REPUBLIQUE ALGERIENNE DEMOCRATIQUE ET POPULAIRE
MINISTERE DE L'ENSEIGNEMENT SUPERIEUR ET DE LA RECHERCHE SCIENTIFIQUE

UNNIVERSITE DES FRERES MENTOURI CONSTANTINE I
FACULTE DES SCIENCES EXACTES
DEPARTEMENT DE PHYSIQUE

N ° d'ordre :……………
Série :…………………...

THESE

PRESENTEE POUR OBTENIR LE DIPLOME DE DOCTORAT
EN SECIENCES PHYSIQUE

SPECIALITE

PHYSIQUE THEORIQUE

THEME

Non-Hermitian and Deformed Hamiltonians

Par

MORCHEDI AMOR

Soutenu le : 02/ 05/ 2019

Devant le jury :

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Non-Hermitian and Deformed Hamiltonians

Morchedi Amor

I am grateful for my adviser Mr. Mebarki for his advices.

Many thanks to my family, in particular my mother,my brother Sofiane.

Greeting to jury members, those have accepted examine my thesis.

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Introduction

In the last two decades, non-Hermitian Hamiltonian systems in particular those having a real energy spectrum have attracted much of interest in theoretical physics, where the reality of spectrum has a deep sense and appeared in a whole class of Hamiltonian systems. The story began in 1993 by C.Bender when he was visiting CEN Saclay, when he interested in the work of Bessis and Zinn-Justin, where they had noticed on the basis of numerical work that some of eigenvalues of cubic anharmonic oscillator [1, 2] seemed to be real and wondered if the spectrum might be entirely real. however, he started his program of research on such class of non-Hermitian Hamiltonians. He argued that the reality of spectrum might be interpreted by the existence of a symmetry, he worked on this purpose a lot until 1998, he published an article with S. Boettcher on [1, 3, 4] where they had defined it as a PT-symmetry. Meaning to say it plays a key role at generalizing quantum mechanics, where Bender and his collaborators [1, 3, 5] adopted all quantum mechanics axioms except the one that restricted the Hamiltonian to be Hermitian, they replaced the latter condition with the requirement that the Hamiltonian must have an exact PT-symmetry. This field of research has been a great deal of interest in the study of non-Hermitian as well as PT- symmetric quantum mechanics, the reason is the fact that this generalization of the standard quantum mechanics to the complex regime renders the spectrum completely real if the PT-symmetry is not spontaneously broken [1], nevertheless this symmetry failed to give a general and clear picture of non-Hermitian Hamiltoniains having a real spectrum, to respond to this crucial question, Mostafazadeh [3, 6] has established that PT symmetry which is neither necessary nor sufficient to assure the reality of the spectrum, where the search for finding a condition which is both necessary and sufficient has led to a notion of pseudo-Hermiticity [6]. moreover if a given pseudo Hermitian Hamitonian could be mapped to a Hermitian one by similarity transformation therefore it is no thing but a quasi-Hermitian Hamiltonian. Since the theory of quantum gravity stipulates in somehow a deformation on the space by choosing the position operator $X = (1 + p^2)x$, this automatically means a change on Heisenberg algebra so this might raise a problem of non Hermiticity of the Hamiltonian.

The purpose of the first three chapters is to study some of deformed Hamiltonian systems, not only by deforming a typical Hermitian Hamiltonian ones but in the large extent those are non-Hermitian in advance, meaning to say defined in the complex regime, so we'll study a deformed-shifted harmonic $p^2 + x^2 + i\epsilon x$ and deformed-cubic anharmonic $p^2 + x^2 + i\epsilon x^3$ oscillators as an examples of quasi-Hermitian Hamiltonians [6, 7, 8, 9, 10].
In 1935, a decade after the invention of quantum theory by Hiesenberg (1925) and Schrodinger (1926), three papers appeared and setting the stage for a new concept, quantum entanglement, which named by Erwin Schrodinger [11] in Germen "*Verschränkung*" that after his famous "cat paradox". Since 1970s entanglement could be observed directly in a laboratory, until 1990s where the experimental development have led to new concepts in information technology, including such topics quantum cryptography,quantum teleportation...etc, in which entanglement play a key role.
Thus, in the last chapter we'll work out for instance an entangled PT-symmetric spin Hamiltonian [12, 13] where we have to focus on the concept of quantum information processing (QIP) in which the quantum entanglement has considered as an interesting issue [14, 15, 16, 17] and we will see how is the magnitude of quantum entanglement perhaps depends on the parameters such the Hermiticity and metric.

Chapter 1

Non-Hermitian and Deformed Hamiltonians

1.1 Non-Hermitian Hamiltonians

One of the axioms of any quantum Hamiltonian system, is the mathematical condition of Dirac Hermiticity

$$H = H^{\dagger} \tag{1.1}$$

Where † stands for a (transpose + complex conjugate), the condition (1) is sufficient to guarantee that the energy spectrum is real and the time evolution is unitary, in fact this condition is not necessary, it is possible to describe a natural process by means of non-Hermitian Hamiltonians [1]. Since the work of Bender et al [2], non-Hermitian quantum systems have been studied extensively, where a diagonalizable non-Hermitian Hamiltonian having a real spectrum may be used to define a unitary quantum system if one modifies the inner product of Hilbert space properly. Bender and his collaborators [3] ,who adopted all axioms of quantum mechanics, except the one that restricted the Hamiltonian to be Hermitian, they replaced the later condition, with the requirement that the Hamiltonian must have an exact PT-symmetry

1.1.1 PT-Symmetric Hamiltonians

Nearly two decades earlier [1, 4], Bender and Boettcher introduced PT-symmetry in the context of non-Hermitian Hamiltonians, they considered it as a response of the question which says; why a given non-hermitian Hamiltonian, may have a real and positive energy spectrum?
First, they figured out that the reality of the spectrum is in part due to PT-symmetry, in which the Hamiltonians that are invariant under PT-Symmetry's

effect may have a real energy spectrum, to make sure, they examined this property on a class of complex Hamiltonians that has the form

$$p^2 + x^2 - (ix)^N, N \ is \ Real \tag{1.2}$$

In the articles [1, 4], Bender and his collaborators showed that the mathematical requirement of Dirac hermiticity (1.1) stands out and it's replaced by a transparent physical condition of space-time reflection

$$H = H^{PT} \tag{1.3}$$

where PT is defined in terms of the space-reflection operator P (parity) and time-reversal operator T. where P is a linear operator whose action on a set of coordinates (x, p) is

$$Px = -xP, \quad Pp = -pP \tag{1.4}$$

furthermore the linearity of P remains the commutation relations (Heisenberg algebra of quantum mechanics) invariant.

$$xp - px = i\hbar \tag{1.5}$$

The time-reversal operator T, leaves x invariant but changes the sign of both of p, and the complex number i

$$Tx = xT, \quad Tp = -TP \tag{1.6}$$

$$Ti = -iT \tag{1.7}$$

T needs to satisfy (1.5), which leads to(1.7), as a result T is an anti-linear operator. Since P and T are reflection operators, their squares are the unit operator

$$P^2 = T^2 = 1 \tag{1.8}$$

Also P and T operators commute with each other

$$[P, T] = 0 \tag{1.9}$$

And finally, the condition (1.3) can be written as follows

$$[PT, H] = 0 \tag{1.10}$$

Often, despite the condition (1.10) is satisfied, it can't guarantee if the Hamiltonian H has to have an entire real and positive energy spectrum!! Thus,the Second thing that Bender and his collaborators [1, 3, 4] were interested in, is how to make sense of PT-symmetric Hamiltonians those have entirely real and positive energy spectrum, by pointing out a further restriction on the PT-symmetry itself, which has to be unbroken.

Unbroken PT-Symmetry

PT-symmetry is said to be unbroken, if the operators PT and H are simultaneously have the same eigenstates. Actually, the condition (1.3) determines a whole branch of non-Hermitian Hamiltonians H that are undergoing to PT-symmetry, but it could not guarantee that they have an entire real spectrum or not, in such case, the PT-symmetry needs to be exact which means unbroken. Let's assume ψ are the eigenstates of the both of H and PT operators, where the PT eigenvalue equation given as

$$PT\psi = \lambda\psi \tag{1.11}$$

and the independent time Schrodinger equation with the energy spectrum E written as

$$H\psi = E\psi \tag{1.12}$$

by multiplying PT on the left of the equation (1.11) and using the property $(PT)^2 = \mathbb{1}$ we obtain

$$\psi = PT\lambda(PT)^2\psi \tag{1.13}$$

PT is anti-linear operator, which implies

$$\psi = (PT)^2\lambda PT\psi \tag{1.14}$$

Use the equation (1.11), we get

$$\psi = (PT)^2\lambda PT\psi = \lambda\lambda^*\psi = \lambda^2\psi \tag{1.15}$$

however λ it cannot be more than a pure phase

$$\lambda = e^{i\beta} \tag{1.16}$$

Next we go back to the Schrodinger eigenvalues equation and multiply it on the left and once again use the property $(PT)^2 = 1$

$$PT\ H\psi = PT\ E\psi = PT\ E(PT)^2\psi \tag{1.17}$$

and use one more time the equation (1.11) we get

$$PT\ H\psi = PT\ E\psi = PT\ EPT\lambda\psi \tag{1.18}$$

PT is an anti linear operator which leads to

$$PTH\psi = E^*\lambda\psi \tag{1.19}$$

Since PT commutes with H the previous equation becomes

$$E\lambda\psi = E^*\lambda\psi \tag{1.20}$$

by simplifying a non zero value λ , then finally obtain that E is real

$$E = E^* \tag{1.21}$$

This result can not be true for any PT-symmetric Hamiltonian even the conditions (1.11) and (1.12) are both satisfied, but there are cases where the PT symmetric Hamiltonian may don't have eigenstates because their corresponding energy eigenvalues are complex. so the eigenvalues are entirely real, if every eigenstate ψ of a PT-symmetric Hamiltonian H is also an eigenstate of PT operator [1, 4, 5] in such case PT it called unbroken.
Conversely, if the previous condition were violated, even with only one eigenstate which can be thought of as an eigenstate of one of H and PT not both, in such case the PT symmetry has to be broken.

Broken PT-Symmetry

PT is said to be broken [1, 3], if some of the eigenstates of a PT-symmetric Hamiltonian H are not simultaneously eigenstates of the PT operator, in this circumstance the the Hamiltonian has no physical significance because its eigenvalues are complex or partially complex unlike the axiom of quantum mechanics which says that the energy spectrum should be real and bounded below where this comes from the fact that the energy is a measurable quantity. The study of the generalized harmonic oscillator Hamiltonians (1.2) is done by Bender and Boettcher [1, 4], where it was showed that when ($N \geq 0$) all of the eigenvalues of these Hamiltonians are entirely real and positive, but when ($N < 0$) there are complex eigenvalues. and they distinguished two regions :

- ($N \geq 0$) : a parametric region of unbroken PT symmetry
- ($N < 0$) : a parametric region of broken PT symmetry

Despite the Hamiltonian H of equation (1.3) has an unbroken PT-symmetry which guarantees the requirement of reality and positivity of the energy spectrum, but this is not sufficient to assure that the PT-symmetric quantum theory is well established and does not suffer form inconsistency with some basic axioms of ordinary quantum mechanics, where any quantum theory needs to be constructed by a Hilbert space vectors associated with inner product having a positive norm and having a unitary time evolution, both can be summarized in one term "*unitarity*", so Bender resolved this problem by reconstructing a Hilbert in accordance with the PT-symmetric Hamiltonian, and he found that the Hamiltonian operator chooses its own Hilbert space in which it prefers to live [3]. and since any Hilbert space is associated with an inner product and this latter has to be dependent on the Hamiltonian itself, so Bender [1] by analogy with Hermitian inner product he introduced what so called PT-symmetric inner product.

PT-Inner Product

Even though a non-Hermitian PT-symmetric Hamiltonian operartor has a positive and real spectrum, but it still has some requirements in order to render it defines a physical theory [18].
In ordinary quantum theories the Hilbert space is specified even before the Hamiltonian is known, that means the inner product also is known *a prior*, and it's represented by Hermitian Hamiltonian, where the Dirac Hermitian conjugation plays a key role in the construction of the inner product, and it's written as

$$(\psi(x), \phi(x)) = \langle\psi(x)|\phi(x)\rangle = \int dx[\psi(x)]^*\phi(x) \tag{1.22}$$

where $\psi(x)$ and $\phi(x)$ are square-integrable functions $L^2(R)$, and the $\langle .|$, $|.\rangle$ are the Dirac notation of state vectors of the Hilbert space [3].
It's clearly shown that the Hermitian-inner product (1.22) is positive-definite, because no matter the state ψ has to be the norm remains positive.

$$(\psi(x), \psi(x)) = \langle\psi(x)|\psi(x)\rangle = \int dx[\psi(x)]^*\psi(x) = |\ \psi(x)\ |^2 \geq 0 \tag{1.23}$$

In PT-symmetric quantum theories the situation is a little bit different, unlike the ordinary quantum mechanics, the novel thing is that the Hilbert space associated with its inner product would have been generalized and modified in its structure, in such a way they become dependent on the Hamiltonian itself.
however, Bender [1] has reformulated the expression of the hermitian-inner product(1.22) corresponding to a Hermitian Hamiltonian ($H = H^\dagger$) by the analogous one which is regarding the PT-symmetric Hamiltonian ($H = H^{PT}$) and it is defined as follows

$$(\psi(x), \phi(x))_{PT} = \langle\psi(x)|\phi(x)\rangle_P = \langle\psi(x)|\, P\, |\phi(x)\rangle = \int_\Gamma dx[PT\psi(x)]\phi(x) \tag{1.24}$$

and

$$PT\psi(x) = [\psi(-x)]^* \tag{1.25}$$

where $\psi(x)$ and $\phi(x)$ are square-integrable functions of $L^2(\Gamma)$ and Γ is the contour in complex-x plane that specifies the PT-symmetric model [3].
by virtue of the symmetry's modification on the Hamiltonian who is respecting the expression (1.24), the energy spectrum might be real, but it still suffering from a sticky problem, where this inner product is indefinite and does not define the physical Hilbert space of the theory!!
Bender [1, 3, 5, 18] has encountered this puzzle of the indefiniteness of norm calculated by a PT-inner product, where the norm in fact should be positive definite, while according to quantum mechanics the probability is nothing but the norm

squared which could never be negative.
In PT-symmetric quantum theories, it's not the case, the PT-inner product is no longer positive, and it's known as indefinite-inner product.
Let's proof that, for a given eigenstates $\phi_n(x)$ of PT-symmetric Hamiltonian H of equation(1.2) and for $(N \geq 0)$, H has unbroken PT-symmetry which results that $\phi_n(x)$ are simultaneously eigenstates of PT operator and satisfy

$$PT\psi(x) = \lambda_n \psi(x) \tag{1.26}$$

where λ_n is no thing but a pure phase (this can be determined in a couple lines) Without loose generality if $(\lambda_n = 1)$, the PT-eigenvalue equation becomes

$$PT\psi(x) = \psi(x) \tag{1.27}$$

Consequently, H has to have a complete set of eigenstates, so for a different eigenvalues, the PT-inner product is surely zero, strictly speaking about orthogonality, but PT-inner product does not hold with the normality property where it gives an alternate sign result [18].

