_l}\widetilde{\alpha}_{l,\mathbf{k}}\Phi_{l,\mathbf{k}}(\boldsymbol{\omega})+\sum_{u=1}^{3}\sum_{j\geq l}\sum_{\mathbf{k}\in\Lambda_j}\delta^{(\cdot)}(\widetilde{\beta}_{j,\mathbf{k},u},\lambda^o_j)\Psi_{j,\mathbf{k},u}(\boldsymbol{\omega}), \tag{5.2.20}$$

be the estimator with optimal (nonrandom) thresholds $\lambda^o_j=\lambda^o_j(\mathbf{N},\mathcal{F})$ and let

$$\widehat{f}(\boldsymbol{\omega})=\sum_{\mathbf{k}\in\Lambda_l}\widetilde{\alpha}_{l,\mathbf{k}}\Phi_{l,\mathbf{k}}(\boldsymbol{\omega})+\sum_{u=1}^{3}\sum_{j\geq l}\sum_{\mathbf{k}\in\Lambda_j}\delta^{(\cdot)}(\widetilde{\beta}_{j,\mathbf{k},u},\lambda_{j,\mathbf{k},u})\Psi_{j,\mathbf{k},u}(\boldsymbol{\omega}), \tag{5.2.21}$$

be an estimate with individual thresholds, which satisfy the following minimal conditions.

Condition 5.5 *Let φ denotes the standard normal density, then*

i) $\sum_{(j,\mathbf{k})\in\mathcal{J}}\left(\frac{\lambda_{j,\mathbf{k},u}}{(\sigma_{j,\mathbf{k},u}\vee\sigma_{\mathbf{N}})}+1\right)\varphi\left(\frac{\lambda_{j,\mathbf{k},u}}{(\sigma_{j,\mathbf{k},u}\vee\sigma_{\mathbf{N}})}\right)=O(N^{2/(2m+1)})$.

ii) $\max_{(j,\mathbf{k})\in\mathcal{J}}\{\lambda_{j,\mathbf{k},u}\}=O(\sqrt{\frac{2\log N}{N^2}})$.

We can see by Neumann (1996), that for hard or soft thresholded estimators based on observations according to (5.2.12) the following relation holds:

$$E\left\{\left(\delta^{(\cdot)}(\overline{\zeta}_{j,\mathbf{k},u},\lambda)-\beta_{j,\mathbf{k},u}\right)^2\right\} \leq C\left\{(\sigma^2_{j,\mathbf{k},u}\vee\sigma^2_{\mathbf{N}})\left(\frac{\lambda}{(\sigma_{j,\mathbf{k},u}\vee\sigma_{\mathbf{N}})}+1\right)\varphi\frac{\lambda}{(\sigma_{j,\mathbf{k},u}\vee\sigma_{\mathbf{N}})} + \min_{(j,\mathbf{k})\in\mathcal{J}}\left(\lambda^2,\beta^2_{j,\mathbf{k},u}\right)\right\}, \tag{5.2.22}$$

uniformly in $\lambda\geq 0$.

From Condition 5.5 we have two particular thresholding schemes defined as

$$\lambda_{j,\mathbf{k},u}=(\sigma_{j,\mathbf{k},u}\vee\sigma_{\mathbf{N}})\sqrt{2\log\left(\sharp\mathcal{J}\right)}, \tag{5.2.23}$$

and

$$\lambda_{j,\mathbf{k},u}=(\sigma_{j,\mathbf{k},u}\vee\sigma_{\mathbf{N}})\sqrt{2\log(\left(\sharp\mathcal{J}\right)/2^l)}. \tag{5.2.24}$$

Now it can be easily shown that both of the proposed thresholding schemes lead to a rate $\left(2\log N/N^2\right)^{2m/(2m+1)}$ for the risk of the estimators $\widehat{f}$.

Theorem 5.3 *Let $\widehat{f},\widehat{f}^o$ be an estimators with individual, optimal thresholds respectively satisfying Conditions 5.1 through 5.4. Then*

i) $\sup_{f\in\mathcal{F}}\left(E\left\{\left\|\widehat{f}^o-f\right\|^2_{L_2(\pi_2)}\right\}\right)=O(N^{-4m/(2m+1)})$.

ii) *if additionally Condition 5.5 is satisfied, then*

$\sup_{f\in\mathcal{F}}\left(E\left\{\left\|\widehat{f}-f\right\|^2_{L_2(\pi_2)}\right\}\right)=O(\left(2\log N/N^2\right)^{2m/(2m+1)})$.

Proof. From Corollary 5.2, we show that

$$\sum_{(j,\mathbf{k})\in\mathcal{J}}E\left\{\left(\delta^{(\cdot)}(\overline{\zeta}_{j,\mathbf{k},u},\lambda_{j,\mathbf{k},u})-\beta_{j,\mathbf{k},u}\right)^2\right\}+\sum_{(j,\mathbf{k})\notin\mathcal{J}}\beta^2_{j,\mathbf{k},u}, \tag{5.2.25}$$

where $\overline{\zeta}_{j,\mathbf{k},u}$ is given by (5.2.12).

By (5.2.5) , the estimate the second term of (5.2.25) is

$$\sup_{f\in\mathcal{F}}\left\{\sum_{(j,\mathbf{k})\in\mathcal{J}}\beta^2_{j,\mathbf{k},u}\right\} = O\left(N^{4(\alpha-1)(m+1/2-1/(p\wedge 2))}\right) = O\left(N^{-4m(2m+1)}\right). \tag{5.2.26}$$

Further, by (5.2.22) the estimate for the first terms of (5.2.25):

$$\sum_{(j,\mathbf{k})\in\mathcal{J}} E\left\{\left(\delta^{(\cdot)}(\overline{\zeta}_{j,\mathbf{k},u},\lambda_{j,\mathbf{k},u})-\beta_{j,\mathbf{k},u}\right)^2\right\} \tag{5.2.27}$$
$$\leq C\sum_{(j,\mathbf{k})\in\mathcal{J}}\left\{(\sigma_{j,\mathbf{k},u}\vee\sigma_{\mathbf{N}})^2\left(\frac{\lambda_{j,\mathbf{k},u}}{\sigma_{j,\mathbf{k},u}\vee\sigma_{\mathbf{N}}}+1\right)\varphi\left(\frac{\lambda_{j,\mathbf{k},u}}{\sigma_{j,\mathbf{k},u}\vee\sigma_{\mathbf{N}}}\right)+\min\left(\beta^2_{j,\mathbf{k},u},\lambda^2_{j,\mathbf{k},u}\right)\right\}.$$

i) Choose an integer j_0 such that $2^{j_0}\asymp N^{2/(2m+1)}$, and let

$$\lambda_{j,\mathbf{k},u}=\begin{cases}0, & \text{if } j\leq j_0,\\ (\sigma_{j,\mathbf{k},u}\vee\sigma_{\mathbf{N}})\sqrt{K(j-j_0)}, & \text{if } j>j_0,\end{cases}$$

for any fixed $K>\log 4$. Then

$$\begin{aligned}&\sum_{(j,\mathbf{k})\in\mathcal{J}}\left[(\sigma_{j,\mathbf{k},u}\vee\sigma_{\mathbf{N}})^2\left(\frac{\lambda_{j,\mathbf{k},u}}{\sigma_{j,\mathbf{k},u}\vee\sigma_{\mathbf{N}}}+1\right)\varphi\left(\frac{\lambda_{j,\mathbf{k},u}}{\sigma_{j,\mathbf{k},u}\vee\sigma_{\mathbf{N}}}\right)\right] \\ &= O(2^{j_0}N^{-2})+\sum_{j>j_0}O\left(N^{-2}2^j\sqrt{K(j-j_0)}e^{-K(j-j_0)/2}\right) \\ &= O\left(N^{-4m/(2m+1)}\right)+O\left(N^{-4m/(2m+1)}\sum_{j>j_0}\sqrt{K(j-j_0)}2^{(j-j_0)(1-K/\log 4)}\right) \\ &= O\left(N^{-4m/(2m+1)}\right),\end{aligned} \tag{5.2.28}$$

with $\widetilde{p}=min\{p,2\}$, we obtain by Jensen's inequality

$$\left(2^{-j}\sum_{\mathbf{k}}\left|\beta_{j,\mathbf{k},u}\right|^{\widetilde{p}}\right)^{1/\widetilde{p}}\leq\left(2^{-j}\sum_{\mathbf{k}}\left|\beta_{j,\mathbf{k},u}\right|^{p}\right)^{1/p}+O\left(2^{-j(m+1/2)}\right),$$

which implies that

$$\begin{aligned}\sum_{\mathbf{k}}\min\left\{\beta^2_{j,\mathbf{k},u},\lambda^2_{j,\mathbf{k},u}\right\} &\leq \lambda^{2-\widetilde{p}}_{j,\mathbf{k},u}\sum_{\mathbf{k}}\left|\beta_{j,\mathbf{k},u}\right|^{\widetilde{p}} \\ &= O\left(N^{-(2-\widetilde{p})}((j-j_0)^{(2-\widetilde{p})/2}2^{-j(m+1/2-1/\widetilde{p})\widetilde{p}}\right) \\ &= O\left(N^{-4m/(2m+1)}(j-j_0)^{(2-\widetilde{p})/2}2^{-(j-j_0)(m+1/2-1/\widetilde{p})\widetilde{p}}\right),\end{aligned}$$

hold uniformly in $f \in \mathcal{F}, j > j_0$. Hence,

$$\sup_{f\in\mathcal{F}} \left\{ \sum_{j>j_0} \sum_{\mathbf{k}} \min\left\{\beta^2_{j,\mathbf{k},u}, \lambda^2_{j,\mathbf{k},u}\right\} \right\} = O\left(N^{-4m/(2m+1)}\right), \tag{5.2.29}$$

ii) We have, according to Condition 5.5 (i), that

$$\sum_{(j,\mathbf{k})\in\mathcal{J}} \left((\sigma_{j,\mathbf{k},u} \vee \sigma_{\mathbf{N}})^2 \left(\frac{\lambda_{j,\mathbf{k},u}}{\sigma_{j,\mathbf{k},u} \vee \sigma_{\mathbf{N}}} + 1 \right) \varphi\left(\frac{\lambda_{j,\mathbf{k},u}}{\sigma_{j,\mathbf{k},u} \vee \sigma_{\mathbf{N}}} \right) \right) = O\left(N^{-4m/(2m+1)}\right). \tag{5.2.30}$$

Let j_* be such that $2^{j_*} = \left(N^2/2\log N\right)^{1/(2m+1)}$. Then we obtain, analogously to the above calculations, that

$$\begin{aligned}
\sum_{j\le j_*} \sum_{\mathbf{k}} \min\left\{\beta^2_{j,\mathbf{k},u}, \lambda^2_{j,\mathbf{k},u}\right\} &= O\left(2^{j_*} N^{-2} 2\log N\right) \\
&= O\left(\left(2\log N/N^2\right)^{2m/(2m+1)}\right),
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j>j_*} \sum_{\mathbf{k}} \min\left\{\beta^2_{j,\mathbf{k},u}, \lambda^2_{j,\mathbf{k},u}\right\} &= O\left(\left(2\log N/N^2\right)^{(2-\widetilde{p})/2} \sum_{j>j_*} \sum_{\mathbf{k}} \left|\beta_{j,\mathbf{k},u}\right|^{\widetilde{p}} \right) \qquad (5.2.31)\\
&= O\left(\left(2\log N/N^2\right)^{(2-\widetilde{p})/2} 2^{-j_*(m+1/2-1/\widetilde{p})\widetilde{p}} \right) \\
&= O\left(\left(2\log N/N^2\right)^{(2-\widetilde{p})/2} 2^{j_*} \left(\log N/N\right)^{\widetilde{p}/2} \right) \\
&= O\left(\left(2\log N/N^2\right)^{2m/(2m+1)} \right).
\end{aligned}$$

5.2.4 Adaptive threshold choice

Although the results of Theorem 5.3 are certainly of some theoretical interest, in particular they are not helpful for practical application. The optimal as well as the long-threshlods depend on a priory assumptions on the set $\mathcal{F}$, or on f itself via the variances of the empirical wavelet coefficients, respectively (see Neumann (1996)).

To make the method applicable, it is necessary to find some completely data-driven rule for the thresholds, which works well over an as wide as possible range of smoothness classes. In analogy to (5.2.23) and (5.2.24) we obtain adaptive thresholds as

$$\widehat{\lambda}_{j,\mathbf{k},u} = \widehat{\sigma}_{j,\mathbf{k},u} \sqrt{2\log\left(\sharp\mathcal{J}\right)}, \tag{5.2.32}$$

and

$$\widehat{\lambda}_{j,\mathbf{k},u} = \widehat{\sigma}_{j,\mathbf{k},u}\sqrt{2\log((\sharp\mathcal{J})/2^l)}, \tag{5.2.33}$$

where $\widehat{\sigma}^2_{j,\mathbf{k},u}$ as an estimate of $var\left\{\widetilde{\beta}_{j,\mathbf{k},u}\right\}$ for all levels $j = j(N)$ with $2^j \gg 1$. A sufficient condition for random thresholds, which ensures the desired rate for the estimator, is the following one.

Condition 5.6 **i)** $\sum_{(j,\mathbf{k})\in\mathcal{J}} P\left(\widehat{\lambda}_{j,\mathbf{k},u} < |\alpha_{\mathbf{N}}|\,(\sigma_{j,\mathbf{k},u} \vee \sigma_{\mathbf{N}})\sqrt{2\log\left(C\left(\sharp\mathcal{J}\right)\right)}\right) = O(N^{2\nu})$ *for any* $C > 0, \nu < \frac{1}{2m+1}$ *and* $\alpha_{\mathbf{N}} \longrightarrow \mathbf{1}$.

ii) $\sum_{(j,\mathbf{k})\in\mathcal{J}} P\left(\widehat{\lambda}_{j,\mathbf{k},u} < DN^{-1}\sqrt{2\log N}\right) = O(N^{-4m/(2m+1)})$ *for any* $D < \infty$.

Let

$$\widetilde{f}(\boldsymbol{\omega}) = \int_{\pi_2} |b_{\mathbf{N}}|^{-1}\, w(\boldsymbol{\omega} - \boldsymbol{\lambda}/b_{\mathbf{N}}) I_{\mathbf{N}}(\boldsymbol{\lambda}) d\boldsymbol{\lambda},$$

be a kernel estimator with nonrandom bandwidth $b_{\mathbf{N}}$ based on the tapered periodogram $I_{\mathbf{N}}(\boldsymbol{\lambda})$.

Lemma 5.2 *Assume Conditions 5.1 through 5.4 holds and let* $\widetilde{f}(\boldsymbol{\omega}) \geq C > 0$ *for all* $\boldsymbol{\omega} \in \pi_2$. *Then*

i) *if* $m > 1/p$ *and* $b_N = O(N^{1-\delta})$ *for any* $\delta > 0$, *then*

$$P\left(|b_{\mathbf{N}}|\, f(\boldsymbol{\omega}) \leq \widetilde{f}(\boldsymbol{\omega}) \leq D \text{ for all } \boldsymbol{\omega} \in \pi_2\right) = 1 - O\left(N^{-4}\right),$$

holds uniformly in $\mathcal{F}$ *for some* $\alpha_{\mathbf{N}} \longrightarrow \mathbf{1}$ *and* $D < \infty$.

ii) *if* $b_N = O(N^{\nu-1})$ *and* $b_N^{-1} = O(N^{1-\delta})$ *for any* $\nu < \frac{1}{2m+1}$ *and* $\delta > 0$, *then*

a) $\sum_{(j,\mathbf{k})\in\mathcal{J}} P\left(\widetilde{f}(\boldsymbol{\omega}) \leq |b_{\mathbf{N}}|\, f(\boldsymbol{\omega})\right) = O\left(N^{2\nu}\right)$, *for any* $\boldsymbol{\omega} \in supp(\Psi_{j,\mathbf{k},u})$

b) $P\left(\widetilde{f}(\boldsymbol{\omega}) > D\right) = O\left(N^{-4}\right)$ *holds uniformly in* $\mathcal{F}$ *for* $\boldsymbol{\omega} \in \pi_2$, $b_{\mathbf{N}} \longrightarrow \mathbf{1}$ *and* $D < \infty$.

Proof. the proof is similarly as that in Neumann (1996).

The performance of the resulting estimator

$$\widehat{\widehat{f}}(\boldsymbol{\omega}) = \sum_{\mathbf{k}\in\Lambda_l} \widetilde{\alpha}_{l,\mathbf{k}}\Phi_{l,\mathbf{k}}(\boldsymbol{\omega}) + \sum_{u=1}^{3}\sum_{j\geq l}\sum_{\mathbf{k}\in\Lambda_j} \delta^{(\cdot)}(\widetilde{\beta}_{j,\mathbf{k},u}, \widehat{\lambda}_{j,\mathbf{k},u})\Psi_{j,\mathbf{k},u}(\boldsymbol{\omega}),$$

is described in the following theorem

Theorem 5.4 *Assume the Conditions 5.1 through 5.4 and 5.6 holds. Then*

$$\sup_{f\in\mathcal{F}}\left(E\left\{\left\|\widehat{\widehat{f}} - f\right\|^2_{L_2(\pi_2)}\right\}\right) = O(\left(2\log N/N^2\right)^{2m/(2m+1)}).$$