$$(\phi_m, \phi_n)_{PT} = \langle \phi_m | \phi_n \rangle_P = \langle \phi_m | P | \phi_n \rangle = (-1)^n \delta_{mn} \tag{1.28}$$

The above result summarizes two properties

- *Orthogonality* $(m \neq n)$: $(\phi_m, \phi_n)_{PT} = \langle \phi_m | \phi_n \rangle_P = \langle \phi_m | P | \phi_n \rangle = 0$
- *Normality* $(m = n)$: $(\phi_m, \phi_n)_{PT} = \langle \phi_m | \phi_n \rangle_P = \langle \phi_m | P | \phi_n \rangle = (-1)^n = \pm 1$

The expression of the P operator in position space [1, 18] in terms of eigenstates.

$$P(x,y) = \delta(x+y) = \sum_n (-1)^n \phi_n(x) \phi_n(-y) \tag{1.29}$$

and satisfies

$$P(x,y)\phi(x) = \int \delta(x+y)\phi(y)dy = \phi(-x) \tag{1.30}$$

$$P^2(x,y) = \delta(x-y) = \mathbb{1} \tag{1.31}$$

The apparent advantage of the PT-inner product [18] is that the associated norm $\langle \psi | P | \psi \rangle$ is independent of phase-factor and is conserved in time , if we consider the evolution with time $U(t) = e^{iHt}$ on $|\psi(x,0)\rangle$ gives $|\psi(x,t)\rangle$ as

$$U(t) |\psi(x,0)\rangle = |\psi(x,t)\rangle \tag{1.32}$$

and

$$PT(|\psi(x,t)\rangle = PT(U(t) |\psi(x,0)\rangle = \langle \psi(x,0)| P.U^{PT} \tag{1.33}$$

Clearly, the norm resulted by a PT-inner product remains conserved with time

$$\begin{aligned} \langle\psi(x,t)| P |\psi(x,t)\rangle &= \langle\psi(x,0)| P.U^{PT}(t).U(t) |\psi(0)\rangle \\ &= \langle\psi(x,0)| P.e^{-iH^{PT}t}.e^{iHt} |\psi(0)\rangle \\ &= \langle\psi(x,0)| P |\psi(x,0)\rangle \end{aligned} \tag{1.34}$$

Despite the Unitarity principle is satisfied, but the lack of positivity renders this PT-inner product indefinite and useless, whereas the definiteness of the inner product is one of the main ingredients of any quantum theory. To resolve this dilemma, Bender [1, 18] has introduced a new operator C where their properties similar to those of the charge conjugation operators in particle physics, where he used it as tricky idea by changing the indefinite PT-inner product into a positive-definite CPT-inner product

CPT-Inner Product

For a PT-symmetric Hamiltonian having an unbroken PT-symmetry, the indefiniteness of the PT-inner product [1, 3, 18] is resulted from the alternation of the sign (+) and (−), in fact the change in the norm sign arises a probabilistic interpretation difficulties and to fix this problem the one should construct a genuine positive-definite inner product which defined by the C-symmetry operator.

- *C-Operator*

Because in any PT-inner product, there is the same number of states having a negative and a positive norms, Bender has introduced in refs [1, 18] a generic symmetry and he termed it as C-symmetry, which stemmed from the work of Dirac where he interpreted the alternation of energy signs of energy spectrum by introducing a new symmetry which do change the matter into anti-matter, and he called it "charge-conjugation operator" C. The C-symmetry of H can be thought as a liner operator represented in position space as a sum over the energy eigenstates of the PT-symmetric Hamiltonian

$$C(x,y) = \sum_n \phi_n(x)\phi_n(y) \tag{1.35}$$

It commutes with the Hamiltonian H and PT operators

$$[C, H] = 0 \tag{1.36}$$

$$[C, PT] = 0 \tag{1.37}$$

and the square of C operator gives the unity operator

$$C^2(x,y) = \delta(x-y) = \mathbb{1} \tag{1.38}$$

In fact the C operator is complex unlike the Parity operator P which is real [1, 18] and they do not commute to each other

$$CP = (PC)^* \tag{1.39}$$

Conversely to the equations (1.31),(1.37), P and C operators are distinct square root of the unity operator $\delta(x-y)$.
The eigenvalues of C are (± 1) this can be proved easily

$$\begin{aligned} C\phi_n(x) &= \int C(x,y)\phi_n(y)dy \\ &= \sum_m \int \phi_m(x)\phi_m(y)\phi_n(y)dy \\ &= \phi_m(x)\sum_m \int \phi_m(y)\phi_n(y)dy \\ &= \phi_m(x)\sum_m (-1)^n \delta_{nm} \\ &= \pm\phi_m(x) \end{aligned} \tag{1.40}$$

- *CPT-Inner product*

For an unbroken PT-symmetry, there is a C operator that satisfies (1.36)(1.37) and (1.38), whereas the resolution of those equations give the expression of C, in which the CPT-inner product has the form

$$(\psi(x), \phi(x))_{CPT} = \langle \psi(x)|\phi(x)\rangle_{CP} = \langle \psi(x)| CP |\phi(x)\rangle = \int_\Gamma dx[CPT\psi(x)]\phi(x) \tag{1.41}$$

In ref [3],CPT-inner product coincides with the positive-definite inner product, where this latter is written in terms of a positive metric η_+

$$(\psi(x), \phi(x))_{CPT} = \langle \psi(x)|\phi(x)\rangle_{\eta_+} = \langle \psi(x)| \eta_+ |\phi(x)\rangle = \int_\Gamma dx\ \eta_+[\psi(x)]^*\phi(x) \tag{1.42}$$

where $\eta_+ = CP$ is a positive-definite metric operator. The completeness condition in position space of set of the energy eigenstates ϕ_n of an unbroken PT-symmetric Hamiltonian H are orthonormal and they given as

$$\sum_n \phi_n(x)[CPT\phi_n(y)] = \delta(x-y) = \mathbb{1} \tag{1.43}$$

We summarize the the subsection (1.1.1) by the next two tables

Hermitian Hamiltonian	PT -symmetric Hamiltonians
E is real $H = H^\dagger$ Time evolution is unitary	E is real if H has an unbroken PT-symmetry $H \neq H^\dagger$ and $H = H^{PT}$ Time evolution is unitary

Table 1.1: Hermitin Hamiltonian vs PT-symmetric Hamiltonian

$\dagger$ -inner product	PT -inner product	CPT -inner product
$\langle\psi(x)\|\phi(x)\rangle$ positive-negative does not depends on H $H = H^\dagger$	$\langle\psi(x)\| P \|\phi(x)\rangle$ indefinite-positive depends implicitly on H $H = H^{PT}$	$\langle\psi(x)\| CP \|\phi(x)\rangle$ positive-definite depends implicitly on H $H = H^{CPT}$

Table 1.2: Comparison between Hermitian, PT and CPT inner products

In ref [19] some disadvantages cited about PT-symmetric quantum theories, for instance ;

- The use of PT-symmetric Hamiltonians doesn't lead to genuine extension of quantum mechanics, rather it provides a new representation of the same theory where the physical Hilbert space is defined using a new inner product.
- Some of the notions developed in PT-symmetric Hamiltonians do not actually play a fundamental role. the primary example is the C operator that used as tool for specifying a particular example of the inner products, called CPT-inner product where the quantities of interest do not involve the C operator.
- PT-symmetry doesn't play any distinctively role, any PT-symmetric or non-PT-symmetric Hamiltonians that has a real spectrum can serve the same purpose, and it would be seen that these operators are a subclass of more general class which called pseudo-hermitian Hamiltonians

1.1.2 Pseudo-Hermitian Hamiltonians

The notion of PT-symmetry can be replaced in general mathematical context known as *pseudo-Hermiticity* [20], which is slightly different from the one used in the earlier studies, the pseudo-hermiticity was introduced by Mostafazadeh [3, 6, 19], where the Hamiltonian H is said to be pseudo-Hermitian with respect a positive-definite Hermitian operator η, if it satisfies [7]

$$H^\dagger = \eta H \eta^{-1} \tag{1.44}$$

where η is called the metric operator which is not unique, but when η is fixed (chosen), H is called η-pseudo Hermitian Hamiltonian [6].
we define the pseudo-Hermitian inner product as follows

$$\langle .|.\rangle = \langle .|\,\eta\,|.\rangle \tag{1.45}$$

we can say, it's kind of hermiticity in non-standard inner product called definite inner product. the study of pseudo-Hermitians Hamiltonians those having a real spectrum is much significant because they have a physical sense, so we restrict ourselves in such category of Hamiltonians.
The necessary and sufficient conditions to guarantee the reality of the spectrum needs two requirements:
The pseudo-Hermiticity condition

$$\eta_+ = \rho^\dagger \rho \tag{1.46}$$

and the existence of a positive-definite metric operator corresponds to the inner product

$$\langle .|.\rangle = \langle .|\,\eta_+\,|.\rangle \tag{1.47}$$

The main problem in pseudo-Hermitian Hamiltonians is to determine the metric operator η_+, in ref [6], there are two approaches help to construct the most general positive-definite inner product

- 1^{st} method employs the approach pursued in the proof of the spectral theorems, and involves constructing a complete bi-orthonormal system $|\psi_n\rangle, |\phi_n\rangle$ where $|\psi_n\rangle$, $|\phi_n\rangle$ are eigenstaes of H and $H^\dagger$ respectively.
- 2^{nd} method uses the fact that any positive definite operator η_+ has a hermitian logarithm *i.e.*, there is a Hermitian operator $Q = -\ln \eta_+$ thus

  $$\eta_+ = e^{-Q} \tag{1.48}$$

  then, apply both the pseudo-Hermiticity relation and *Baker-Campbell-Hausdroff* formula.
  this technique is much used in this work and it will be shown with details in the appendices.

Particular case in PT-symmetric Hamiltonians the role of η_+ is played by the combined operator PC and by definition

$$\eta_+ = e^{Q} P \tag{1.49}$$

The fact that P revers the sign of Q it means ($PQ = -QP$) thus the positive-definite metric is no thing but

$$\eta_+ = e^{-Q} \tag{1.50}$$

and the corresponding positive-definite inner product of equation (1.47) can be written in terms of Q

$$\langle .|\, \eta_+ \,|.\rangle = \langle .|\, e^{-Q} \,|.\rangle \tag{1.51}$$

This inner product corresponds to the choice $\eta = PC$, where C required to fulfill [19]

$$C^2 = 1,\ \ [C, H] = 0,\ \ [C, PT] = 0, \tag{1.52}$$

The one must solve these operator equations to find

$$C = e^{Q} P \tag{1.53}$$

for a Hermitian operator Q it is clearly seems that the CPT-inner product coincides with $\langle .|\, e^{-Q} \,|.\rangle$, therefore this procedure provides means of computing a metric operator of the form $\eta_+ = e^{-Q}$.
If the one fixes a particular metric η_+ [8] which is also Hermitian operator, the Hamiltonian H is said to be η_+ pseudo Hermitian [2]
Mostafazadeh has derived from Pseudo Hermitian Hamiltonian a sub class which has apparent property and it considered as a generalized version of PT-symmetric Hamiltonians which called Quasi-Hermitian Hamiltonians [3, 6].

1.1.3 Quasi-Hermitian Hamiltonians

The Quasi-hermitian is a pseudo hermitian Hamiltonian in nature but it could be mapped to hermitian one by a similarity transformation [6]

$$h = \eta^{1/2} H \eta^{-1/2} \tag{1.54}$$

where H is pseudo Hermitian Hamiltonian defined by the relation $H^\dagger = \eta H \eta^{-1}$ and belongs to Hilbert space $\mathcal{H}_H$, where h is a hermitian Hamiltonian in an equivalent Hilbert space $\mathcal{H}_h$ In such case H is said to be quasi-Hermitian Hamiltonian [7].

1.2 Deformed Heisenberg Algebra

The interest to deformed algebra was renewed after investigations in string theory and quantum gravity which suggest the existence of nonzero minimal in uncertainty in position following from the generalized uncertainty principle (GUP). It was shown that GUP and nonzero minimal uncertainty in position can be obtained from a modified Heisenberg algebra, where the more general case of it includes non-zero uncertainties in momenta as well as position, this general case is far more difficult to handle, since neither a position nor a momentum space representation

is viable. Instead on has to resort to generalized Bargmann-Fock space representation [9]. The construction of generalized Bargmann-Fock Hilbert spaces was realized by making use of algebraic techniques developed in the field of quantum groups. it was found that it has so far only in examples been possible to prove. let's confine ourselves in the case of minimal uncertainty in position, by taking the minimal uncertainty in momentum to vanish, for instance one-dimensional deformed Heisenberg algebra, where in the right hand side of it a term proportional to squared momentum is added [10].
In this section we study the one-dimensional deformed Heisenberg algebra, where the right hand side of it is quadratic on the momentum, in which we prefer to mention just the case of continuous momentum space representation rather than the quasi-position representation. finally we probe the question of existence of minimal length is much easier to handle in a momentum space representation.