Proof. Using the monotonicity of $\delta^{(\cdot)}(\widetilde{\beta}_{j,\mathbf{k},u},.)$ in the second argument we get, with $\lambda_{j,\mathbf{k},u}^{(l)} = |\alpha_{\mathbf{N}}|\left(\sigma_{j,\mathbf{k},u} \vee \sigma_{\mathbf{N}}\right)\sqrt{2\log\left(C\left(\sharp\mathcal{J}\right)\right)}$ and $\lambda_{j,\mathbf{k},u}^{(h)} = DN^{-1}\sqrt{2\log N}$, that

$$\left(\delta^{(\cdot)}(\widetilde{\beta}_{j,\mathbf{k},u},\widehat{\lambda}_{j,\mathbf{k},u}) - \beta_{j,\mathbf{k},u}\right)^2$$
$$\leq \begin{cases} \left(\widetilde{\beta}_{j,\mathbf{k},u} - \beta_{j,\mathbf{k},u}\right)^2 + \left(\delta^{(\cdot)}(\widetilde{\beta}_{j,\mathbf{k},u},\lambda_{j,\mathbf{k},u}^{(l)}) - \beta_{j,\mathbf{k},u}\right)^2, & \text{if } \widehat{\lambda}_{j,\mathbf{k},u} < \lambda_{j,\mathbf{k},u}^{(l)}, \\ \left(\delta^{(\cdot)}(\widetilde{\beta}_{j,\mathbf{k},u},\lambda_{j,\mathbf{k},u}^{(l)}) - \beta_{j,\mathbf{k},u}\right)^2 + \left(\delta^{(\cdot)}(\widetilde{\beta}_{j,\mathbf{k},u},\lambda_{j,\mathbf{k},u}^{(h)}) - \beta_{j,\mathbf{k},u}\right)^2, & \text{if } \lambda_{j,\mathbf{k},u}^{(l)} < \widehat{\lambda}_{j,\mathbf{k},u} < \lambda_{j,\mathbf{k},u}^{(h)}, \\ \left(\delta^{(\cdot)}(\widetilde{\beta}_{j,\mathbf{k},u},\lambda_{j,\mathbf{k},u}^{(h)}) - \beta_{j,\mathbf{k},u}\right)^2 + \left(\beta_{j,\mathbf{k},u}\right)^2, & \text{if } \widehat{\lambda}_{j,\mathbf{k},u} > \lambda_{j,\mathbf{k},u}^{(h)}, \end{cases}$$

which implies

$$\begin{aligned} E\left\{\left\|\widehat{\widehat{f}} - f\right\|_{L_2(\pi_2)}^2\right\} &\leq \sum_{\mathbf{k}} E\left\{(\widetilde{\alpha}_{l,\mathbf{k}} - \alpha_{l,\mathbf{k}})^2\right\} + \sum_{(j,\mathbf{k})\in\mathcal{J}} E\left\{\left(\delta^{(\cdot)}(\widetilde{\beta}_{j,\mathbf{k},u},\lambda_{j,\mathbf{k},u}^{(l)}) - \beta_{j,\mathbf{k},u}\right)^2\right\} \\ &\quad + \sum_{(j,\mathbf{k})\in\mathcal{J}} E\left\{\left(\delta^{(\cdot)}(\widetilde{\beta}_{j,\mathbf{k},u},\lambda_{j,\mathbf{k},u}^{(h)}) - \beta_{j,\mathbf{k},u}\right)^2\right\} \\ &\quad + \sum_{(j,\mathbf{k})\in\mathcal{J}} E\left\{1_{\left(\widehat{\lambda}_{j,\mathbf{k},u} < \lambda_{j,\mathbf{k},u}^{(l)}\right)}\left(\widetilde{\beta}_{j,\mathbf{k},u} - \beta_{j,\mathbf{k},u}\right)\right\}^2 \\ &\quad + \sum_{(j,\mathbf{k})\in\mathcal{J}} P\left(\widehat{\lambda}_{j,\mathbf{k},u} > \lambda_{j,\mathbf{k},u}^{(h)}\right)\beta_{j,\mathbf{k},u}^2 + \sum_{(j,\mathbf{k})\notin\mathcal{J}} \beta_{j,\mathbf{k},u}^2 \\ &= T_1 + \ldots + T_6. \end{aligned}$$

Since both thresholding schemes, $\left(\lambda_{j,\mathbf{k},u}^{(l)}\right)$ and $\left(\lambda_{j,\mathbf{k},u}^{(h)}\right)$, satisfy Condition 5.5, we obtain by (ii) of Theorem 5.3 that

$$T_1 + T_2 + T_3 + T_6 = O(\left(2\log N/N^2\right)^{2m/(2m+1)}).$$

Using Hölder's inequality we obtain by (5.2.16) that

$$\begin{aligned} T_4 &< \sum_{(j,\mathbf{k})\in\mathcal{J}} \left(P\left(\widehat{\lambda}_{j,\mathbf{k},u} > \lambda_{j,\mathbf{k},u}^{(l)}\right)\right)^{1-\delta} E\left(\left|\widetilde{\beta}_{j,\mathbf{k},u} - \beta_{j,\mathbf{k},u}\right|^{2/\delta}\right)^{\delta} \\ &= O(N^{-2}) \sum_{(j,\mathbf{k})\in\mathcal{J}} \left(P\left(\widehat{\lambda}_{j,\mathbf{k},u} > \lambda_{j,\mathbf{k},u}^{(l)}\right)\right)^{1-\delta} \\ &= O(N^{-2}) \sum_{(j,\mathbf{k})\in\mathcal{J}} \left(P\left(\widehat{\lambda}_{j,\mathbf{k},u} > \lambda_{j,\mathbf{k},u}^{(l)}\right)\right)^{1-\delta} \left(\sharp\mathcal{J}\right)^{\delta} \\ &= O(N^{-2}N^{2\nu(1-\delta)}N^{2\delta}) \\ &= O(N^{-4m/(2m+1)}), \end{aligned}$$

holds, if $0 < \delta \leq (1/(2m+1) - \nu) / (1-\nu)$. Finally, we obviously have that

$$T_5 = O(N^{-4m/(2m+1)}).$$

5.3 Wavelet-thresholding for bispectrum estimation

In this section we propose a wavelet-thresholding estimator of the bispectra for a wide class of stationary random fields. Like one dimensional case, we show that this estimator reaches minimax rate on Sobolev spaces, which is not attained by linear (kernel or spiline) estimators whenever a certain amount of inhomogeneity in the smoothness of the bispectrum is present.

5.3.1 Wavelet estimator

Let $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^2}$ be a stationary random fields with bispectrum defined as

$$f_3(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2) = \frac{1}{(2\pi)^4} \sum_{\mathbf{h}_1, \mathbf{h}_2 \in \mathbb{Z}^2} C_3(\mathbf{h}_1, \mathbf{h}_2) e^{-i(\mathbf{h}_1.\boldsymbol{\omega}_1 + \mathbf{h}_2.\boldsymbol{\omega}_2)}, \tag{5.3.1}$$

A naive estimator of $f_3(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2)$ is the tapered biperiodogram:

$$I_{\mathbf{N}}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2) = \frac{1}{(2\pi)^4 H_3^T} d_{\mathbf{N}}(\boldsymbol{\omega}_1) d_{\mathbf{N}}(\boldsymbol{\omega}_2) d_{\mathbf{N}}(-\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2), \tag{5.3.2}$$

where $H_3^T = \sum_{\mathbf{t}=\mathbf{0}}^{\mathbf{N-1}} \prod_{i=1}^{3} h_i(\frac{\mathbf{t}}{\mathbf{N}}), h_i, i = 1, 2, 3,$ are the taper functions. It is well known that, under quite general assumptions, $I_{\mathbf{N}}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2)$ is asymptotically unbiased for $f_3(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2)$ and that the use of a smooth data tapers $h_i, 1 \leq i \leq 3$, reduces the finite sample bias of the biperiodogram. However the biperiodogram is anticonsistent: this variance is proportional to the sample size $\mathbf{N} = (N, N)$. In order to ensure consistency, kernel methods use adequate Kernels with well chosen bandwidth to smooth the biperiodogram. Alternatively, we attempt to construct wavelet-thresholding estimator of the bispectrum, which outperform linear traditional ones.

More precisely, we will consider the following model:

$$I_{\mathbf{N}}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2) = f_3(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2) + e_{\mathbf{N}}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2). \tag{5.3.3}$$

Unlike the traditional one dimensional model in wavelet estimation the errors, $e_{\mathbf{N}}$ in this model is not Gaussian nor i.i.d.

For $f_3 \in \widetilde{L}^2(\pi_4)$ we have the representation

$$f_3(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2) = \sum_{(\mathbf{k}_1,\mathbf{k}_2)\in\Lambda_l^2} \alpha_{l,\mathbf{k}_1,\mathbf{k}_2}\Phi_{l,\mathbf{k}_1,\mathbf{k}_2}(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2) + \sum_{u=1}^{7}\sum_{j\geq l}\sum_{(\mathbf{k}_1,\mathbf{k}_2)\in\Lambda_j^2} \beta_{j,\mathbf{k}_1,\mathbf{k}_2,u}\Psi_{j,\mathbf{k}_1,\mathbf{k}_2,u}(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2).$$

In this section, we show that wavelet-thresholding estimators of bispectra $f_3(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2)$, attain near-optimal minimax rate of convergence in the two dimensional Sobolev ball

$$\mathbf{W}_{m,p}(C) = \left\{ \|f\|_{L_p(\boldsymbol{\pi}^2)} + \left\|\frac{\delta^m f}{\delta x_1^m}\right\|_{L_p(\pi_4)} + \left\|\frac{\delta^m f}{\delta x_2^m}\right\|_{L_p(\pi_4)} \leq C \right\}.$$

Such an estimator is obtained by using a four-dimensional wavelet decomposition of the tapered biperiodogram, threshold the obtained empirical wavelet coefficients and then reconstruct the estimator from the thresholded coefficients.

The empirical wavelet-coefficient of the bispectra are:

$$\begin{aligned}
\widetilde{\alpha}_{l,\mathbf{k}_1,\mathbf{k}_2} &= \int_{\pi_2}\int_{\pi_2} I_{\mathbf{N}}(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2)\Phi_{l,\mathbf{k}_1,\mathbf{k}_2}(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2)d\boldsymbol{\omega}_1 d\boldsymbol{\omega}_2,\\
\widetilde{\beta}_{j,\mathbf{k}_1,\mathbf{k}_2,u} &= \int_{\pi_2}\int_{\pi_2} I_{\mathbf{N}}(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2)\Psi_{j,\mathbf{k}_1,\mathbf{k}_2,u}(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2)d\boldsymbol{\omega}_1 d\boldsymbol{\omega}_2.
\end{aligned}$$

So the wavelet estimator is

$$\widehat{f}_3(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2) = \sum_{(\mathbf{k}_1,\mathbf{k}_2)\in\Lambda_l^2} \widetilde{\alpha}_{l,\mathbf{k}_1,\mathbf{k}_2}\Phi_{l,\mathbf{k}_1,\mathbf{k}_2}(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2) + \sum_{u=1}^{7}\sum_{j\geq l}\sum_{(\mathbf{k}_1,\mathbf{k}_2)\in\Lambda_j^2} \delta(\widetilde{\beta}_{j,\mathbf{k}_1,\mathbf{k}_2,u},\boldsymbol{\lambda}^{\mathbf{N}})\Psi_{j,\mathbf{k}_1,\mathbf{k}_2,u}(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2),$$

where $\delta(.)$ denotes soft or hard-thresholding and the threshold value $\lambda^{\mathbf{N}} > 0$.

Further, we denote by $\gamma_{j,\mathbf{k}_1,\mathbf{k}_2}$ one of the coefficients $\alpha_{j,\mathbf{k}_1,\mathbf{k}_2}$, $\beta_{j,\mathbf{k}_1,\mathbf{k}_2,u}$, by $\widetilde{\gamma}_{j,\mathbf{k}_1,\mathbf{k}_2}$ one of the coefficients $\widetilde{\alpha}_{j,\mathbf{k}_1,\mathbf{k}_2}, \widetilde{\beta}_{j,\mathbf{k}_1,\mathbf{k}_2,u}$ and the variance of these components will be denoted by $\sigma_{j,\mathbf{k}_1,\mathbf{k}_2}$, and by $\varphi_{j,\mathbf{k}_1,\mathbf{k}_2}$ the associated wavelet basis function. We denote also by $\gamma^{r,i}_{j,\mathbf{k}_1,\mathbf{k}_2}$ the real and imaginary parts of $\gamma_{j,\mathbf{k}_1,\mathbf{k}_2}$ and similarly for $\widetilde{\gamma}^{r,i}_{j,\mathbf{k}_1,\mathbf{k}_2}$. The variance of these components will be denoted by $\widetilde{\sigma}^{r,i}_{j,\mathbf{k}_1,\mathbf{k}_2}$.

5.3.2 The minimaxity of the estimator

Let

$$\mathcal{J}_\delta := \mathcal{J}_\delta(\mathbf{N}) = \{l \leq j, 2^j \leq N^{1-\delta}, \delta > 0\}.$$

where $(1-\delta)\, r(m,p) \geq \frac{m}{m+1}$, $r(m,p) = m+1+\frac{2}{\widetilde{p}}$ and $\widetilde{p} = \min(p,2)$.

By Conditions 5.1-5.4 the problem in model (5.3.3) is transferred to the following Gaussian regression

$$\zeta^{r,i}_{j,\mathbf{k}_1,\mathbf{k}_2} = \gamma^{r,i}_{j,\mathbf{k}_1,\mathbf{k}_2} + \widetilde{\sigma}^{r,i}_{j,\mathbf{k}_1,\mathbf{k}_2}\varepsilon_{j,\mathbf{k}_1,\mathbf{k}_2}, j \in \mathcal{J}_\delta, \mathbf{k}_1, \mathbf{k}_2 \in \Lambda^2_j, \tag{5.3.4}$$

where $\varepsilon_{j,\mathbf{k}_1,\mathbf{k}_2} \sim \mathcal{N}(0,1)$ are i.i.d. The near-minimaxity of the estimator is based on estimation of the third order cumulants (the empirical wavelet coefficients of the biperiodogram). Thus, similar results have been obtained for the estimation of spectrum.

Proposition 5.2 *For any $\rho > 0$, let $\mathcal{J}_{\delta,\rho} = \{l \leq j, 2^j \leq N^{1-\delta}, 2^j \geq N^{2\rho}\}$ and assume Conditions 5.1 through 5.4 holds. Then*

$$\left| cum\left(\frac{\widetilde{\gamma}^{r,i}_{j,\mathbf{k}_1,\mathbf{k}_2} - \gamma^{r,i}_{j,\mathbf{k}_1,\mathbf{k}_2}}{\widetilde{\sigma}^{r,i}_{j,\mathbf{k}_1,\mathbf{k}_2}}\right)\right| \leq (p!)^{3+3\gamma}\,(K_1 N)^{-2\mu(p-2)},$$

for appropriate K_1 and $\mu > 0$, $p \geq 3$.

Proof. the proof is similarly as that in Theorem 1 in Touati and Pesquet (2002).

Theorem 5.5 *Suppose that Conditions 5.1 through 5.4 holds and the threshold satisfies*

$$\widetilde{\sigma}^{r,i}_{j,\mathbf{k}_1,\mathbf{k}_2}\,[2\log(\sharp\,(\mathcal{J}_\delta))]^{1/2} \leq \boldsymbol{\lambda}^{\mathbf{N}}_{j,\mathbf{k}_1,\mathbf{k}_2} \leq KN^{-1}\sqrt{2\log(N)},$$

on $\mathcal{J}_\delta$, where K is a constant. Then,

$$\sup_{f_3 \in \mathbf{W}_{m,p}(C)} \left(E\left\{\left\|\widehat{f}_3 - f_3\right\|^2_{L_2(\pi_4)}\right\}\right) = O\left(\left(\frac{2\ln(N)}{N^2}\right)^{\frac{m}{m+1}}\right).$$

Proof. Using Proposition 5.2 and Theorem 1 in Touati and Pesquet (2002).

5.3.3 Further improvement of the estimator

The estimator $\widehat{f}_3$ reaches the desired near-optimal rate $\left(\frac{2\log(N)}{N^2}\right)^{\frac{m}{m+1}}$, but there are two obvious possibilities to improve it further for finite sample sizes.

First, in contrast to the usual kernel estimator of f_3, wavelet estimators are not translation-invariant. If we shift the biperiodogram by a certain amount (s_1, s_2), apply non linear thresholding and shift the estimate back by (s_1, s_2), this new estimator $\widehat{f}_3^{(s_1,s_2)}$ will differ from the unsifted variant $\widehat{f}_3$ in most cases. The only shift lengths which do not alter the estimator $\widehat{f}_3$ are multiples of the shift length of the wavelet basis at the coarsest scale, i.e. $\frac{(2\pi)^4}{2^l}$. On the other hand, there is no reason to assume that any of the possible shifts are always superior to the other ones. To weaken the effect of not being stationary wavelet transform and define, with shifts $s_{i,j} = (s_i, s_j)$ where $s_i = \frac{i(2\pi)^4}{2^l I}, i = 0, ..., I-1$, the new estimator

$$\widehat{f}_3^*(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2) = \frac{1}{I^2} \sum_{i,j=0}^{I-1} \widehat{f}_3^{(s_{i,j})}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2),$$

then, we obtain by Jensen's inequality that

$$\left\| \widehat{f}_3^* - f_3 \right\|_{L_p(\pi_4)}^2 \leq \frac{1}{I^2} \sum_{i,j=0}^{I-1} \left\| \widehat{f}_3^* - f_3 \right\|_{L_p(\pi_4)}^2, \tag{5.3.5}$$

where strict inequality holds if $\widehat{f}_3^{(s_{i,j})} \neq \widehat{f}_3^{\left(s_{i',j'}\right)}$ for any $(i,j) \neq (i',j')$. In particular $\widehat{f}_3^*$ also satisfy the result in theorem 5.5. Moreover, in view of the possibly strict inequality in (5.3.5) we hope to get a significant improvement for finite sample sizes.