1.2.1 Deformed Harmonic Oscillator in momentum space

Consider a modified one dimensional Heisenberg algebra generated by position X and momentum P Hermitian operators obeying the commutation relation

$$[X, P] = i\hbar f(P) \tag{1.55}$$

where f is a function of deformation and we assume that it is strictly positive ($f > 0$), even function [10].
In momentum representation both operators acting on a square integrable functions $\psi(p) \in \mathcal{L}(-a, a; f), (a \leq \infty)$, often we assume that the deformed function given as follows

$$f(P) = 1 + \beta P^2 \tag{1.56}$$

where $\beta > 0$, in this choice $a = \infty$ and the operators X , P are dense on the domain $\mathcal{L}(-\infty, \infty)$. It is highly expected to be a good choice for leading to a non zero minimal length. However equation (1.55) becomes

$$[X, P] = i\hbar(1 + \beta P^2) \tag{1.57}$$

in fact the Heisenberg algebra can be represented on momentum space wave functions $\psi(p) := \langle\phi|\psi\rangle$ the operators X and P acting on $\psi(p)$ as

$$P.\psi(p) = p\ \psi(p) \tag{1.58}$$

$$X.\psi(p) = i\hbar(1 + \beta p^2)\frac{\partial\psi(p)}{\partial p} \tag{1.59}$$

and the deformed inner product

$$\langle\phi|\psi\rangle = \int_{-\infty}^{\infty} \frac{dp}{1 + \beta p^2}\ \phi^*(p)\ \psi(p) \tag{1.60}$$

The norm of ψ is given by

$$\|\psi\|^2 = \int_{-\infty}^{\infty} \frac{dp}{1+\beta p^2} \; |\psi|^2 \tag{1.61}$$

The deformed completeness relation need preserve a factor $(1+\beta p^2)^{-1}$ to cancel the corresponding factor of the operator X this changes the identity operator as

$$1 = \int_{-\infty}^{\infty} \frac{dp}{1+\beta p^2} \; |p\rangle\langle p| \tag{1.62}$$

therefore the inner product of momentum eigenstates is

$$\langle p|p'\rangle = (1+\beta p^2)\delta(p-p') \tag{1.63}$$

We define the average value of operator A

$$< A >_\psi = \int_{-\infty}^{\infty} \frac{dp}{1+\beta p^2} \; \psi^*(p) \; A \; \psi(p) \tag{1.64}$$

The uncertainty of A could be extracted from the following relation

$$\Delta_\psi(A)^2 \; =< A^2 >_\psi - < A >^2_\psi \tag{1.65}$$

For norm states $\|\psi\|^2 = < I >_\psi = 1$

1.2.2 The minimal Length

The aim of this subsection is to find a nonzero minimal uncertainty in position $\Delta_\psi(X) \geq \Delta(X)_{min} = l_0$, which is called also *nonzero minimal length.* In the general, if the deformed function f is arbitrary, the existence of minimal length is not surly guaranteed. let us use Schwartz inequality of position and momentum operators X, P and equations (1.64),(1.65) help to express the generalized uncertainty principle (GUP)

$$\Delta_\psi(X)^2\Delta_\psi(P)^2 \geq \frac{1}{4} < [X,P] >^2_\psi \tag{1.66}$$

Since the commutator $[X,P] = i\hbar f(P)$ so

$$\Delta_\psi(X)^2\Delta_\psi(P)^2 \geq \frac{\hbar^2}{4} < f(P) >^2_\psi \tag{1.67}$$

where $f(P) = 1+\beta p^2$ obviously the first term of $f(P)$ which equals one corresponds to zero minimal length this means non-deformed Heisenberg algebra (ordinary quantum mechanics), but the second who contributes in the calculation of the minimal length [10]. Now if we plunging in the $f(P) = 1 + \beta P^2$ into the inequality we get

$$\Delta_\psi(X)^2 \Delta_\psi(P)^2 \geq \frac{\hbar^2 \beta^2}{4} < P^2 >_\psi^2 \tag{1.68}$$

we use the equation (1.65), therefore

$$< P^2 >_\psi = \Delta_\psi(p)^2 + < p >_\psi^2 \tag{1.69}$$

we substitute it in the expression (1.68), and apply the square root on the equality, we obtain

$$\Delta X \Delta P \geq \frac{\hbar}{2}(1 + \beta(\Delta P)^2 + \beta < p >^2) \tag{1.70}$$

In ref [2, 9] shows that the minimal position uncertainty is

$$\Delta x_{min}(< p >) = \hbar\sqrt{\beta}\sqrt{1 + \beta < p >^2} \tag{1.71}$$

so that the smallest value of it corresponds to

$$\Delta x_0 = \hbar\sqrt{\beta} \tag{1.72}$$

1.2.3 The Eigenfunctions of the position operator

The eigenvalue equation of the position operator X, on momentum space

$$i\hbar(1 + \beta p^2)\frac{\partial \psi_\lambda(p)}{\partial p} = \lambda \psi_\lambda(p) \tag{1.73}$$

The solution is given in [9] as

$$\psi_\lambda(p) = A\ exp(-i\frac{\lambda}{\hbar\sqrt{\beta}}(\sqrt{\beta}p)) \tag{1.74}$$

The constant A is determined by the normalization property of ψ_λ

$$1 = AA^* \int_{-\infty}^{\infty} \frac{1}{1 + \beta p^2} = AA^* \cdot \frac{\pi}{\sqrt{\beta}} \tag{1.75}$$

thus

$$\psi_\lambda(p) = \sqrt{\frac{\sqrt{\beta}}{\pi}}\ exp(-i\frac{\lambda}{\hbar\sqrt{\beta}}(\sqrt{\beta}p)) \tag{1.76}$$

1.2.4 Deformed Hamiltonian Operator

without loosing generality let us work out the example of deformed harmonic oscillator [20], let us assume the commutation relation taken as

$$[X, P] = i(1 + \beta P^2),\ \ \hbar = 1 \tag{1.77}$$

This modification on Heisenberg algebra inevitably lead to change on the Hamiltonian itself

$$H = P^2 + X^2 \tag{1.78}$$

and

$$P = p,\ \ X = (1 + \beta p^2)x,\ \ [x, p] = i \tag{1.79}$$

then

$$H = p^2 + (1 + \beta p^2)x(1 + \beta p^2)x \tag{1.80}$$

Thus

$$H = p^2 + x^2 - 6\, i\beta\, x \cdot p + 2\,\beta\, x^2 \cdot p^2 - 2\,\beta - 6\,\beta^2 p^2 - 6\, i\beta^2 x \cdot p^3 + \beta^2 x^2 \cdot p^4 \tag{1.81}$$

and use the factorization method to write H in terms of ladder operators, where $p = \frac{i}{\sqrt{2}}(a^\dagger - a)$ $x = \frac{1}{\sqrt{2}}(a^\dagger + a)$ therefore

$$\begin{aligned} H = &-\frac{1}{2}\,(a^\dagger - a)^2 + \frac{1}{2}\,(a + a^\dagger)^2 + 3\,\beta\,(a + a^\dagger)(a^\dagger - a) - \frac{1}{2}\,\beta\,(a + a^\dagger)^2 (a^\dagger - a)^2 \\ &- 2\,\beta + 3\,\beta^2 (a^\dagger - a)^2 - \frac{3}{2}\,\beta^2 (a + a^\dagger)(a^\dagger - a)^3 + \frac{1}{8}\,\beta^2 (a + a^\dagger)^2 (a^\dagger - a)^4 \end{aligned} \tag{1.82}$$

However, the energy eigenvalues are easily obtained

$$E_n = 2n + 1 + \beta(n^2 + n + \frac{1}{2}) + \frac{3}{4}\beta^2(\frac{2}{3}n^3 + n^2 + \frac{4}{3}n + \frac{1}{2}) + o(\beta^3) \tag{1.83}$$

Chapter 2

Deformation on Shifted Harmonic Oscillator

2.1 Deformation and non-Hermiticity

One of the achievements of modern physics, is how to unify General Relativity (GR) and Quantum Mechanics (QM), so the attempts at quantifying gravity have not been as successful [3], in fact that they are incompatible, inconsistent because they are different in scale, this actually motivated physicists to do some modification or generalization either on GR or QM, although none of these modifications leads to a consistent physical theory, in this work we confine ourselves in two disconnected generalizations of QM, the first concerns the generalization of Bender and his collaborators in the context of non-Hermitian Hamiltonians [1, 3, 4, 18] in particular PT symmetric and pseudo hermitian hamiltonians, the second started by Synder's paper [9, 10] and which has considered as a starting point of a whole branch of Quantum mechanics called by deformed QM , it is much expected that the consequence of a theory of quantum gravity is non zero minimal length in position or momentum or both where deformed Heisenberg algebra does the job, the sticky problem is that modification on Heisenberg algebra change the Hamiltonian and might renders it non Hermitian in some problems in the real contour, but in a complex contour they are strictly distinct to each other.
in this chapter we study the problem of shifted harmonic oscillator under the influence of deformation ($\beta > 0$) we define the modified metric and consequently the energy spectrum of deformed Hamiltonian H_d.

2.2 Deformed shifted Harmonic oscillator

2.2.1 PT Shifted Harmonic Oscillator

in the case of ordinary shifted Harmonic oscillator $H = p^2 + x^2 + i\epsilon x$, it is PT symmetric [1, 3, 7].
The general way to represent the C operator [1] is by expressing it in terms of fundamental dynamical operators x and p by the following expression

$$C = e^{Q(x,p)}P \tag{2.1}$$

where $Q(x,p)$ is even on x and odd on p this restriction comes from the conditions of equation (1.52). In ref [1, 3] as shown in appendix B, $Q(x,p)$ can be determined by using perturbation method, in shifted harmonic oscillator

$$Q(x,p) = -\epsilon p \tag{2.2}$$

by consequence the metric

$$\eta = e^{-Q(x,p)} = e^{\epsilon p} \tag{2.3}$$

Since PT-symmetric Hamiltonians are subclass of quasi-Hermitians ones? so there exist a Hermitian Hamiltonian h which can be mapped from H and satisfies similarity relation.

$$h^\dagger = h = \eta^{1/2} H \eta^{-1/2} \tag{2.4}$$

bulging in expression of h we get

$$h = e^{\epsilon p/2}\left(p^2 + x^2 + i\epsilon x\right) e^{-\epsilon p/2} \tag{2.5}$$

by simplification it becomes

$$h = p^2 + x^2 + \frac{\epsilon^2}{4} \tag{2.6}$$

Thus the energy spectrum given is in ref [1] by

$$E_n = 2n + 1 + \frac{\epsilon^2}{4} \tag{2.7}$$

2.2.2 Deformed shifted Harmonic oscillator

we define a deformed shifted harmonic oscillator as follows

$$H_d = p^2 + X^2 + i\epsilon X \tag{2.8}$$

where ϵ is real as stated before

$$X = (1 + \beta p^2)x, \;\; [X, p] = i(1 + \beta p^2) \tag{2.9}$$

we note that there is no deformation on the momentum.
The Hamiltonian H_d is pseudo Hermitian [2], it satisfies

$$H_d^{\dagger} = \eta_t H_d \eta_t^{-1} \tag{2.10}$$

and it's quasi-Hermitian because η_t mapping H_d to h_d by the similarity relation

$$h_d^{\dagger} = \eta_t^{1/2}{}_d H_d \eta_t^{-1/2}{}_d = h_d \tag{2.11}$$

The main problem here is how to calculate this deformed metric η_t, which is no thing but a correction on the previous one of ordinary shifted harmonic oscillator by a Dyson operator $\eta_d = e^{-\beta p^2}$ representing the deformation effect and the corresponding metric is

$$\eta_t = e^{-\beta p^2 - Q(x,p)} \tag{2.12}$$

we substitute it on the expression of η_t in the equation (2.11) we get

$$h_d = e^{-\beta p^2/2 - Q(x,p)/2} H_d e^{\beta p^2/2 + Q(x,p)/2} = h_d^{\dagger} \tag{2.13}$$

where $Q(x, p) = -\epsilon p$.
The appendix B contains all the details of calculation, however the expression of h_d obtained is

$$\begin{aligned} h_d = {} & -4\, i\beta\, xp + \beta^2 p^2 + p^2 - \beta + 2\, i\beta^3 xp^5 - 2\, i\beta^2 xp^3 + 5\, \beta^3 p^4 + 2\, \beta\, x^2 p^2 + \beta^2 x^2 p^4 \\ & - \beta^4 p^6 + x^2 + (\beta^3 p^5 - i\beta\, xp^2 - \beta\, p - 2\, \beta^2 p^3 - i\beta^2 xp^4)\epsilon + (-1/4\, \beta^2 p^4 + 1/4)\epsilon^2 \end{aligned} \tag{2.14}$$

using the factorization method and substitute x,p by the ladder operators $a^{\dagger}$ and a, therefore the energy spectrum is given as follows

$$\begin{aligned} h_d = {} & 2 \cdot \beta \cdot (\texttt{‘a-‘} + \texttt{‘a+‘})\,(\texttt{‘a+‘} - \texttt{‘a-‘}) - 1 \cdot 2^{-1} \cdot \beta^2 \cdot (\texttt{‘a+‘} - \texttt{‘a-‘})^2 - 1 \cdot 2^{-1} \cdot \\ & (\texttt{‘a+‘} - \texttt{‘a-‘})^2 - \beta - 4^{-1} \cdot \beta^3 \cdot (\texttt{‘a-‘} + \texttt{‘a+‘})\,(\texttt{‘a+‘} - \texttt{‘a-‘})^5 - 2^{-1} \cdot \beta^2 \\ & \cdot (\texttt{‘a-‘} + \texttt{‘a+‘})\,(\texttt{‘a+‘} - \texttt{‘a-‘})^3 + 5 \cdot 4^{-1} \cdot \beta^3 \cdot (\texttt{‘a+‘} - \texttt{‘a-‘})^4 - 2^{-1} \cdot \beta \cdot \\ & (\texttt{‘a-‘} + \texttt{‘a+‘})^2 (\texttt{‘a+‘} - \texttt{‘a-‘})^2 + 8^{-1} \cdot \beta^2 (\texttt{‘a-‘} + \texttt{‘a+‘})^2 \cdot (\texttt{‘a+‘} - \texttt{‘a-‘})^4 \\ & + 8^{-1} \cdot \beta^4 \cdot (\texttt{‘a+‘} - \texttt{‘a-‘})^6 + 2^{-1} \cdot (\texttt{‘a-‘} + \texttt{‘a+‘})^2 + (8^{-1} \cdot i \cdot \beta^3 \cdot 2^{2^{-1}} \cdot \\ & (\texttt{‘a+‘} - \texttt{‘a-‘})^5 + 4^{-1} \cdot i \cdot \beta \cdot 2^{2^{-1}} \cdot (\texttt{‘a-‘} + \texttt{‘a+‘})\,(\texttt{‘a+‘} - \texttt{‘a-‘})^2 - 2^{-1+2^{-1}} \\ & \cdot i \cdot \beta\,(\texttt{‘a+‘} - \texttt{‘a-‘}) + 2^{-1} \cdot i \cdot \beta^2 \cdot 2^{2^{-1}} \cdot (\texttt{‘a+‘} - \texttt{‘a-‘})^3 - 8^{-1} \cdot i \cdot \beta^2 \cdot 2^{1 \cdot 2^{-1}} \\ & \cdot (\texttt{‘a-‘} + \texttt{‘a+‘})\,(\texttt{‘a+‘} - \texttt{‘a-‘})^4)\epsilon + \left(-16^{-1} \cdot \beta^2 \cdot (\texttt{‘a+‘} - \texttt{‘a-‘})^4 + 4^{-1}\right)\epsilon^2 \end{aligned} \tag{2.15}$$

Therefore the deformed Energy pseudo [6] spectrum of such Hamiltonian is

$$E_n = 1+2n+\frac{\epsilon^2}{4}+\left(n^2+n+\frac{1}{2}\right)\beta+\frac{1}{2}\left(n^3+\frac{3}{2}\,n^2+2n+\frac{7}{4}\right)\beta^2+o(\beta^2,\epsilon^2) \quad (2.16)$$

2.3 Conclusion

The deformation on a PT-symmetric Hamiltonians type has been studied, for instance PT-shifted-harmonic oscillator $p^2 + x^2 + i\epsilon x$, after doing the deformation, it behaves as quasi-Hermitian Hamiltonian and by virtue of similarity transformation,in which using a deformed metric, we got a Hermitian hamiltonian has a real energy spectrum which is entirely real and depends on two parameters (α, β).