Secondly, note that the bispectrum f_3 satisfies the symmetries below, whereas they are not satisfied by $\widehat{f}_3^*$ if compactly supported wavelets different from the Haar wavelets are used

$$f_3(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2) = \overline{f_3(-\boldsymbol{\omega}_1, -\boldsymbol{\omega}_2)} = f_3(\boldsymbol{\omega}_2, \boldsymbol{\omega}_1) = f_3(-(\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2), \boldsymbol{\omega}_2), \tag{5.3.6}$$

In order to construct an estimator which satisfies the symmetries above we take the mean of eight symmetric nearly optimal estimators:

$$\begin{aligned} \widehat{f}_3^{**}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2) \quad = \quad & \frac{1}{8}[\widehat{f}_3^*(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2) + \widehat{f}_3^*(\boldsymbol{\omega}_2, \boldsymbol{\omega}_1) + \overline{\widehat{f}_3^*(-\boldsymbol{\omega}_1, -\boldsymbol{\omega}_2)} + \overline{\widehat{f}_3^*(-\boldsymbol{\omega}_2, -\boldsymbol{\omega}_1)} \\ & + \widehat{f}_3^*(-(\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2), \boldsymbol{\omega}_1) + \widehat{f}_3^*(-(\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2), \boldsymbol{\omega}_2) \\ & + \overline{\widehat{f}_3^*(\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2, -\boldsymbol{\omega}_1)} + \overline{\widehat{f}_3^*(\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2, -\boldsymbol{\omega}_2)}]. \end{aligned}$$

Hence, we have again by Jensen's inequality, and the fact that f_3 satisfies (5.3.6), that the new estimator $\widehat{f}_3^{**}$ satisfies

$$\left\| \widehat{f}_3^{**} - f_3 \right\|_{L^2(\pi_4)}^2 \leq \left\| \widehat{f}_3^* - f_3 \right\|_{L^2(\pi_4)}^2,$$

where strict inequality holds if two of the eight estimators above are different.

Conclusion

In accordance with the stated objectives in the introduction, the study that we have conducted in this thesis has allowed us to contribute in enriching the approach of spectral analysis for random fields for a domain that was widely required in all types of applications in physics, array processing, seismic data processing and from multichannel EEG digital signal processing. This approach which was also the extension of spectral analysis of time series, has been studied by several authors including Rosenblatt, Guyon, Robinson and more.

This work is based on two types of analysis: Fourier analysis (Part I) and wavelet analysis (Part II). It is helpful to recall that we are interested in the structures probabilistic and inference statistics for random fields. Previously (Part I), we had to treat models which are capable of taking into account the non Gaussianity and spatiality dependence, and more specifically we describe the spatial subdiagonal bilinear process with respect to its transfer functions, and we give conditions ensuring the existence of regular second order stationary and ergodic solutions. Then we consider the third order structure and Yule-Walker equations (chapter 1).

In addition we presented (Chapter 2) estimation of spectral density for nonlinear models and upon which our study is to answer one of the basic problems of the analysis of this models is that the information contained in the spectrum is insufficient. We have considered the bispectral and trispectral density estimate and we have studied the asymptotic normality, then we have proposed the higher order spectral density estimation. This study was strengthened by the parameter estimation which is based on a functional of the spectrum and bispectrum, in Chapter 3.

During the last decade, wavelet analysis has expanded in different fields of science. In order to be applied this analysis to the discrete random fields taking into account the mathematical aspect of our study, we have developed an approach to treat the wavelet transform; it also seems important to study its structure and probabilistic inference. Finally, in addition to theoretical developments that we have proposed in this thesis, an interesting future research direction would be to develop this study in many fields and under several conditions especially applied to different fields of science which should deserve attention.

Appendix 2.1

Let $X = [X(i,j)]$ be an $(p_1+1)(p_2+1)$ matrix of observation on spatial series in plane

$$X = \begin{pmatrix} X(i,j) & X(i,j-1) & \cdots & X(i,j-p_2) \\ X(i-1,j) & X(i-1,j-1) & \cdots & X(i-1,j-p_2) \\ \vdots & \vdots & & \vdots \\ X(i-p_1,j) & X(i-p_1,j-1) & \cdots & X(i-p_1,j-p_2) \end{pmatrix}.$$

We let $\underline{X}(i,j) = vec(X)'$ denote the $P \times 1$ vectorisation of the matrix X with $P = (p_1+1)(p_2+1)$. Then

$$\underline{X}(i,j) = \begin{bmatrix} \underline{X}_i(i,j) & \underline{X}_{i-1}(i,j) & \cdots & \underline{X}_{i-p_1}(i,j) \end{bmatrix}'$$

where $\underline{X}_{i-k}(i,j) = (X(i-k,j), X(i-k,j-1), ..., X(i-k,j-p_2))', k = 0, ..., p_1$. These imply that in two dimensions, $SSBL_d(\mathbf{p},\mathbf{q},\mathbf{P},\mathbf{Q})$ models (1.3.2) can write in matrix form as

$$\begin{aligned} \underline{X}(i,j) &= A_1\underline{X}(i-1,j) + A_2\underline{X}(i,j-1) + B\underline{e}(i,j) \\ &+ \sum_{k_1=1}^{Q_1}\sum_{k_2=0}^{Q_2}\left[C^{(1)}_{k_1k_2}\underline{X}(i,j-1) + D^{(1)}_{k_1k_2}\underline{X}(i-1,j)\right] e_{i-k_1,j-k_2} \\ &+ \sum_{k_2=1}^{Q_2}\left[C^{(2)}_{0k_2}\underline{X}(i,j-1) + D^{(2)}_{0k_2}\underline{X}(i-1,j)\right] e_{i,j-k_2} \end{aligned}$$

where $\underline{X}(i-1,j) = \begin{bmatrix} \underline{X}_{i-1}(i,j) & \cdots & \underline{X}_{i-p_1}(i,j) & \underline{0} \end{bmatrix}', \underline{X}(i,j-1) = \begin{bmatrix} \underline{X}_{j-1}(i,j) & \underline{0} & \cdots & \underline{0} \end{bmatrix}'$ and

$$A_1 = \begin{bmatrix} \Gamma_1 & \Gamma_2 & \cdots & \Gamma_{p_1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & & \mathbf{0} & \\ \vdots & & \ddots & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix}_{(p_1+1)\times(p_1+1)}, A_2 = \begin{bmatrix} \Gamma_0 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & & \mathbf{0} \\ \vdots & & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \end{bmatrix}_{(p_1+1)\times(p_1+1)},$$

$$
\begin{aligned}
\Gamma_0 &= \begin{bmatrix} a_{01} & a_{02} & \cdots & a_{0p_2} & 0 \\ 1 & 0 & & 0 & \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}_{(p_2+1)\times(p_2+1)}, \Gamma_k = \begin{bmatrix} a_{k0} & a_{01} & \cdots & a_{kp_2} \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}_{(p_2+1)\times(p_2+1)}, \\
k &= 1, ..., p_1 \\
\mathbf{0} &= \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}_{(p_2+1)\times(p_2+1)}, \mathbf{1} = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}_{(p_2+1)\times(p_2+1)}
\end{aligned}
$$

and

$$
B = \begin{bmatrix} B_0 & B_1 & \cdots & B_{p_1} \\ \mathbf{0} & \mathbf{0} & & \mathbf{0} \\ \vdots & & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \end{bmatrix}_{(p_1+1)\times(p_1+1)},
$$

$$
B_0 = \begin{bmatrix} 1 & b_{01} & \cdots & b_{0p_2} \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}_{(p_2+1)\times(p_2+1)}, B_k = \begin{bmatrix} b_{k0} & b_{01} & \cdots & b_{kp_2} \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}_{(p_2+1)\times(p_2+1)}, k = 1, ..., p_1
$$

$$
C^{(1)}_{k_1k_2} = \begin{bmatrix} \Sigma_0^{(k_1k_2)} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & & \\ \vdots & & \ddots & \vdots \\ \mathbf{0} & \cdots & & \mathbf{0} \end{bmatrix}_{(p_1+1)\times(p_1+1)}, C^{(2)}_{0k_2} = \begin{bmatrix} \Sigma_0^{(0k_2)} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & & \\ \vdots & & \ddots & \vdots \\ \mathbf{0} & \cdots & & \mathbf{0} \end{bmatrix}_{(p_1+1)\times(p_1+1)},
$$

$$
\Sigma_0^{(k_1k_2)} = \begin{bmatrix} c_{01}^{(k_1k_2)} & \cdots & c_{0p_2}^{(k_1k_2)} & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}_{(p_2+1)\times(p_2+1)}, \Sigma_0^{(0k_2)} = \begin{bmatrix} c_{01}^{(0k_2)} & \cdots & c_{0p_2}^{(0k_2)} & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}_{(p_2+1)\times(p_2+1)},
$$

and

$$D_{k_1k_2}^{(1)} = \begin{bmatrix} \Pi_1^{(k_1k_2)} & \Pi_2^{(k_1k_2)} & \cdots & \Pi_{p_1}^{(k_1k_2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & & \mathbf{0} & \\ \vdots & & \ddots & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}_{(p_1+1)\times(p_1+1)},$$

$$D_{0k_2}^{(2)} = \begin{bmatrix} \Pi_1^{(0k_2)} & \Pi_2^{(0k_2)} & \cdots & \Pi_{p_1}^{(0k_2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & & \mathbf{0} & \\ \vdots & & \ddots & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}_{(p_1+1)\times(p_1+1)},$$

$$\Pi_i^{(k_1k_2)} = \begin{bmatrix} c_{i0}^{(k_1k_2)} & c_{i1}^{(k_1k_2)} & \cdots & c_{ip_2}^{(k_1k_2)} \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}_{(p_2+1)\times(p_2+1)},$$

$$\Pi_i^{(0k_2)} = \begin{bmatrix} c_{i0}^{(0k_2)} & c_{i1}^{(0k_2)} & \cdots & c_{ip_2}^{(0k_2)} \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}_{(p_2+1)\times(p_2+1)},$$