Chapter 3

Deformation on Cubic-Anharmonic Oscillator

3.1 Introduction

The PT-symmetric Hamiltonian $H = p^2 + x^2 + i\epsilon x^3$ which called cubic an harmonic Oscillator, was first studied in detail by Bender and Boettcher [7], this Hamiltonian was shown to have a real , positive spectrum? but it surfers from probabilistic interpretation problem mainly caused by the form of PT-inner product. Bender [1] invented the operator C, which helps to define a new inner product known as CPT-inner product where it is definite positive, often the difficulty is how to determine C because the one needs to know the eigenvalues and eigenvectors of the Hamiltonian jones2005, in such case of Hamiltonian the perturbative expansion is the only issue. Let us expand the example to a deformed QM, the deformed Hamiltonian becomes more complicated, which is no thing but a quasi-Hermitian one, and the determination of a positive definite metric η_t which satisfies $H_d^\dagger = \eta_t H \eta_d^{-1}$, where $\eta_t = e^{-\beta p^2}\ e^{Q(x,p)}$. The aim of this chapter is to calculate that metric which helps to determine the energy spectrum gotten by a mapping from a pseudo Hermitian H_d to a Hermitian one h_d using a similarity relation $h_d = \eta_t^{1/2} H_d \eta_t^{1/2}$ of the modified metric η_t

3.2 Deformed Cubic Anharmonic Oscillator

3.2.1 PT Cubic Anharmonic Oscillator

The non deformed Cubic Anharmonic Oscillator $H = p^2 + x^2 + i\epsilon x^3$, is PT symmetric [1, 3, 7].

The general way to represent the C operator [1] is by expressing it in terms of fundamental dynamical operators x and p by the following expression

$$C = e^{Q(x,p)}P \tag{3.1}$$

where $Q(x,p)$ is even on x and odd on p this restriction comes from the conditions of equation (1.52). In ref [3, 8, 6, 7], $Q(x,p)$ can be determined by using perturbation method, in cubic anharmonic oscillator.

$$Q(x,p) = \epsilon \left(-2\cdot 3^{-1}\cdot p^3 - 1\cdot 2^{-1}\cdot AntiCommutator\left(x^2,p\right)\right) + \epsilon^3 \left(p + 16\cdot 15^{-1}\cdot p^5 + 5\cdot 6^{-1}\cdot AntiCommutator\left(x^2,p^3\right) + 2^{-1}\cdot AntiCommutator\left(x^4,p\right)\right) \tag{3.2}$$

Substitute $Q(x,p)$ in $\eta = e^{-Q(x,p)}$ we get

$$\eta = exp{-}(\epsilon \left(-2\cdot 3^{-1}\cdot p^3 - 1\cdot 2^{-1}\cdot AntiCommutator\left(x^2,p\right)\right) + \epsilon^3 \left(p + 16\cdot 15^{-1}\cdot p^5 + 5\cdot 6^{-1}\cdot AntiCommutator\left(x^2,p^3\right)\right) + 2^{-1}\cdot AntiCommutator\left(x^4,p\right)\big)) \tag{3.3}$$

The mapped Hermitian Hamiltonian h is gotten by similarity relation.

$$h^{\dagger} = \eta^{1/2} H \eta^{-1/2} \tag{3.4}$$

by substituting η, it becomes

$$h = e^{Q(x,p)/2}\left(p^2 + x^2 + i\epsilon x\right) e^{-Q(x,p)/2} \tag{3.5}$$

where $Q(x,p)$ written as equation (3.2).
Then use Baker-Campbell-Hausdroff [6] formula, we get a long expression of h as shown in appendix C. Therefore the corresponding Energy spectrum can be simply calculated in using factorization method, it's expression shown in ref [3, 8, 7]

$$E_n = 2n + 1 + \epsilon^2\left(\frac{11}{8} + \frac{15}{4}n + \frac{15}{4}n^2\right) + o(\epsilon^4) \tag{3.6}$$

3.2.2 Deformed Cubic anharmonic oscillator

The deformed cubic anharmonic oscillator can be written as follows

$$H_d = p^2 + X^2 + i\epsilon X^3 \tag{3.7}$$

where ϵ is real and

$$X = (1+\beta p^2)x, \;\; [X,p] = i(1+\beta p^2) \tag{3.8}$$

The Hamiltonian H_d is pseudo Hermitian [2], it satisfies

$$H_d^\dagger = \eta_t H_d \eta_t^{-1} \tag{3.9}$$

and it's quasi-Hermitian because η_t mapping H_d to h_d by the similarity relation

$$h_d^\dagger = \eta_t^{1/2} H_d \eta_t^{-1/2} = h_d \tag{3.10}$$

The calculation of the metric η_t, is much complicated because we have to take into a count a modification done by deformation effect which represented by Dyson operator $\eta_d = e^{-\beta p^2}$, by consequence the metric takes the form

$$\eta_t = e^{-\beta p^2 - Q(x,p)} \tag{3.11}$$

we substitute it on the expression of h_t in the equation (2.11) we get

$$h_d = e^{-\beta p^2/2 - Q(x,p)/2} H_d e^{\beta p^2/2 + Q(x,p)/2} = h_d^\dagger \tag{3.12}$$

Therefore the expression of h_d is so long and it's given in equation (C.27) of Appendix C. The deformed energy spectrum is calculated by using the factorization method where we substitute x,p by the ladder operators $a^\dagger$ and a, we obtain

$$\begin{aligned} E_n &= 1 + 2\,n + \left(n^2 + 1/2 + n\right)\beta + \left(3/8 + 1/2\,n^3 + 3/4\,n^2 + n\right)\beta^2 \\ &+ \left(\frac{15}{4}\,n + \frac{15}{4}\,n^2 + \frac{11}{8}\right)\epsilon^2 + \left(\frac{21}{4} + \frac{27}{2}\,n + \frac{27}{2}\,n^2 + 6\,n^3\right)\epsilon^2\beta + o(\beta^3, \epsilon^3) \end{aligned} \tag{3.13}$$

3.3 Conclusion

The deformation on a PT-symmetric Hamiltonians, in the case of cubic an-harmonic oscillator $p^2 + x^2 + i\epsilon x^3$, where after doing the deformation it becomes quasi-Hermitian Hamiltonian. By following the same steps as well as chapter 02, we got the mapped Hermitian hamiltonian and it's energy spectrum which is entirely real and depends on two parameters (α, β).

Chapter 4

Spin Entangled PT-Symmetric Hamiltonian in a Curved Space

4.1 Entanglement and Non-Hermiticity

Recently, many interest has been developed to the systems with non-hermitian Hamiltonians and their applications to solve some of the physical problems [12, 13] especially in the non linear and quantum optics. The later has a tight relationship to quantum information processing (QIP)and implementation [14, 15, 16, 17]. Furthermore,one of the interesting issues in QIP is the quantum entanglement and many contributions have been presented in the literature so far. The most important ones are those related to a large class of PT-symmetric Hamiltonian where the rate of the generated quantum entanglement was characterized by some set of parameters. This rate can be improved and optimized by the variation of an appropriate parameters. Recently, much extensive attention has been paid to relativistic (inertial and non inertial) and gravitational field effects in the context of QIP [21, 22, 23, 24, 25, 26, 27, 28]. It is worth to mention that in QIP, the spin of particles is often used as a qubit regardless of the momentum state of the particle. However, spin and momenta are not separable in general in the relativistic motion. Thus ,the spin alone cannot be used as a qubit in a relativistic moving observers [16, 17]. Moreover, the curvature of the space-time has an important effect on the spin entropy production [29, 30, 31, 32, 33]. This means that even if the state of the spin is pure at one point of space-time, becomes mixed in another point. the main goal here is to show the effect of gravity on PT-symmetric Hamiltonian systems during a time evolution especially the creation of quantum entanglement.

4.2 Spin Entangled PT-Symmetric Hamiltonian

In ordinary quantum mechanics,we require that the Hamiltonian of the system has to be Hermitian in order to generate a real spectrum and therefore the corresponding time evolution operator is unitary. It turns out that the Hermiticity condition is not necessary and with a class of the so called CPT-symmetric Hamiltonian one can ensure spectrum reality. For the spin $\frac{1}{2}$ particle, the most general Hamiltonian H, which commutes with CPT (C, P and T stand for charge conjugation,parity and time reversal operators), has the following form:

$$H = \begin{pmatrix} re^{i\theta} & se^{i\chi} \\ se^{-i\chi} & re^{-i\theta} \end{pmatrix} \tag{4.1}$$

where $\frac{r}{s}\sin\theta < 1$ The parity P and charge conjugation C operators are shown to have the expression

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.2}$$

and

$$C = \frac{1}{\cos\phi}\begin{pmatrix} i\sin\phi & e^{i\chi} \\ e^{-i\chi} & -i\sin\phi \end{pmatrix} \tag{4.3}$$

where $\sin\phi = \frac{r}{s}\sin\theta$.

Note that the PT-symmetric Hamiltonian which was used in reference. [13] is not general . One can show that the general PT-symmetric Hamiltonian of (1) can be written as

$$H = r\cos(\theta)1_{2\times 2} + s\,\vec{n}\cdot\vec{\sigma} \tag{4.4}$$

with

$$\vec{n} = (\cos\chi, -\sin\chi, i\sin\phi) \tag{4.5}$$

Now consider a wave packet of a spin $\frac{1}{2}$ particle with a mass m in a Schwarzschild space-time such that

$$ds^2 = -fc^2dt^2 + \frac{1}{f}dr^2 + r^2(d\theta^2 + \sin^2\phi^2) \tag{4.6}$$

where $f = 1 - \frac{r_s}{r}$ and r_s is the Shwarchild radius. As it was pointed out in ref. [22], that at this radius , the space-time has an event horizon, on which the coordinates system (t, r, θ, ϕ) breaks down and therefore the time coordinate t is known as a killing time. In this case the accelerated observer doesn't suffer from the Hawking radiation, because the state of the quantum field represented in the Fock space is defined with this time and not the crucial time. one can show that the momentum of the wave packet center in the Minkowski space (local initial frame) is given by

$$q^a(x) = e_\mu{}^a\,[m\,u^\mu] \tag{4.7}$$

where u^μ is it four-vector of velocity normalized as $u^\mu u_\mu = c^2$ and e^a_μ the verbeins. One can choose

$$e_0{}^t = \frac{1}{c\sqrt{f(r)}} \tag{4.8}$$

$$e_1{}^r = \sqrt{f(r)} \tag{4.9}$$

$$e_2{}^\theta = \frac{1}{r} \tag{4.10}$$

$$e_3{}^\phi = \frac{1}{r\sin\phi} \tag{4.11}$$

Now, if we make an infinitesimal transformation $x^\mu \to x'^\mu = x^\mu + \delta x^\mu$; ($\delta x^\mu = u^\mu d\tau$, (τ is the proper time), the wave packet center (in the local frame) transforms as $q^a \to q'^a = q^a + \delta q^a$ with

$$\delta q^a = \lambda^b{}_b\,d\tau = [m\,a^b(x) + \chi^b{}_c\,q^c] \tag{4.12}$$

where

$$a^b(x) = e_\mu{}^b\,[u^\nu\,\nabla_\nu\,u^\mu] \tag{4.13}$$

Is the acceleration of the external force (gravitational force) and

$$\chi^b{}_c = u^\mu\,[e^\nu{}_c(x)\,\nabla_\nu\,e^a{}_\nu(x)] \tag{4.14}$$

(∇_ν is the co-variant derivatives) represents the change in the charge in the local inertial frame along $u^\mu(x)$ due to the space-time curvature. For a wave packet moving along a circular trajectory of radius ($r > r_s$) with a constant velocity $r\dfrac{d\phi}{dt} = v\sqrt{f}$ on the equatorial plane $\theta = \dfrac{\pi}{2}$, the 4-velocity components $u^t(x)$ and $u^\phi(x)$ of the wave packet center are

$$u^t(x) = \frac{\cosh\xi}{\sqrt{f(r)}} \tag{4.15}$$

and

$$u^\phi(x) = \frac{c\sinh\xi}{r} \tag{4.16}$$

where ξ is the rapidity in the local inertial frame defined by

$$\tanh\xi = \frac{v}{c} \tag{4.17}$$

(v is the velocity). In this case (as it was given in ref. [22]) straightforward calculation leads to :

$$\xi^0{}_1 = \xi^1{}_0 = -\frac{c\, r_s \cosh\xi}{2\, r^2 \sqrt{f(r)}} \tag{4.18}$$

and

$$\xi^1{}_3 = -\xi^3{}_1 = \frac{c\, r_s \sinh\left(\sqrt{f(r)}\right)}{r} \tag{4.19}$$

and therefore

$$\lambda^0{}_1 = \lambda^1{}_0 = -L \tanh\xi \tag{4.20}$$

and

$$\lambda^1{}_3 = -\lambda^3{}_2 = L \tag{4.21}$$

where

$$L = \frac{c \coth^2\xi \sinh\xi}{r}\,[1 - \frac{r_s}{2\, f(r)}] \tag{4.22}$$

The finite Lorentz Transformation $\Lambda^a{}_b$ are in this case

$$\Lambda^a{}_b(x_f, x_i) = T\, exp^{\int_{\tau_i}^{\tau_f} \lambda^a{}_b(x(\tau))d\tau} \tag{4.23}$$

when the wave packet center moves along a path $x^\mu(\tau)$ from $x_i{}^\mu = x^\mu(\tau_i)$ to $x_f{}^\mu = x^\mu(\tau_f)$. T is the ordering operator. After a proper time $\tau_p = \tau_f - \tau_i$ of the particle , the momentum eigenstate $|p^a, \sigma\rangle$ (σ stands for spin up $\uparrow$ or down $\downarrow$) transforms by a Wigner rotation $D^{\frac{1}{2}}(W(\Lambda(x_f, x_i), p))$ where

$$W^a{}_b(\Lambda, p) = [L^{-1}(p)L] \tag{4.24}$$

with

$$L^0{}_0(p) = \frac{p_0}{mc} \tag{4.25}$$

$$L^0{}_i(p) = L^i{}_0 = \frac{p_i}{mc} \tag{4.26}$$

$$L^i{}_k(p) = \delta^i{}_k + (\frac{p^0}{m\,c} - 1)\,\frac{p^i\, p_k}{|\, p\,|^2} \tag{4.27}$$

here $i, k = 1..3$. this spin rotation is reduced to rotation about the y-axis (i=2)

$$D^{\frac{1}{2}}(W(\Lambda, p)) = exp^{[-i\frac{\sigma_y}{2}\,\Theta(p^a)\,\tau_p]} = \begin{pmatrix} cos\frac{\theta}{2} & sin\frac{\theta}{2} \\ -sin\frac{\theta}{2} & cos\frac{\theta}{2} \end{pmatrix} \tag{4.28}$$

where σ_y is the Pauli matrix:

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{4.29}$$

and

$$\Theta(p^a) = [1 - \frac{p^3}{p^0 + mc} \tanh \xi] \tag{4.30}$$

4.3 Quantum Entanglement Entropy

at the initial τ_i, we have the following wave packet of a composed a bi-parties spin system $|\psi_i\rangle$ such that

$$|\psi_i\rangle = \int d^3p_1 \int d^3p_2 f(p_1) f(p_2) \, |p_2, \uparrow\rangle \otimes |p_1, \downarrow\rangle \tag{4.31}$$