$i = 1, ..., p_1$, and we can write in the form

$$\underline{X}(\mathbf{t}) = B(\mathbf{t})\underline{e}(\mathbf{t}) + \mathcal{A}(\mathbf{t})\underline{X}(\mathbf{t}-\mathbf{e}_1) + \mathcal{B}(\mathbf{t})\underline{X}(\mathbf{t}-\mathbf{e}_2).$$

i.e.

$$\underline{X}(i,j) = B(i,j)\underline{e}(i,j) + \mathcal{A}(i,j)\underline{X}(i-1,j) + \mathcal{B}(i,j)\underline{X}(i,j-1)$$

where

$$\mathcal{A}(\mathbf{t}) = \left[A_1 + \sum_{k_2=1}^{Q_2} C_{0k_2}^{(1)} e_{i,j-k_2} + \sum_{k_1=1}^{Q_1}\sum_{k_2=0}^{Q_2} C_{k_1k_2}^{(2)} e_{i-k_1,j-k_2}\right]$$

$$\mathcal{B}(\mathbf{t}) = \left[A_2 + \sum_{k_2=1}^{Q_2} D_{0k_2}^{(1)} e_{i,j-k_2} + \sum_{k_1=1}^{Q_1}\sum_{k_2=0}^{Q_2} D_{k_1k_2}^{(2)} e_{i-k_1,j-k_2}\right]$$

Appendix 2.2

Theorem 5.6 *Let* $(e(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^2}$ *be a strictly stationary ergodic random field and let* $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^2}$ *be as defined in* $(4.4.2)$. *Assume that* $\mathcal{A}(\mathbf{t})$ *and* $\mathcal{B}(\mathbf{t})$ *are spectral radius diagonable matrices with* $\|P^{-1}\mathcal{A}(\mathbf{t})P\|_\infty = \mathcal{C}(\mathbf{t}), \|P^{-1}\mathcal{B}(\mathbf{t})P\|_\infty = \mathcal{D}(\mathbf{t})$. *Assume further that*

$$\sup E\{\log(\mathcal{C}(\mathbf{t}))\} < 0 \text{ and } \sup E\{\log(\mathcal{D}(\mathbf{t}))\} < 0.$$

Then for every $\mathbf{t} = (i,j) \in \mathbb{Z}^2$ *and* $\mathbf{r} = (r_1, r_2)$

$$\sum_{r_1+r_2\geq 1} T^{r_1,r_2}(i,j)\, B\underline{e}(i-r_1, j-r_2),$$

converges absolutely a.s., where the transition matrix $T^{r_1,r_2}(i,j)$ *is defined as follows:*

1. $T^{0,1}(i,j) = \mathcal{A}(i,j)$ *and* $T^{1,0}(i,j) = \mathcal{B}(i,j)$
2. $T^{r_1,r_2}(i,j) = \mathcal{A}(i,j)\, T^{r_1,r_2-1}(i,j-1) + \mathcal{B}(i,j)\, T^{r_1-1,r_2}(i-1,j)$
3. $T^{0,0}(i,j) = I_{d\times d}$, *and*
4. $T^{-r_1,r_2}(i,j) = T^{r_1,-r_2}(i,j) = O_{d\times d}$.

Further, if

$$\underline{X}(i,j) = \sum_{r_1+r_2=0}^{\infty} T^{r_1,r_2}(i,j)\, B\underline{e}(i-r_1, j-r_2), \tag{2.3.3}$$

then $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^2}$ *is a strictly stationary process satisfying* $(4.4.2)$. *Conversely, if* $(X(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^2}$ *is a strictly stationary process satisfying* $(4.4.2)$, *for some strictly stationary and ergodic sequence* $(e(\mathbf{t}))_{\mathbf{t}\in\mathbb{Z}^2}$ *with* $\sup E\{\log(\mathcal{C}(\mathbf{t}))\} < 0$ *and* $\sup E\{\log(\mathcal{D}(\mathbf{t}))\} < 0$. *Then* $\underline{X}(i,j)$, *satisfying* $(4.4.2)$.

Proof. The proof is similar as that of Theorem 2 in Chanda (1991).

Bibliography

[1] Alekseev, V. G. (2010). On nonparametric estimates of higher order spectral densities. Theory Prob. Appl. Vol. 54. No. 3, 499-504.

[2] Alekseev, V. G. (1990). On certain properties of statistical estimates of higher order spectral densities. Theory Prob. Appl. Vol. 35. No. 3.

[3] Alekseev, V. G. (1990). On the uniform convergence of estimates of the spectral density of a homogeneous Gaussian random field, Theory Prob. Appl. Vol. 35. No. 3.

[4] Alekseev, V. G. (1973). Empirical spectral analysis of homogeneous and isotropic random fields. Theory Prob. Appl. Vol. 35, 2.

[5] Amirmazlaghani, M., Amindavar, H. (2007). Image denoising using two-dimensional GARCH model. Systems. Signals and image processing, 397-400.

[6] Antoine, J. P., Murenzi, R., Vanderghynst, P., Twareque Ali, S. (2004). Two dimensional wavelets and their relatives. Cambridge Press.

[7] Basu, S., Reinsel, G. C. (1992). A note on the properties of spatial Yule-Walker estimators. Journal of statistical computation and simulation. Vol. 41, Issue. 3, 243-255.

[8] Basu, S., Reinsel, G. C. (1993). Properties of the spatial unilateral first-order ARMA model. *Adv. Appl. Prob.* **25,** 631-648.

[9] Benedetto, J. J., Frazier, M. W. (1994). Wavelets. Mathematics and applications. CRC Press.

[10] Berg, A., Politis, D. N. (2009). Higher-order accurate polyspectral estimation with flat-top lag-windows. Ann Inst Stat Math 61, 477-98.

[11] Bibi, A. (2003). On the covariance structure of time-varying bilinear models. Stoch. Anal. App. Vol. 21, 25-60.

[12] Bibi, A. (2006). Evolutionary transfer functions of bilinear processes with time-varying coefficients. Computers Math. Applic. Vol. 52, 331-338.

[13] Bibi, A., Kimouche, K. (2012). On $\mathbb{L}_2$ structure of bilinear models on $\mathbb{Z}^d$. C. R. Acad. Sci. Paris. Ser, I 350, 427-432.

[14] Bradley, R. C. (1992). On the spectral density and asymptotic normality of weakly dependent random fields. Journal of Theoretical Probability. Vol. 5. No. 2.

[15] Brillinger, D. R. (1965). An introduction to polyspectra. Ann. Math. Statistics. 36, 1351–1374.

[16] Brillinger, D. R. (1975). Statistical inference for stationary point processes, stochastic processes and related topics. In proceeding of the summer research institute on statistical inference for stochastic processes. New York, 55-99.

[17] Brillinger, D. R. (1981). Time Series: Data Analysis and Theory. Mc GrawHill, New York.

[18] Broughton, S. A., Bryan, K. (2008). Discrete Fourier analysis and wavelets. Applications to signal and image processing, A John Wiley and Song, Inc. density function by wavelets with an application to competing risks. Indian statistical institute. Delhi centre.

[19] Chanda, K. C. (1991). Stationarity and central limit theorem associated with bilinear time series models. J. Time ser. Anal. Vol. 12. No. 4.

[20] Chanda, K. C. (1993). Asymptotic properties of serial covariances for nonlinear stationary processes. J. of multivariate Ana. 47, 163-171.

[21] Chanda, K. C. (2005). Large sample properties of spectral estimators for a class of stationary nonlinear processes. J. Times ser. Anal. 26, 1-16.

[22] Chesneau, C., Dewan, I., Doosti, H. (2011). Nonparametric estimation of two dimensional continuous-discrete. J. international of mathematics and mathematical sciences, 1-21.

[23] Chesneau, C. (2011). Adaptive wavelet estimation of a based density for strongly mixing sequences. J. international of mathematics and mathematical sciences.

[24] Chesneau, C. (2012). On the adaptive wavelet deconvolution of a density for strong mixing sequences. Journal of the Korean statistical society, 1-14.

[25] Chesneau, C., Doosti, H. (2012). Wavelet linear density estimation for GARCH model under various dependence structures. JIRSS. Vol. 11, 1-21.

[26] Chian, C., Morettin, P. A. (1998). A wavelet analysis for time series. Nonparametric Statistics. Vol. 10, 1-46.

[27] Choi, B. (1997). A recursive algorithm for solving the spatial Yule-Walker equations of causal spatial AR models. Statistics and Probability Letters. 33, 241-251.

[28] Choi, B. (2001). The asymptotic joint distribution of the Yule-Walker estimators of a causal multidimensional AR process. Com. Stat. Theory Math. 30 (4), 601-614.

[29] Chui, C. K. (1992). An introduction to wavelets. Academic press.

[30] Clouet, J. E., Fouque, J. R., Postel, M. (1995). Estimation of local power spectral densities for non-stationary signals using wavelet transform. Mathematics and Computers in Simulation 38, 183-188.

[31] Crujeiras, R. M., Fernandez-Casal, R. (2009). On the estimation of the spectral density for continuous spatial processes. Statistics. 1-14.

[32] Dai, Y., Billard, L. (1998). A space time bilinear model and its identification. J. Time Ser. Analysis. Vol. 19, 657-79.

[33] Dai, Y., Billard, L. (2003). Maximum likelihood estimation in space time bilinear models. J. Time Ser. Analysis. Vol. 24, 26-44.

[34] Daubechies, I. (1990). The wavelet transform, time-frequency localization and signal analysis. IEEE. Trans. Inform. Theory. 36, 961–1005.

[35] Dijkerman, R. W., Mazumdar, R. R. (1994). Wavelet representation of stochastic processes and multiresolution stochastic models. IEEE trans. Signal Process, 1640-1652.

[36] Dimitriou-Fakalou, C. (2009). Modified Gaussian likelihood estimators for ARMA models on $\mathbb{Z}^d$. Stoch. Proc. App. Vol. 119, 4149-4175.

[37] Doukhan, P., Truquet, L. (2007). A fixed point approach to model random fields. Alea 3, 111-132.

[38] Failla, G., Pappatico, M., Cundari, G. A. (2011). A wavelet-based spectrum for non-stationary processes. Mechanics Research Communications 38, 361– 367.

[39] Gabr, M. M. (1988). On the third-order moment structure and bispectral analysis of some bilinear time series. Journal of Time Series Analysis. 9, 11-20.

[40] Gaetan, C., Guyon, X. (2010). Spatial statistics and modeling. Springer.

[41] Gaetan, C., Guyon, X. (2008). Modélisation et statistique spatiales. Springer.

[42] Giraitis, L., Surgailis, D. (2002). ARCH-Type bilinear models with double long memory. Stoch. Proc. and their App. 100, 275-300.

[43] Glindemann, A. Lane, R. G., Dainty, J. C. (1992). Bispectral parameter estimation using least square. ESO Conference on High-Resolution Imaging by Interferometry II. Germany.

[44] Grossmann, A., Morlet, J. (1985). Decomposition of functions into wavelets of constant shape, and related transforms. In: Mathematics, Physics, Lectures on recent results. I., 135–66.

[45] Guo, J. H., Billard, L. (1998). Some inference results for causal autoregressive processes on a plane. J. Time Ser. Analysis. Vol. 19, 681-691.

[46] Guyon, X. (1982). Parameter estimation for a stationary process on a d-dimensional lattice. Biometrika 69, 95-106.

[47] Guyon, X. (1995). Random fields on the network. Springer-Verglag. New York.

[48] Hannan, E. J. (1982). A note on bilinear time series models. Stoc. Proc. App., 12, 221-224.

[49] Harbaugh, J. W., Preston, F. W. (1968). Fourier series analysis in geology, in: B.J.L. Berry, S.F. Marble(Eds), Spatial analysis: A Reader in statistical Geography, Prentice-Hall, Englewood Cliffs, NJ.

[50] Härdle, W., Kerkyacharian, G., Picard, D., Tsybakov, A. (1998). Lectures notes in statistics. Wavelet. Approximation and statistical applications. New York, Springer Verglas. Vol. 129.

[51] Hongwei, L., Qiansheng, C. (1998). Convergence properties of higher order moment and cumulant estimates, Journal of electronics. Vol. 15. 3, 240- 247.

[52] Hosseinion, N., Doosti, H., Nirumand, H. A. (2007). Wavelet based estimators of the integrated density derivatives for a mixing sequences. Proceeding of the 9th Islamic countries conference on statistical sciences.

[53] Huang, G., Chen, X. (2009). Wavelets-based estimation of multivariate evolutionary spectra and its application to nonstationary downburst winds. Engineering Structures 31, 976-989.

[54] Jain, A. K. (1981). Advances in mathematical models for image processing. Proc. IEEE 69, 502-528.

[55] Kharfouchi, S., Kimouche, K. (2011). Central limit associated with bilinear random fields. Random Oper. Stoch. Equ. 19, 157-163.

[56] Kim, P. T. (1988). Consistency estimation of the fourth-order cumulant spectral density. J. Times series analysis. Vol. 12. N. 1, 63-71.

[57] Leblanc, F. (1996). Wavelet linear density estimator for a discrete times stochastic process: L_p-Losses, Statistics and probability letters. Vol. 27, 71-84.

[58] Li, H., Cheng, Q. (1998). Convergence properties oh higher-order moment and cumulant estimates. J. Of electronics. Vol. 15. No. 3, 240-247.

[59] Lii, K. S., Rosenblatt, M. (1990). Asymptotic normality of cumulant spectral estimates. Journal of Theoretical Probability. Vol. 3, No. 2.

[60] Lii, K. S., Rosenblatt, M. (1993). Bispectra and Phase of Non-Gaussian Linear Processes. Journal of Theoretical Probability. Vol. 6. No. 3.

[61] Major, P. (1981). Multiple Wiener-Itô integrals. Lecture Notes in mathematics. No. 849, Springer.

[62] Mallat, S. (2009). A wavelet tour of signal processing (3rd ed.). Amesterdam: Elsiver.

[63] Mary, E. (1993). Wavelet transform of stochastic processes with stationary increments and its application to fractional Brownian motion. IEEE Trans. Inform. Theory, 260-264.

[64] Mandelbrot, B. B. (1975). Stochastic models for the Eart's relief. the shape and fractal dimension of the coastlines, and the number-area rule for islands. Proc. Nat. Acad. Sci. USA 72, 3825-3828.

[65] Mandelbrot, B. B. (1975). On the geometry of homogeneous turbulence, with stress on the fractal dimension of the iso-surface of scalars. J. Fluid Mech. 72, 401-417.

[66] Matern, B. (1960). Spatial variation. Comm. Swed. Forestry Res, Inst. 49, 1-144.

[67] Masry, E. (1998). Covariance and spectral properties of the wavelet transform and discrete wavelet coefficients of second order random fields. Signal Processing. Vol. 64, 131-143.

[68] Meyer, Y. (1992). Wavelet and operators. Cambridge.

[69] Miller, C. (1995). A CLT for the periodogram of a ρ^*-mixing random fields. Stochastic Processes and their Applications. 60, 313-330.

[70] Mokkadem, A. (1987). Sur un modèle autorégressif non linéaire, ergodicité et ergodicité géométrique. J.S.T.A., Vol. 8, 195-204.

[71] Mondal, D., Percival, D. B. (2009). Wavelet variance analysis for random fields.

[72] Moor, M. (1988). Spatial linear processes. Comm. Statist, -Stochastic Models, 4 (1), 45-75.

[73] Naveau, P., Brockwell, P., Nychka, D. (1999). Dependence structure of random wavelet coefficient in terms of cumulant. Proceeding, Vol. 2. Curve and surface fitting.

[74] Neumann. M. H. (1996). Spectral density estimation via nonlinear wavelet methods for stationary non Gaussian times series, Journal of Time Series Analysis. Vol. 17. 6, 601-633.

[75] Nikias, C. L., Petropulu, A. P. (1993). Higher-order spectra analysis. A nonlinear signal processing Framework. prentice Hall, Englewood Cliffs, NJ.

[76] Noiboar, A., Cohen, I. (2007). Anomaly detection based on wavelet domain GARCH random field modelling. IEEE Trans. Geoscience and remotesensing. Vol. 45. No. 5.

[77] Percival, D. B. (2000). Wavelet methods for time series analysis. Cambridge.

[78] Perera, G. (1997). Geometry of $\mathbb{Z}^d$ and the central limit Theorem for weakly dependent random fields. Journal of Theoretical Probability, Vol. 10, No. 3.

[79] Pinsky, M. A. (2002). Introduction to Fourier analysis and wavelets. American mathematical society.

[80] Pierson, W. J. Tick, L. J. (1957). Stationary random processes in meteorology and oceanography. Bull. Inst. Int . Statist. 35. 2, 271-281.

[81] Priestly, M. B. (1964). The analysis of two dimensional stationary processes with discontinuous spectra. Biometrika, 51, 1 and 2, 195.

[82] Rachdi, M., Sabre, R. (2008). Mixed-Spectra analysis for stationary random fields. Springer,1-26.