We have assumed that there is no correlation between the two momenta subsystems and the normalization condition reads

$$\int d^3p |\ f(p)\ |^2 = 1 \tag{4.32}$$

Now, after a proper time $\tau_p = \tau_f - \tau_i$ the system will evolve and becomes

$$|\psi_f\rangle = k \sum_{\sigma_1,\sigma_2} \int \int d^3p_1 d^3p_2 f(p_1) f(p_2) A_{\sigma_1\sigma_2} \, |p_2, \sigma_2\rangle \, |p_1, \sigma_1\rangle \tag{4.33}$$

where the normalization constant k is

$$k = [|\ \alpha\ |^2 + |\ \beta\ |^2 + |\ \gamma\ |^2 + |\ \delta\ |^2]^{-1/2} \tag{4.34}$$

and

$$A_{\uparrow\uparrow} = \alpha D^{(2)}_{\uparrow\uparrow} D^{(1)}_{\uparrow\uparrow} + \beta D^{(2)}_{\uparrow\uparrow} D^{(1)}_{\downarrow\uparrow} + \gamma D^{(2)}_{\downarrow\uparrow} D^{(1)}_{\uparrow\uparrow} + \delta D^{(2)}_{\downarrow\uparrow} D^{(1)}_{\downarrow\uparrow} \tag{4.35}$$

$$A_{\uparrow\downarrow} = \alpha D^{(2)}_{\uparrow\uparrow} D^{(1)}_{\uparrow\downarrow} + \beta D^{(2)}_{\uparrow\uparrow} D^{(1)}_{\downarrow\downarrow} + \gamma D^{(2)}_{\downarrow\uparrow} D^{(1)}_{\uparrow\downarrow} + \delta D^{(2)}_{\downarrow\uparrow} D^{(1)}_{\downarrow\downarrow} \tag{4.36}$$

$$A_{\downarrow\uparrow} = \alpha D^{(2)}_{\uparrow\downarrow} D^{(1)}_{\uparrow\uparrow} + \beta D^{(2)}_{\uparrow\downarrow} D^{(1)}_{\downarrow\uparrow} + \gamma D^{(2)}_{\downarrow\downarrow} D^{(1)}_{\uparrow\uparrow} + \delta D^{(2)}_{\downarrow\downarrow} D^{(1)}_{\downarrow\uparrow} \tag{4.37}$$

$$A_{\downarrow\downarrow} = \alpha D^{(2)}_{\uparrow\downarrow} D^{(1)}_{\uparrow\downarrow} + \beta D^{(2)}_{\uparrow\downarrow} D^{(1)}_{\downarrow\downarrow} + \gamma D^{(2)}_{\downarrow\downarrow} D^{(1)}_{\uparrow\downarrow} + \delta D^{(2)}_{\downarrow\downarrow} D^{(1)}_{\downarrow\downarrow} \tag{4.38}$$

where

$$\alpha = a^{(2)}_{\downarrow} a^{(1)}_{\uparrow} \tag{4.39}$$

$$\beta = a^{(2)}_{\downarrow} b^{(1)}_{\uparrow} \tag{4.40}$$

$$\gamma = b^{(2)}_{\downarrow} a^{(1)}_{\uparrow} \tag{4.41}$$

$$\delta = b_{\downarrow}^{(2)} b_{\uparrow}^{(1)} \tag{4.42}$$

with

$$a_{\uparrow} = a_{\uparrow}(\tau_p) = \cos\omega\tau_p - \sin\phi\sin\omega\tau_p \tag{4.43}$$

$$b_{\uparrow} = b_{\uparrow}(\tau_p) = ie^{i\chi}\cos\omega\tau_p \tag{4.44}$$

$$a_{\downarrow} = a_{\downarrow}(\tau_p) = \cos\omega\tau_p + \sin\phi\sin\omega\tau_p \tag{4.45}$$

$$b_{\downarrow} = b_{\downarrow}(\tau_p) = -ie^{-i\chi}\sin\omega\tau_p \tag{4.46}$$

and

$$\omega = s\cos\phi \tag{4.47}$$

Here $D^{(1)}_{\sigma_1\sigma_2}$ (resp. $D^{(1)}_{\sigma_2\sigma_2}$) stands for the two dimensional Wigner rotation matrix representation $D^{\frac{1}{2}}_{\sigma_1\sigma_2}$ for the subsystem 1 (resp. subsystem 2).
Straightforward but tedious calculation gives the following expression of the pure state matrix density $\hat{\rho}$

$$\hat{\rho} = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix} \tag{4.48}$$

where

$$\begin{array}{l} \rho_{11} = \overline{\mid A_{\uparrow\uparrow} \mid^2},\ \rho_{22} = \overline{\mid A_{\uparrow\downarrow} \mid^2},\ \rho_{33} = \overline{\mid A_{\downarrow\uparrow} \mid^2},\ \rho_{44} = \overline{\mid A_{\downarrow\downarrow} \mid^2} \\ \rho_{12} = \rho_{21}^* = \overline{A_{\uparrow\uparrow}A_{\uparrow\downarrow}^*},\ \rho_{13} = \rho_{31}^* = \overline{A_{\uparrow\uparrow}A_{\downarrow\uparrow}^*},\ \rho_{14} = \rho_{41}^* = \overline{A_{\uparrow\uparrow}A_{\downarrow\downarrow}^*} \\ \rho_{23} = \rho_{32}^* = \overline{A_{\uparrow\downarrow}A_{\downarrow\uparrow}^*},\ \rho_{24} = \rho_{42}^* = \overline{A_{\uparrow\downarrow}A_{\downarrow\downarrow}^*},\ \rho_{34} = \rho_{43}^* = \overline{A_{\downarrow\uparrow}A_{\downarrow\downarrow}^*} \end{array} \tag{4.49}$$

$\overline{a}$ denote the average over the momentum distribution

$$\overline{X} =< X >= \int d^3p \mid f(p) \mid^2 X \tag{4.50}$$

The reduced matrix tr_2 of the subsystem 2 takes the form

$$\hat{\rho} = k^2 \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix} \tag{4.51}$$

where

$$\begin{aligned} \eta_{11} = \frac{1}{2}(\mid \alpha \mid^2 + \mid \gamma \mid^2) < \cos^2\frac{\theta}{2} > +(\mid \beta \mid^2 + \mid \delta \mid^2) < \sin^2\frac{\theta}{2} > \\ - \frac{1}{2}(\alpha\beta^* + \alpha^*\beta + \gamma\delta^* + \gamma^*\delta) < \sin\theta > \end{aligned} \tag{4.52}$$

$$\eta_{22} = (\mid \alpha \mid^2 + \mid \gamma \mid^2) < \sin^2 \frac{\theta}{2} > +(\mid \beta \mid^2 + \mid \delta \mid^2) < \cos^2 \frac{\theta}{2} >$$
$$+ \frac{1}{2}(\alpha\beta^* + \alpha^*\beta + \gamma\delta^* + \gamma^*\delta) < \sin\theta > \tag{4.53}$$

And

$$\eta_{12} = (\mid \alpha \mid^2 + \mid \beta \mid^2 - \mid \gamma \mid^2 - \mid \delta \mid^2)\langle \sin\theta \rangle$$
$$+ (\alpha\gamma^* + \alpha^*\gamma + \beta\delta^* + \beta^*\delta)\langle \cos^2 \frac{\theta}{2} \rangle + \frac{1}{2}(\mid \alpha \mid^2 + \mid \delta \mid^2)\langle \sin\theta \rangle \langle \cos\theta \rangle$$
$$+ 2\alpha\gamma^* \langle \sin^2 \frac{\theta}{2} \rangle^2 + (\alpha\delta^* + \beta\gamma^*)\langle \sin\theta \rangle \langle \sin^2 \frac{\theta}{2} \rangle + 2\beta\delta^* \langle \sin^2 \frac{\theta}{2} \rangle \langle \cos^2 \frac{\theta}{2} \rangle \tag{4.54}$$

Therefore the Von Newman quantum entanglement entropy S is

$$S = -\lambda_+ \log_2 \lambda_+ - \lambda_- \log_2 \lambda_- \tag{4.55}$$

where $\lambda_\pm$ are eigenvalues of the reduced matrix of (4.51),with

$$\lambda_\pm = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \Omega^2} \tag{4.56}$$

and

$$\Omega = 4(\mid \eta_{12} \mid^2 \eta_{11}\eta_{22})k^4 \tag{4.57}$$

It is important to mention that since $\mid \beta \mid^2 = \mid \gamma \mid^2$ and $\beta = \gamma^*$ the eigenvalues of the reduced matrices $tr_1\hat{\rho}$ and $tr_2\hat{\rho}$ of a subsystems 1 and 2 are the same. for a numerical analysis and to illustrate the effect of the gravity and the non-Hermiticity on the spin quantum entanglement ,one has to choose a proper normal distribution such as a Gaussian one and evaluate the corresponding averages as a function of the various parameters (r, s, θ, r_s) and optimize $\frac{dS}{d\tau}$ to get a good performance (work under investigation)

4.4 Conclusion

Throughout this chapter, we have studied the late time (time evolution)effect of both the space-time curvature and non-Hermiticity of a general PT-symmetric spin Hamiltonian on a two initially separable spin $\frac{1}{2}$ subsystems where we have assumed that there is no momentum correlation . It turns out that it is possible to generate a bi-parties spin quantum entanglement quantified in the von Newman entropy. In fact, the amount of created quantum entanglement can be magnified and becomes maximal or reduced (decoherence) depending on the various parameters and physical quantities of the system (Hermiticity and metric). Thus, one can adjust the composed system various parameters and control the quantum entanglement capability (variation of the concurrence per unit of time) and try to make it optimal. Moreover, we expect that the result depends strongly on the metric of the space-time especially near the horizon (the Scharzschild radius). This formalism can be easily applied to other multi-parties states such as Bells, Werner,Greenberg-Homer-Zeilinger(GHZ) and W states and study the effect of the momentum correlation on the quantum entanglement phenomenon. This approach can be also extended to arbitrary spin system (e.g. spin 1) and momentum-spin quantum entanglement witness etc. Finally, numerical analysis will allow to understand how the quantum entanglement (of this kind of systems)evolves and whither it can persist(robustness)over time (work in progress)

Appendix A

Calculation Methods

A.1 Perturbation Theory

The perturbation theory [1, 3, 8] is mainly used to construct a metric operator for a given quasi-Hermitian Hamiltonian,which helps to find the spectrum energy easily,often this method concerned the Hamiltonian those have a complex potentials,in the ref [3] Mostafazadeh has figured out it as a tool to calculate the metric operator which helps to find the energy spectrum,according the following steps :
1/ Decompose the Hamiltonian H into the form

$$H = H_0 + \epsilon H_1 \tag{A.1}$$

where H_0 and H_1 are respectively Hermitian and anti-Hermitian and ϵ-independent operators.
2/ For a definite positive operator η,which has a unique logarithm to introduce a Hermitian $Q = -\ln\eta$, so that

$$\eta = e^{-Q} \tag{A.2}$$

The pseudo-Hermiticity relation:

$$H^{\dagger} = e^{-Q}\, H\, e^{Q} \tag{A.3}$$

and in terms of Backer-Campbell-Hausdroff identity the expression is getting much easier

$$\begin{aligned} H^{\dagger} = e^{-Q} H e^{Q} &= H + \sum_{l=1}^{\infty} \frac{1}{l!}[H, Q]_l \\ &= H + [H, Q] + \frac{1}{2!}[[H, Q], Q] + \frac{1}{3!}[[[H, Q], Q], Q] + ... \end{aligned} \tag{A.4}$$

where

$$[H, Q]_l = [[...[[H, Q], Q], ...], Q \tag{A.5}$$

3/ The expansion of Q in a power series in ϵ in such way it is even on x and odd in p

$$Q = \sum_{j=1}^{\infty} Q_j \epsilon^j \tag{A.6}$$

4/ Insert the expression of $H^\dagger$ and Q in a relation of hermiticity, and do combination term by term we get:

$$\begin{aligned}
&[H_0, Q_1] = -2H_1 \\
&[H_0, Q_2] = 0 \\
&[H_0, Q_3] = -\frac{1}{6}[H_1, Q_2] \\
&[H_0, Q_4] = -\frac{1}{6}([[H_1, Q_1], Q_2] + [[H_1, Q_2], Q_1]) \\
&... \\
&...
\end{aligned} \tag{A.7}$$

5/ Solve the above equations for Q_j by iteration
6/ In refs [1, 3],the examples $p^2 + x^2 - i\epsilon x$ and $p^2 + x^2 + i\epsilon x^3$, have a satisfied recurrence equations of Q_j by taking $Q_{2i} = 0$ an putting the ansatz

$$Q_{2i+1} = \sum_{j,k=0}^{i+1} C_{ijk} \cdot AntiCommutator\left(x^{2j}, p^{2k+1}\right) \tag{A.8}$$

Where C_{ijk} are real constants. Mainly, the problem in this work is consists to find the expression of Q in terms of Q_j that have to be determined by iteration.
7/Write the pseudo-Hermitian relation in terms of a positive definite metric operator η in such a way

$$h = h^\dagger = \eta^{\frac{1}{2}} H \eta^{-\frac{1}{2}} \tag{A.9}$$

The Hamiltonians h and H are respectively Hermitian and pseudo hermitian.