[83] Rachdi, M., Sabre, R. (1998). Le choix optimal de la largeur de fenêtre spectrale pour un champs aléatoire. Traitement du signal. Vol. 15. 6, 569–575.

[84] Rice. J. (1979). On the Estimation of the Parameters of a Power Spectrum. Journal of multivariate analysis 9, 378-392.

[85] Robinson. P. M. (2006). Nonparametric spectrum estimation for spatial data. London School of Economics.

[86] Rosenblatt, M., Van Ness, J. W. (1965). Estimation of the bispectrum. Springer Science.

[87] Rosenblatt, M. (1985). Stationary sequences and random fields. Birkhäuser.

[88] Ruch, D. K., Van fleet, P. J. (2009). Wavelet theory, An elementary approach with applications. A John Wiley and Song, Inc.

[89] Rudzkis, R., Saulis, L., Statulevicius, V. (1978). A general lemma on probabilities of large deviations. Lits. Math. J. Vol. 18, 1, 226-238.

[90] Shao, X., Wu, W. B. (2007). Asymptotic spectral theory for nonlinear times series, The annals os statistics. Vol. 35. 4, 1773-1801.

[91] Subba Rao, T. and Gabr, M. M. (1984). An introduction to bispectral analysis and bilinear time series models, In Lecture Notes in Statistics. No 24. Springer-Verglag, Berlin.

[92] Subba Rao, T., Indukumar, K. C. (1996). Spectral and wavelet methods for the analysis of nonlinear and nonstationary times series. J. Franklin Inst. Vol. 333 (B), 425-452.

[93] Swami, A., Giannakis, G. B., Zhou, G. (1997). Bibliography on higher-order statistics. Signal processing, Vol. 60, 65-126.

[94] Sylvester, R. (1974). Coastal Engeigneering. I: Generation, Propagation, and Influence of Waves, Developments in Geatechnical Engineering. Elsiver, New York.

[95] Terdik, G., Subba Rao, T. (1989). On Wiener-Itô representation and the best linear predictions for bilinear time series. J. App. Prob. Vol. 26, 274-286.

[96] Terdik, G. (1991). Bilinear state space realization for polynomial sthocastic systems. Computers Math. Applic. Vol. 22, No. 7, 69-83.

[97] Terdik, G. (1991). The exact bispectra for bilinear realizable processes with hermit degree 2. Adv. Appl. Prob. Vol. 23, 798-808.

[98] Terdik, G. (2000). Bilinear stochastic models and related problems of nonlinear time series: A frequency domain approach. Lecture Notes in Statistics. No. 124. Springer.

[99] Terdik, G. (2011). Long-range dependence in third order and bispectrum singularity. Periodica Mathematics Hungaria, Vol. 62 (1), 103-119.

[100] Tewfik, A. H. (1992). Correlation structure of the discrete wavelet coefficient of fractional Brownian motion, IEEE Trans. Inform. Theory, 904-909.

[101] Tjostheim, D. (1978). Statistical spatial series modelling I. Adv. App. Prob. Vol. 10, 130-154.

[102] Tjostheim, D. (1983). Statistical spatial series modelling II. Some further results on unilateral lattice processes. Adv. Appl. Probab. 15, 562-584.

[103] Touati, S., Pesquet, J. C. (2002). Wavelet-thresholding for bispectrum estimation. European Signal and Image Processing Conference (EUSIPCO 2002). Toulouse, France. 107-110.

[104] Van Ness, J. W. (1966). Asymptotic normality of bispectral estimates. The annals of Mathematical Statistics. Vol. 35 (5), 1257-1272.

[105] Vidal-Sanz, J. (2009). Automatic spectral density estimation for random fields on a lattice via bootstrap, TEST: An Official Journal of the Spanish Society of Statistics and Operations Research. Springer. Vol. 18 (1), 96-114.

[106] Vidakovic, B. (1999). Statistical modeling by wavelets. A John Wiley and Song. Inc.

[107] Walnut, D. F. (2002). An introduction to wavelet analysis. Birkhäuser.

[108] Wang, H. B., Wei, B. C. (2004). Separable lower triangular bilinear model. J. App. Prob.41, 221-235.

[109] Yao, Q., Brockwell, P. (2006). Gaussian Maximum likelihood estimation for ARMA models II: Spatial processes. Bernoulli. Vol. 12(3), 403-429.

[110] Whitle, P. (1954). On stationary processes in the plane. Biometrika. 41, 434-449.

[111] Yuan, J., Subba Rao, T. (1993). Spectral estimation for random fields with application to Markov modelling and texture classification. Theory and applications. Academic Press. Boston, 179-210.

[112] Yuan, J., Subba Rao, T. (1993). Higher order spectral estimation for random fields. Multidimensional Systems and Signal Processing. 4, 7-22.

[113] Yuan, J. (1999). Testing Gaussianity and linearity for random fields in the frequency domain. J. Times series analysis. Vol. 21. No. 6.

Abstract

This thesis is devoted mainly to the study of spectral analysis of random fields, which based on the Fourier analysis and wavelet analysis. Among the numerous random fields in the literature, we have chosen to explore a particular class of models which are capable of taking into account the non Gaussianity character and spatiality behavior. Principally we study the L_2-structure of some SBL models and we establish the spectral density estimation, then we obtained the bispectral and higher order spectral density estimation in which these results can be used to discriminate between linear and nonlinear models.

We show also that the estimator of the parameter obtained as minimum of a particular quadratic form which depends on the second and third spectra is consistent and asymptotically normal under certain assumptions.

However, In the second part of this thesis, we are interested to examine the fundamental concepts needed in the study of the wavelet transform and random fields. Finally, we consider the nonlinear wavelet estimators of the spectral density and we continued investing in estimation by proposing wavelet-thresholding estimator of the bispectrum.

Keywords:

- Fourier analysis
- Wavelet analysis
- Random fields
- Spectral density
- SBL models

الملخــــص

كرست هذه الأطروحة أساسا إلى دراسة التحليل الطيفي للحقول العشوائية، والذي يرتبط أساسا على تحليل فورييه وتحليل المويجات. من بين العديد من الحقول العشوائية المقترحــة في المجــال ، اخترنا تقصي فئة معينة من النماذج التي هي قادرة على الأخذ بعين الاعتبار الطابع الغير جوسياني (non gaussianité) والسلوك المكاني ، حيث نقوم أساسا بدراسة الهيكل الاحتمالي لبعض النماذج SBL و كذا تقدير الكثافة الطيفية، ثم الحصول علي تقدير الكثافة الطيفية الثنائية و من الدرجة العليا والتي يمكن استخدامهم في التمييز بين النماذج الخطية وغير الخطية.

كما نعرض أيضا في هذه الأطروحة بأن تقدير المتغير الذي تم الحصول عليه من خلال الحد الأدنى لشكل خاص من الدرجة الثانية و الذي يعتمد على الأطياف الثنائية والثلاثية، ثابت و بشكل مقارب طبيعي في ظل افتراضات معينة.

نحن مهتمون في الجزء الثاني من هذه الأطروحة لدراسة المفاهيم الأساسية اللازمة في دراسة تحويل المويجات والحقول العشوائية، حيث نرى في الأخير تقديرات الغير خطية للكثافة الطيفية بطريقة المويجات و واصلنا الدراسة باقتراح تقدير العتبة للكثافة الطيفية الثنائية .

الكلمــــات المفتاحيــــة :

- تحليل فورييه
- تحليل المويجات
- حقول عشوائية
- الكثافة الطيفية
- النماذج SBL

Résumé

Cette thèse est consacrée essentiellement à l'étude de la densité spectrale dans les champs aléatoires non linéaires, qui basé à l'analyse de Fourier et l'analyse en ondelettes. Parmi les nombreux domaines aléatoires dans la littérature, nous avons choisi d'explorer une classe particulière de modèles qui sont capables de prendre en compte le caractère de non gaussianité et le comportement de spatialité. Principalement nous avons étudié la structure- L_2 de certains modèles SBL et nous avons établi l'estimation de la densité spectrale, alors la fonction de la densité bispectrale et la densité spectrale d'ordre supérieur sont obtenues dans lesquels ces résultats peuvent être utilisés pour distinguer entre les modèles linéaires et les modèles non linéaires.

Nous avons également montré que l'estimateur du paramètre obtenu en moins d'une forme particulièrement quadratique qui dépend du spectre de deuxième et troisième ordre est consistent et asymptotiquement normal sous certaines hypothèses.

Cependant, dans la deuxième partie de cette thèse, nous nous intéressons à examiner les concepts fondamentaux nécessaires à l’étude de la transformée en ondelettes et les champs aléatoires. Enfin, nous avons considéré les estimateurs non linéaires de la densité spectrale par méthode d’ondelettes, et nous avons continué l’investissement dans l’estimation en proposant l’estimateur de seuillage du bispectre par la même méthode.

Mots clés

- L’analyse de Fourier
- L’analyse en ondelettes
- Champs aléatoires
- La densité spectrale
- Les modèles SBL