A.2 PT-Symmetric 2×2 matrix Hamiltonian

we work out the example of 2×2 matrix Hamiltonian [1, 18]

$$H = \begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix} \tag{A.10}$$

where r, s and θ are real parameters, it is shown clearly that this Hamiltonian is not Hermitian,moreover it is PT-symmetric, and the PT-operator consists of the parity operator

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{A.11}$$

And the time reversal T which is no thing but a conjugation of charge operator.To determine the the Energy eigenvalues $\epsilon_{\uparrow\downarrow}$, we should resolve the Schrodinger equation.

$$H\psi_{\uparrow\downarrow} = \epsilon_{\uparrow\downarrow}\psi_{\uparrow\downarrow} \tag{A.12}$$

The Hamiltonian characterized by two parametric regions, comes from the calculation of the determinant

$$H - \epsilon_{\uparrow\downarrow}\mathbb{1} = 0 \tag{A.13}$$

where as $\mathbb{1}$ stands for a unit matrix,so we rewrite the equation (A.12) as

$$\begin{vmatrix} re^{i\theta} - \epsilon_{\uparrow\downarrow} & s \\ s & re^{-i\theta} - \epsilon_{\uparrow\downarrow} \end{vmatrix} = 0 \tag{A.14}$$

The calculation of (A.13) gives us, as we precede two distinguished regions, the first when $s^2 < r^2 \sin^2\theta$ in which the calculation of energy gives complex conjugate pair,and as a result the PT-symmetry is broken, but if $s^2 \geqslant r^2 \sin^2\theta$ which corresponds the region of unbroken PT-symmetry where, the energy eigenvalues are

$$\epsilon_{\uparrow\downarrow} = r\cos\theta \pm \sqrt{s^2 - r^2\sin^2\theta} \tag{A.15}$$

and the eigenstates are

$$|\epsilon_{\uparrow}\rangle = \frac{1}{\sqrt{2\cos\alpha}} \begin{pmatrix} e^{i\frac{\alpha}{2}} \\ e^{-i\frac{\alpha}{2}} \end{pmatrix}$$

and

$$|\epsilon_{\downarrow}\rangle = \frac{i}{\sqrt{2\cos\alpha}} \begin{pmatrix} e^{-i\frac{\alpha}{2}} \\ -e^{-i\frac{\alpha}{2}} \end{pmatrix} \tag{A.16}$$

where $\sin\alpha = \frac{r}{s}\sin\theta$, it is easily verified that $\langle\epsilon_{\uparrow\downarrow}|\epsilon_{\uparrow\downarrow}\rangle_{PT} = \pm 1$ and that $\langle\epsilon_{\downarrow\uparrow}|\epsilon_{\uparrow\downarrow}\rangle_{PT} = 0$ we conclude that the resulting vector space spanned by energy eigenstates has a metric signature $(+,-)$ The condition $s^2 > r^2\sin^2\theta$ is the region where the PT-symmetry is unbroken. The Hamiltonian H has a real spectrum that okay with the axiom of any quantum theory, but the fact that it has a metric signature $(+,-)$ that makes the corresponding inner product indefinite, Bender in ref [1], has introduced a C operator which was the analogous of the charge conjugation of particle physics,but they are different in nature ,so by virtue of C the construction of a positive definite inner product could be possible,under which the the axiom of unitarity of quantum mechanics remains preserved, in the case of the Hamiltonian (A.9) the operator C is

$$C = \frac{1}{\cos\alpha}\begin{pmatrix} i\sin\alpha & 1 \\ 1 & -i\sin\alpha \end{pmatrix} \tag{A.17}$$

Note that C commute with H and PT operators and satisfies the following relations

$$C^2 = \mathbb{1}$$

$$C\left|\epsilon_{\uparrow\downarrow}\right\rangle = \pm\left|\epsilon_{\uparrow\downarrow}\right\rangle \tag{A.18}$$

It is precisely clear that the eigenvalues of C are the signs of the PT norms, then the CPT inner product becomes positive definite as

$$\langle\epsilon_{\uparrow\downarrow}|\epsilon_{\uparrow\downarrow}\rangle_{CPT} = 1 \tag{A.19}$$

That is to say the two dimensional complex vector of Hilbert space spanned by $|\epsilon_{\uparrow\downarrow}\rangle$, associated with inner product $\langle\epsilon_{\uparrow\downarrow}|\epsilon_{\uparrow\downarrow}\rangle_{CPT}$, has a Hermitian structure with a signature $(+,+)$, in which we denote $\langle\ .\ |$ as a CPT-conjugate of $|\ .\ \rangle$, and next relations are taken with this statement.
Because the completeness condition is no thing but the outer product of the dyads where summation of all them represent the identity operator.

$$\left|\epsilon_{\uparrow}\right\rangle\left\langle\epsilon_{\uparrow}\right| + \left|\epsilon_{\downarrow}\right\rangle\left\langle\epsilon_{\downarrow}\right| = \mathbb{1} \tag{A.20}$$

Thus leads to write the operator C in terms of the outer product of dyads, as the following

$$C = \left|\epsilon_{\uparrow}\right\rangle\left\langle\epsilon_{\uparrow}\right| - \left|\epsilon_{\downarrow}\right\rangle\left\langle\epsilon_{\downarrow}\right| \tag{A.21}$$

Note again that the bras $\langle\ .\ |$ are CPT-states.
Mainly,the reality of energy spectrum of a given PT-symmetric Hamiltonian,will never happen to be if the PT-symmetry is broken,so it should be unbroken, but this is not sufficient because The PT-symmetric quantum theory needs to be hold under the axiom of unitarity ,to resolve this dilemma Bender [1] suggested a new operator and use it to modify the positive definite inner product and it so called CPT-inner product,and the PT-symmetric Hamiltonian H is becomes Hermitian with respect to CPT-inner product.
Finally, we point out that the above ingredients of section (A.2)are used in part of the calculations of chapter (4)

Appendix B

Calculations in Chapter 2

B.1 Deformed Shifted Harmonic Oscillator

It has the form

$$H_d = P^2 + X^2 + i\epsilon X \tag{B.1}$$

where the deformed coordinates are

$$P = p \qquad X = (1 + \beta p^2)x \tag{B.2}$$

by substituting their expressions in H we find

$$H_d = P^2 + (1 + \beta p^2)x(1 + \beta p^2)x + i\epsilon(1 + \beta p^2)x \tag{B.3}$$

Obviously This Hamiltonian is non-Hermitian and it belongs to the a class of pseudo-Hermitian Hamiltonians, where their energy spectrum could be found by virtue of a metric operator η_t which renders them Hermitian, also it serves to remain the Hamiltonian reconciled with axiom of unitarity, which is the heart of any quantum theory, this happens by a mapping process from a Hilbert space basis of H_d to an extensive one of h_d, where this later is perfectly Hermitian and it satisfies the following relation.

$$h_d = h_d^\dagger = \eta_t^{\frac{1}{2}}\, H_d\, \eta_t^{-\frac{1}{2}} \tag{B.4}$$

B.2 Symbolic Computation with Maple

B.2.1 The Metric Operator η_t

The metric of a Shifted deformed harmonic oscillator H_d is

$$\eta_t = \eta_d\, \eta = e^{-\beta p^2} e^{-Q(x,p)} \tag{B.5}$$

where $\eta_d = e^{-\beta p^2}$ and $\eta = e^{-Q(x,p)}$ are respectively correspond to the deformation and displacement effect, furthermore η_d is called a Dyson [2] operator which came from the ansatz of weighted-inner product of deformed Quantum Mechanics.
The case $\beta = 0$ the metric changes where, $\eta_t = \eta = e^{-Q(x,p)}$ and the Hamiltonian H_d is no more than H and it takes the form

$$H = H_0 + H_1 = p^2 + x^2 + i\epsilon x$$

where

$$\begin{aligned} H_0 &= p^2 + x^2 \\ H_1 &= ix \end{aligned} \tag{B.6}$$

In refs [1, 2], the use of perturbation method on H gives

$$H^\dagger = \eta H \eta^{-1} = e^{-Q(x,p)} H e^{Q(x,p} = H + \sum_{l=1}^{\infty} \frac{1}{l!}[H, Q]$$

by Inserting $Q(x,p) = \sum_{j=1}^{\infty} Q_j \epsilon^j$ into it,and the simplification, get the following recurrence equations

$$\begin{aligned} &[H_0, Q_1] = -2H_1 \\ &[H_0, Q_2] = 0 \\ &[H_0, Q_3] = -\frac{1}{6}[H_1, Q_1] \\ &\ldots \\ &\ldots \end{aligned} \tag{B.7}$$

It remains only one equation $[H_0, Q_1] = -2ix$, because H has the form $p^2+x^2-i\epsilon x$, the following assumption is convenient

$$Q_1(x,p) = ap + bx^2 + cp^3 + dx^4 \tag{B.8}$$

where $Q_1(x,p)$ is necessarily odd on p and even on x,this constraint came from the PT-symmetry conditions [1, 3]

$$[C, PT] = 0 \tag{B.9}$$

$$[C, H] = 0 \tag{B.10}$$

$$C^2 = \mathbb{1} \tag{B.11}$$

where the metric operator η expressed in terms of P (parity) and C operators as follows

$$C = \eta^{-1} P = e^{Q(x,p)} P \tag{B.12}$$

by Going back again to the equation(B.8) and inserting it into the first equation(B.7), we get

$$a[H_0,p]+b[H_0,x^2]+c[H_0,p^3]+d[H_0,x^4]=-2ix$$

The expansion and the comparison lead to the values $a=-1$, $b=c=d=0$ which implies $Q_1=ap=-p$, therefore $Q=\epsilon p=-\epsilon p$ by consequence

$$\eta=e^{-Q(x,p)}=e^{\epsilon p} \tag{B.13}$$

In the case $\beta\neq 0$ on a shifted harmonic oscillator,it gets more complicated as shown in the relation(B.5), and according to the equation (B.13) the corresponding metric operator becomes

$$\eta_t=\eta_d\,\eta=e^{-\beta p^2+\epsilon p} \tag{B.14}$$

Use relation (12) and substitute in it the expression of η_t

$$h_d=h_d^\dagger=e^{\frac{-\beta}{2}p^2+\frac{\epsilon}{2}p}\left(p^2+(1+\beta p^2)x(1+\beta p^2)x+i\epsilon(1+\beta p^2)x\right)e^{\frac{\beta}{2}p^2-\frac{\epsilon}{2}p} \tag{B.15}$$

with some simplification we get

$$h_d=h_d^\dagger=-4i\beta xp+\beta^2p^2+p^2-\beta+2i\beta^3xp^5-2i\beta^2xp^3+5\beta^3p^4+2\beta x^2p^2 \\ +\beta^2x^2p^4-\beta^4p^6+x^2+\left(\beta^3p^5-i\beta xp^2-\beta p-2\beta^2p^3-i\beta^2xp^4\right)\epsilon+\left(-4^{-1}\beta^2p^4+4^{-1}\right)\epsilon^2 \tag{B.16}$$

B.2.2 The Energy spectrum E_n

Let's assume that `a-` `a+` are the annihilation and creation operators, and their effect on the Fock states.

$$\texttt{a-}\,|\phi_n\rangle=\sqrt{n}\,|\phi_{n-1}\rangle \tag{B.17}$$

$$\sqrt{n}\,\langle\phi_{n-1}| \tag{B.18}$$

$$\sqrt{n+1}\,|\phi_{n+1}\rangle \tag{B.19}$$

$$\sqrt{n+1}\,\langle\phi_{n+1}| \tag{B.20}$$

The commutator

$$[\texttt{a-},\texttt{a+}]=1 \tag{B.21}$$

After Maple simplify the expression of h_d it takes this form

$$h_d=-4i\beta xp+\beta^2p^2+p^2-\beta+2i\beta^3xp^5-2i\beta^2xp^3+5\beta^3p^4+2\beta x^2p^2+\beta^2x^2p^4-\beta^4p^6 \\ +x^2+\left(\beta^3p^5-i\beta xp^2-\beta p-2\beta^2p^3-i\beta^2xp^4\right)\epsilon \\ +\left(-4^{-1}\beta^2p^4+4^{-1}\right)\epsilon^2 \tag{B.22}$$

Plugging in $x = \frac{1}{\sqrt{2}}(\texttt{a-} + \texttt{a+})$ and $p = \frac{i}{\sqrt{2}}(\texttt{a+} - \texttt{a-})$ into h_d we get

$$h_d = 2 \cdot \beta \cdot (`\texttt{a-}` + `\texttt{a+}`)(`\texttt{a+}` - `\texttt{a-}`) - 1 \cdot 2^{-1} \cdot \beta^2 \cdot (`\texttt{a+}` - `\texttt{a-}`)^2 - 1 \cdot 2^{-1} \cdot (`\texttt{a+}` - `\texttt{a-}`)^2 - \beta - 4^{-1} \cdot \beta^3 \cdot (`\texttt{a-}` + `\texttt{a+}`)(`\texttt{a+}` - `\texttt{a-}`)^5 - 2^{-1} \cdot \beta^2 \cdot (`\texttt{a-}` + `\texttt{a+}`)(`\texttt{a+}` - `\texttt{a-}`)^3 + 5 \cdot 4^{-1} \cdot \beta^3 \cdot (`\texttt{a+}` - `\texttt{a-}`)^4 - 2^{-1} \cdot \beta \cdot (`\texttt{a-}` + `\texttt{a+}`)^2 (`\texttt{a+}` - `\texttt{a-}`)^2 + 8^{-1} \cdot \beta^2 (`\texttt{a-}` + `\texttt{a+}`)^2 \cdot (`\texttt{a+}` - `\texttt{a-}`)^4 + 8^{-1} \cdot \beta^4 \cdot (`\texttt{a+}` - `\texttt{a-}`)^6 + 2^{-1} \cdot (`\texttt{a-}` + `\texttt{a+}`)^2 + (8^{-1} \cdot i \cdot \beta^3 \cdot 2^{2^{-1}} \cdot (`\texttt{a+}` - `\texttt{a-}`)^5 + 4^{-1} \cdot i \cdot \beta \cdot 2^{2^{-1}} \cdot (`\texttt{a-}` + `\texttt{a+}`)(`\texttt{a+}` - `\texttt{a-}`)^2 - 2^{-1+2^{-1}} \cdot i \cdot \beta (`\texttt{a+}` - `\texttt{a-}`) + 2^{-1} \cdot i \cdot \beta^2 \cdot 2^{2^{-1}} \cdot (`\texttt{a+}` - `\texttt{a-}`)^3 - 8^{-1} \cdot i \cdot \beta^2 \cdot 2^{1 \cdot 2^{-1}} \cdot (`\texttt{a-}` + `\texttt{a+}`)(`\texttt{a+}` - `\texttt{a-}`)^4)\epsilon + \left(-16^{-1} \cdot \beta^2 \cdot (`\texttt{a+}` - `\texttt{a-}`)^4 + 4^{-1}\right)\epsilon^2 \tag{B.23}$$

by definition

$$E_n = \langle \phi, n | \, h_d \, | \phi, n \rangle \tag{B.24}$$

which leads to the expression of E_n in function of n,ϵ and β

$$E_n = 1 + 2n + 4^{-1}\epsilon^2 + \left(n^2 + 2^{-1} + n\right)\beta + \left(2n + 2^{-1}n^3 + 7 \cdot 8^{-1} + 3 \cdot 4^{-1}n^2\right)\beta^2 - \left(3 \cdot 16^{-1} + 3 \cdot 8^{-1}n^2 + 3 \cdot 8^{-1}n\right)\beta^2\epsilon^2 - \left(5n + 15 \cdot 8^{-1} + 15 \cdot 4^{-1}n^2 + 5 \cdot 2^{-1}n^3\right)\beta^4 \tag{B.25}$$

for $(\beta, \epsilon) << 1$

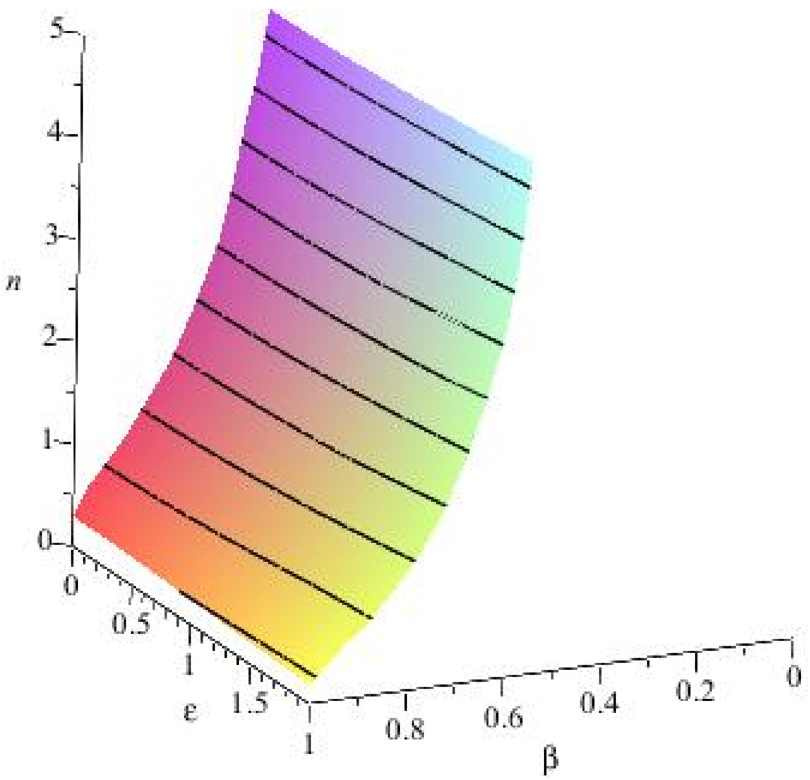
$$E_n = 1 + 2n + \frac{\epsilon^2}{4} + \left(n^2 + n + \frac{1}{2}\right)\beta + \frac{1}{2}\left(n^3 + \frac{3}{2}\,n^2 + 2n + \frac{7}{4}\right)\beta^2 + o(\beta^2) + o(\epsilon^2) \tag{B.26}$$

Particular Cases

1. $\epsilon \neq 0$ and $\beta \neq 0$

$$\begin{aligned} E_n = {} & 1 + 2n + 4^{-1}\epsilon^2 + \left(n^2 + 2^{-1} + n\right)\beta + \left(2n + 2^{-1}n^3 + 7\cdot 8^{-1} + 3\cdot 4^{-1}n^2\right)\beta^2 \\ & - \left(3\cdot 16^{-1} + 3\cdot 8^{-1}n^2 + 3\cdot 8^{-1}n\right)\beta^2\epsilon^2 \\ & - \left(5n + 15\cdot 8^{-1} + 15\cdot 4^{-1}n^2 + 5\cdot 2^{-1}n^3\right)\beta^4 \end{aligned} \tag{B.27}$$

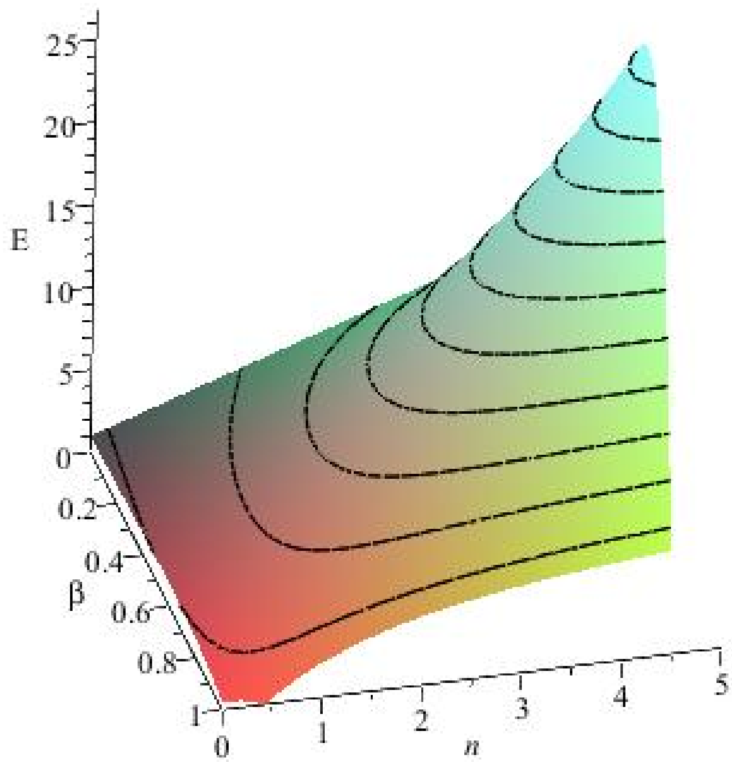
Figure B.1: Deformed Shifted Harmonic Oscillator



2. $\epsilon = 0$ and $\beta \neq 0$

$$E_n = 1 + 2n + \left(n^2 + 2^{-1} + n\right)\beta + \left(2n + 2^{-1}n^3 + 7\cdot 8^{-1} + 3\cdot 4^{-1}n^2\right)\beta^2 - \left(5n + 15\cdot 8^{-1} + 15\cdot 4^{-1}n^2 + 5\cdot 2^{-1}n^3\right)\beta^4 \tag{B.28}$$

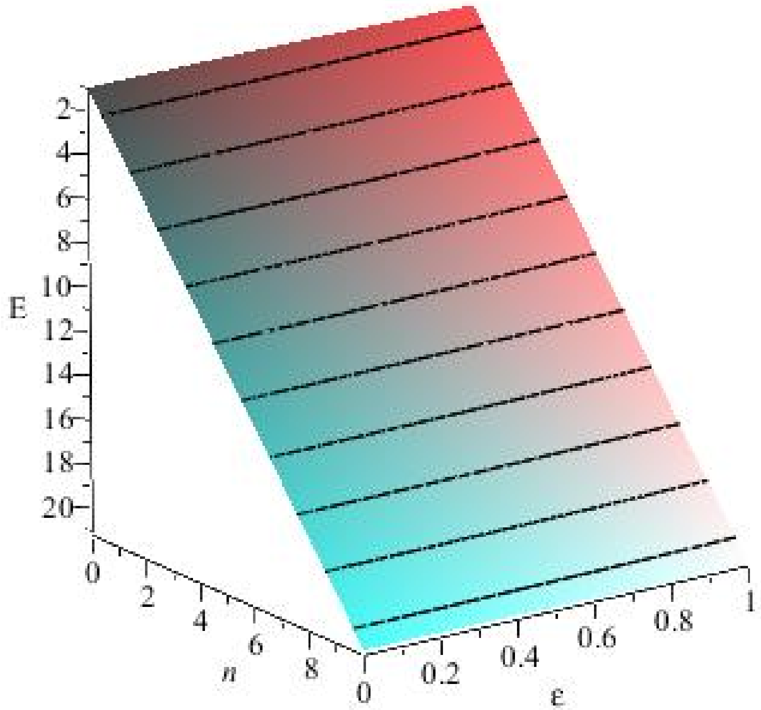
Figure B.2: Deformed Harmonic Oscillator



3. $\epsilon \neq 0$ and $\beta = 0$

$$E_n = 1 + 2n + 4^{-1}\epsilon^2 \tag{B.29}$$

Figure B.3: Shifted Harmonic Oscillator



Appendix C

Calculations in Chapter 3

C.1 Deformed Cubic Anharmonic Oscillator

which is given as follows

$$H_d = P^2 + X^2 + i\epsilon X^3 \tag{C.1}$$

where ϵ is a constant and choosing the deformation on the coordinates as

$$P = p$$

and

$$X = (1 + \beta p^2)x \tag{C.2}$$

We substitute their expressions in H we get

$$H_d = P^2 + (1 + \beta p^2)x(1 + \beta p^2)x + i\epsilon(1 + \beta p^2)x(1 + \beta p^2)x(1 + \beta p^2)x \tag{C.3}$$

This Hamiltonian seems non-Hermitian,moreover strictly speaking about pseudo Hermitian Hamiltonian, to find it's energy spectrum, first we use the mapping process showed in ref [3, 8], by virtue of the metric operator η_t which serves as an operator of transformation from H_d to a new Hamiltonian h_d by doing extension on the Hilbert space in which the new transformed Hamiltonian h_d is Hermitian however the spectrum could be found easily. That in one hand it characterized by a real energy spectrum, and another hand it preserves the axiom of Unitarity. The Hermitian h_d which has gotten by the mapping process, will satisfy the pseudo-Hermiticity relation

$$h_d = h_d^\dagger = \eta_t^{\frac{1}{2}}\, H_d\, \eta_t^{-\frac{1}{2}} \tag{C.4}$$

C.2 Symbolic Computations with Maple

C.2.1 The Metric Operator η_t

The metric of cubic An-harmonic deformed oscillator H is

$$\eta_t = \eta_d\, \eta = e^{-\beta p^2} e^{-Q(x,p)} \tag{C.5}$$

where $\eta_d = e^{-\beta p^2}$ and $\eta = e^{-Q(x,p)}$ are respectively correspond to the deformation and the cubic interaction term effect, furthermore η_d came from the ansatz of weighted-inner product of deformed Quantum Mechanics.
The case $\beta = 0$ the metric changes where, $\eta_t = \eta = e^{-Q(x,p)}$ and the Hamiltonian H_d is no more than H and it takes the form

$$H = H_0 + H_1 = p^2 + x^2 + i\epsilon x^3$$

where

$$\begin{aligned} H_0 &= p^2 + x^2 \\ H_1 &= +ix^3 \end{aligned} \tag{C.6}$$

In refs [1, 3, 7, 8], the perturbation method on H gives

$$H^\dagger = \eta H \eta^{-1} = e^{-Q(x,p)} H e^{Q(x,p} = H + \sum_{l=1}^{\infty} \frac{1}{l!}[H, Q] \tag{C.7}$$

by inserting $Q(x,p) = \sum_{j=1}^{\infty} Q_j \epsilon^j$ into it and doing some simplifications, to obtain the following recurrence equations

$$\begin{aligned} &[H_0, Q_1] = -2H_1 \\ &[H_0, Q_2] = 0 \\ &[H_0, Q_3] = -\frac{1}{6}[H_1, Q_1] \\ &... \\ &... \end{aligned} \tag{C.8}$$

we get $[H_0, Q_1] = -2ix$ and $[H_0, Q_3] = -\frac{1}{6}[[H_1, Q_1]$ the convenient assumption on Q_1 and Q_3 are

$$Q_1(x,p) = \sum_{j=0}^{1} \sum_{k=0}^{1} C_{0,j,k} \cdot AntiCommutator\left(x^{2j}, p^{2k+1}\right) \tag{C.9}$$

and

$$Q_3(x,p) = \sum_{j=0}^{2} \sum_{k=0}^{2} B_{0,j,k} \cdot AntiCommutator(x^{2j}, p^{2k+1} \tag{C.10}$$

where $Q_1(x,p)$ and $Q_3(x,p)$ are thought of as odd on p and even on x let's determine the expression of $Q_1(x,p)$, by developing (C.9) to get

$$Q_1(x,p) = 2 \cdot C_{0,0,0}p + 2 \cdot C_{0,0,1}p^3 + C_{0,1,0}AntiCommutator\left(x^2, p\right)$$
$$+C_{0,1,1}AntiCommutator\left(x^2, p^3\right) \qquad \text{(C.11)}$$

by simplification we find

$$C_{0,0,0} = C_{0,1,1} = 0$$
$$C_{0,0,1} = -\frac{1}{3}$$
$$C_{0,1,0} = -\frac{1}{2}$$

thus

$$Q_1(x,p) = -2 \cdot 3^{-1} \cdot p^3 - \frac{1}{2}AntiCommutator\left(x^2, p\right) \qquad \text{(C.12)}$$

It 's clearly shown that the calculation of $Q_3(x,p)$ is much complicated, because there are many terms raised up by the effect of anti-commutators. So we work out the expression of equation (C.10) which gives

$$Q_3(x,p) = 2 \cdot B_{0,0,0} \cdot p + 2 \cdot B_{0,0,1} \cdot p^3 + 2 \cdot B_{0,0,2} \cdot p^5 + B_{0,1,0} \cdot AntiCommutator\left(x^2, p\right)$$
$$+ B_{0,1,1} \cdot AntiCommutator\left(x^2, p^3\right) + B_{0,1,2} \cdot AntiCommutator\left(x^2, p^5\right)$$
$$+ B_{0,2,0} \cdot AntiCommutator\left(x^4, p\right) + B_{0,2,1} \cdot AntiCommutator\left(x^4, p^3\right)$$
$$+ B_{0,2,2} \cdot AntiCommutator\left(x^4, p^5\right) \qquad \text{(C.13)}$$

Simplify once again then we obtain the following constants C_{ijk}

$$B_{0,0,0} = B_{0,2,0} = 1/2$$
$$B_{0,0,2} = 8/15$$
$$B_{0,1,1} = 5/6$$

and

$$B_{0,2,2} = B_{0,2,1} = B_{0,1,2} = B_{0,0,1} = B_{0,1,0} = 0$$

therefore

$$Q_3(x,p) = p + 16 \cdot 15^{-1} \cdot p^5 + 5 \cdot 6^{-1} \cdot AntiCommutator\left(x^2, p^3\right)$$
$$+ 2^{-1} \cdot AntiCommutator\left(x^4, p\right) \qquad \text{(C.14)}$$

Note that Q is defined as follows

$$Q(x,p) = \epsilon Q_1 + \epsilon^3 Q_3$$

Substitute Q_1 and Q_3 we obtain

$$Q(x,p) = \epsilon \left(-2 \cdot 3^{-1} \cdot p^3 - 1 \cdot 2^{-1} \cdot AntiCommutator\left(x^2, p\right)\right) + \epsilon^3 \left(p + 16 \cdot 15^{-1} \cdot p^5 + 5 \cdot 6^{-1} \cdot AntiCommutator\left(x^2, p^3\right) + 2^{-1} \cdot AntiCommutator\left(x^4, p\right)\right) \quad \text{(C.15)}$$

Finally the expression of θ in function of Q_1 , Q_3 and the parameter ϵ becomes

$$\eta = e^{-Q(x,p)}$$

that implies

$$\eta = exp-(\epsilon \left(-2 \cdot 3^{-1} \cdot p^3 - 1 \cdot 2^{-1} \cdot AntiCommutator\left(x^2, p\right)\right) + \epsilon^3 \left(p + 16 \cdot 15^{-1} \cdot p^5 + 5 \cdot 6^{-1} \cdot AntiCommutator\left(x^2, p^3\right)\right) + 2^{-1} \cdot AntiCommutator\left(x^4, p\right)) \quad \text{(C.16)}$$

Apply (A.9) to find the Hermitian Hamiltonian

$$h = p^2+x^2+2^{-1}\cdot i\cdot\epsilon\left(4^{-1}\cdot\epsilon\cdot Commutator\left(x^3, Commutator\left(p^2, AntiCommutator\left(x^2,p\right)\right)\right)\right.$$

$$+2^{-1}\cdot\epsilon^3\left(5\cdot 6^{-1}\cdot Commutator\left(x^3, Commutator\left(p^2, AntiCommutator\left(x^2,p^3\right)\right)\right)\right.$$

$$+2^{-1}\cdot Commutator\left(x^3, Commutator\left(p^2, AntiCommutator\left(x^4,p\right)\right)\right)$$

$$-i\cdot\epsilon\left(-6\cdot x^2+2\cdot\left(2\cdot i\cdot x\left(i+2\cdot p\cdot x\right)+2\cdot i\cdot p\cdot x^2\right)x\right)-64\cdot 3^{-1}\cdot\left(3\cdot i\cdot p^2+2\cdot p^3\cdot x\right)x^2$$

$$+2^{-1}\cdot\epsilon^3\left(16\cdot 3^{-1}\cdot i\left(4\cdot i\left(3\cdot i\cdot x\left(2\cdot i\cdot p+2\cdot p^2\cdot x\right)+3\cdot i\cdot p^2\cdot x^2\right)\right.\right.$$

$$+2\cdot\left(4i\cdot x\left(3i\cdot p^2+2p^3\cdot x\right)+4i\cdot p^3\cdot x^2\right)x-4^{-1}\epsilon\cdot Commutator\left(p^2, AntiCommutator\left(x^2,p\right)\right)$$

$$+5\cdot 6^{-1}\cdot Commutator\left(x^3, Commutator\left(x^2, AntiCommutator\left(x^2,p^3\right)\right)\right)$$

$$+i\cdot\epsilon\left(1\cdot 2^{-1}\cdot\epsilon\left(-2\cdot i\cdot x\left(-6\cdot x^2+2\cdot\left(2\cdot i\cdot x\left(i+2\cdot p\cdot x\right)+2\cdot i\cdot p\cdot x^2\right)x\right)\right.\right.$$

$$-2\cdot i\cdot\left(2\cdot i\cdot x\left(i+2\cdot p\cdot x\right)+2\cdot i\cdot p\cdot x^2\right)x^2+2^{-1}\cdot\epsilon^3\left(16\cdot 3^{-1}\cdot i\cdot x\left(-12\cdot x\left(2\cdot i\cdot p+2\cdot p^2\cdot x\right)\right.\right.$$

$$-12\cdot p^2\cdot x^2+2\cdot\left(4\cdot i\cdot x\left(3\cdot i\cdot p^2+2\cdot p^3\cdot x\right)+4\cdot i\cdot p^3\cdot x^2\right)x+16\cdot 3^{-1}\cdot i\cdot\left(4\cdot i\cdot x\left(3\cdot i\cdot p^2+2\cdot p^3\cdot x\right)\right.$$

$$+4\cdot i\cdot p^3\cdot x^2 x^2+5\cdot 6^{-1}\cdot Commutator\left(x^3, Commutator\left(x^3, AntiCommutator\left(x^2,p^3\right)\right)\right)$$

$$+2^{-1}\cdot\epsilon^3\left(5\cdot 6^{-1}\cdot Commutator\left(p^2, AntiCommutator\left(x^2,p^3\right)\right)\right.$$

$$+2^{-1}\cdot Commutator\left(p^2, AntiCommutator\left(x^4,p\right)\right)+1\cdot 2^{-1}\cdot\epsilon\left(-2\cdot i\left(2\cdot i\cdot p+2\cdot p^2\cdot x\right)\right.$$

$$-1\cdot 2^{-1}\cdot Commutator\left(x^2, AntiCommutator\left(x^2,p\right)\right)$$

$$+1\cdot 2^{-1}\cdot\epsilon^3\left(2\cdot i\cdot x+16\cdot 3^{-1}\cdot i\left(4\cdot i\cdot p^3+2\cdot p^4\cdot x\right)\right.$$

$$+5\cdot 6^{-1}\cdot Commutator\left(x^2, AntiCommutator\left(x^2,p^3\right)\right)$$

$$+1\cdot 2^{-1}\cdot Commutator\left(x^2, AntiCommutator\left(x^4,p\right)\right.$$

$$-2^{-1}\cdot Commutator\left(p^2, Commutator\left(x^3, AntiCommutator\left(x^2,p\right)\right)\right)$$

$$+2^{-1}\cdot i\cdot\epsilon\left(2^{-1}\cdot\epsilon\left(-2\cdot i\left(-4\cdot i\cdot x\cdot p^3-2\cdot i\cdot p\left(2\cdot i\cdot p+2\cdot p^2\cdot x\right)\right)-4\cdot p^2\left(i+2\cdot p\cdot x\right)\right.\right.$$

$$+1\cdot 2^{-1}\cdot\epsilon^3\left(6\cdot i+12\cdot p\cdot x+16\cdot 3^{-1}\cdot i\left(-4\cdot i\cdot x\cdot p^5-2\cdot i\cdot p\left(4\cdot i\cdot p^3+2\cdot p^4\cdot x\right)\right)\right.$$

$$+32\cdot 3^{-1}\cdot p^4\left(i+2\cdot p\cdot x\right)+5\cdot 6^{-1}\cdot Commutator\left(p^2, Commutator\left(x^3, AntiCommutator\left(x^2,p^3\right)\right)\right)$$

$$+2^{-1}\cdot Commutator\left(p^2, Commutator\left(x^3, AntiCommutator\left(x^4,p\right)\right)\right)$$

$$-8^{-1}\cdot\epsilon\cdot Commutator\left(p^2, Commutator\left(p^2, AntiCommutator\left(x^2,p\right)\right)\right)$$
$$+4^{-1}\cdot\epsilon^3\left(5\cdot 6^{-1}\cdot Commutator\left(p^2, Commutator\left(p^2, AntiCommutator\left(x^2,p^3\right)\right)\right)\right.$$
$$+2^{-1}\cdot Commutator\left(p^2, Commutator\left(p^2, AntiCommutator\left(x^4,p\right)\right)\right)$$
$$+4^{-1}\cdot\epsilon\left(-8\cdot p^3-1\cdot 2^{-1}\cdot Commutator\left(p^2, Commutator\left(x^2, AntiCommutator\left(x^2,p\right)\right)\right)\right)$$
$$+4^{-1}\cdot\epsilon^3\left(4\cdot p+64\cdot 3^{-1}\cdot p^5+5\cdot 6^{-1}\cdot Commutator\left(p^2, Commutator\left(x^2, AntiCommutator\left(x^2,p^3\right)\right)\right.\right.$$
$$+2^{-1}\cdot Commutator\left(p^2, Commutator\left(x^2, AntiCommutator\left(x^4,p\right)\right)\right)$$

$$+1\cdot 2^{-1}\cdot i\cdot\epsilon\left(1\cdot 2^{-1}\cdot\epsilon\left(-2\cdot i\cdot x\left(-4\cdot x+4\cdot i\cdot(i+2\cdot p\cdot x)\,x\right)+4\cdot(i+2\cdot p\cdot x)\,x^2\right)\right.$$
$$+2^{-1}\cdot\epsilon^3\left(16\cdot 3^{-1}\cdot i\cdot x\left(-24\cdot i\cdot p-24\cdot p^2\cdot x+8\cdot i\cdot\left(3\cdot i\cdot p^2+2\cdot p^3\cdot x\right)x\right)\right.$$
$$+5\cdot 6^{-1}\cdot Commutator\left(x^2, Commutator\left(x^3, AntiCommutator\left(x^2,p^3\right)\right)\right)$$
$$-8^{-1}\cdot\epsilon\cdot Commutator\left(x^2, Commutator\left(p^2, AntiCommutator\left(x^2,p\right)\right)\right)$$
$$+4^{-1}\cdot\epsilon^3\left(5\cdot 6^{-1}\cdot Commutator\left(x^2, Commutator\left(p^2, AntiCommutator\left(x^2,p^3\right)\right)\right)\right.$$
$$+2^{-1}\cdot Commutator\left(x^2, Commutator\left(p^2, AntiCommutator\left(x^4,p\right)\right)\right)$$
$$+i\cdot\epsilon\left(2^{-1}\cdot\epsilon\left(-2\cdot i\cdot x\left(2\cdot i\cdot p+2\cdot p^2\cdot x\right)-2\cdot i\cdot p^2\cdot x^2\right.\right.$$
$$-2^{-1}\cdot Commutator\left(x^3, AntiCommutator\left(x^2,p\right)\right)$$
$$+1\cdot 2^{-1}\cdot\epsilon^3\left(3\cdot i\cdot x^2+16\cdot 3^{-1}\cdot i\cdot x\left(4\cdot i\cdot p^3+2\cdot p^4\cdot x\right)+16\cdot 3^{-1}\cdot i\cdot p^4\cdot x^2\right.$$
$$+5\cdot 6^{-1}\cdot Commutator\left(x^3, AntiCommutator\left(x^2,p^3\right)\right)$$
$$+1\cdot 2^{-1}\cdot Commutator\left(x^3, AntiCommutator\left(x^4,p\right)\right)$$
$$+4^{-1}\cdot\epsilon^3\left(16\cdot 3^{-1}\cdot i\left(-24\cdot i\cdot p-24\cdot p^2\cdot x+8\cdot i\cdot\left(3\cdot i\cdot p^2+2\cdot p^3\cdot x\right)x\right)\right.$$
$$+5\cdot 6^{-1}\cdot Commutator\left(x^2, Commutator\left(x^2, AntiCommutator\left(x^2,p^3\right)\right)\right)+i\cdot\epsilon\cdot x^3$$
$$-2^{-1}\cdot i\cdot\epsilon\left(-4\cdot x+4\cdot i\cdot(i+2\cdot p\cdot x)\,x\right)\quad\text{(C.17)}$$

The simplification and expansion of commutators and anti-commutator gives

$$h=p^2+x^2+\left(3\,ix-3\,x^2p\right)\epsilon+\left(3/2\,x^4-6\,ixp^3-9\,p^2-2+3\,x^2p^2-6\,ixp\right)\epsilon^2$$
$$+\left(12\,ix^3+15\,ixp^2+4\,p^3+6\,p-3\,ix-5\,x^2\cdot p^3-6\,x^4p+ix^5+2\,p^5+2\,ixp^4\right.$$
$$+3\,x^2p+ix^3p^2)\epsilon^3+(-8\,x^2p^4+78\,ixp+18-\frac{45}{2}\,x^4-63\,x^2p^2-9\,ix^5p+32\,ixp^3$$
$$-3/2\,x^6+30\,ix^3p+32\,p^2-15/2\,x^4p^2+16\,ixp^5+26\,x^2-14\,ix^3p^3+40\,p^4)\epsilon^4$$
$$+(\frac{45}{2}\,x^6p-480\,x^2p+48\,x^4p^3-\frac{135}{2}\,ix^5-288\,ix^3p^2+192\,ix)\epsilon^5\quad\text{(C.18)}$$

According to the equation (C.4), we conclude that the expression of h_d is

$$h_d = \eta_t^{\frac{1}{2}} H_d \eta_t^{-\frac{1}{2}} = \eta_d^{\frac{1}{2}} (\eta^{\frac{1}{2}} H \eta^{-\frac{1}{2}}) \eta_d^{-\frac{1}{2}} = \eta_d^{\frac{1}{2}} h \eta_d^{-\frac{1}{2}}$$

to get a deformed version of h of expression (C.17), x should be changed by X and h_d becomes

$$h_d = p^2 + X^2 + \left(3\,iX - 3\,X^2 p\right)\epsilon + \left(3/2\,X^4 - 6\,iXp^3 - 9\,p^2 - 2 + 3\,X^2p^2 - 6\,iXp\right)\epsilon^2 + \left(12\,iX^3 + 15\,iXp^2 + 4\,p^3 + 6\,p - 3\,iX - 5\,x^2 \cdot p^3 - 6\,X^4 p + iX^5 + 2\,p^5 + 2\,iXp^4 + 3\,X^2 p + iX^3 p^2\right)\epsilon^3 + (-8\,X^2p^4 + 78\,iXp + 18 - \frac{45}{2}\,X^4 - 63\,X^2p^2 - 9\,iX^5 p + 32\,iXp^3 - 3/2\,X^6 + 30\,iX^3 p + 32\,p^2 - 15/2\,X^4p^2 + 16\,iXp^5 + 26\,X^2 - 14\,iX^3p^3 + 40\,p^4)\epsilon^4 + (\frac{45}{2}\,X^6 p - 480\,X^2 p + 48\,X^4p^3 - \frac{135}{2}\,iX^5 - 288\,iX^3p^2 + 192\,iX)\epsilon^5 \quad \text{(C.19)}$$

. if we substitute $X = (1+\beta p^2)x$, then h_d gets much complicated as the following expression

$$h_d = p^2 + \left(\left(1+\beta\,p^2\right)x\right)^2 + \left(3\,i\left(1+\beta\,p^2\right)x - 3\,\left(\left(1+\beta\,p^2\right)x\right)^2 p\right)\epsilon + \left(3/2\,\left(\left(1+\beta\,p^2\right)x\right)^4 - 6\,i\left(1+\beta\,p^2\right)xp^3 - 9\,p^2 - 2 + 3\,\left(\left(1+\beta\,p^2\right)x\right)^2 p^2 - 6\,i\left(1+\beta\,p^2\right)xp\right)\epsilon^2 + \left(12\,i\left(\left(1+\beta\,p^2\right)x\right)^3 + 15\,i\left(1+\beta\,p^2\right)xp^2 + 4\,p^3 + 6\,p - 3\,i\left(1+\beta\,p^2\right)x - 5\,x^2 \cdot p^3 - 6\,\left(\left(1+\beta\,p^2\right)x\right)^4 p + i\left(\left(1+\beta\,p^2\right)x\right)^5 + 2\,p^5 + 2\,i\left(1+\beta\,p^2\right)xp^4 + 3\,\left(\left(1+\beta\,p^2\right)x\right)^2 p + i\left(\left(1+\beta\,p^2\right)x\right)^3 p^2\right)\epsilon^3 + (-8\,\left(\left(1+\beta\,p^2\right)x\right)^2 p^4 + 78\,i\left(1+\beta\,p^2\right)xp + 18 - \frac{45}{2}\,\left(\left(1+\beta\,p^2\right)x\right)^4 - 63\,\left(\left(1+\beta\,p^2\right)x\right)^2 p^2 - 9\,i\left((1+\beta\,p^2)x\right)^5 p + 32\,i\left(1+\beta\,p^2\right)xp^3 - 3/2\,\left(\left(1+\beta\,p^2\right)x\right)^6 + 30\,i\left(\left(1+\beta\,p^2\right)x\right)^3 p + 32\,p^2 - 15/2\,\left(\left(1+\beta\,p^2\right)x\right)^4 p^2 + 16\,i\left(1+\beta\,p^2\right)xp^5 + 26\,\left(\left(1+\beta\,p^2\right)x\right)^2 - 14\,i\left(\left(1+\beta\,p^2\right)x\right)^3 p^3 + 40\,p^4)\epsilon^4 + (\frac{45}{2}\,\left(\left(1+\beta\,p^2\right)x\right)^6 p - 480\,\left(\left(1+\beta\,p^2\right)x\right)^2 p + 48\,\left(\left(1+\beta\,p^2\right)x\right)^4 p^3 - \frac{135}{2}\,i\left(\left(1+\beta\,p^2\right)x\right)^5 - 288\,i\left(\left(1+\beta\,p^2\right)x\right)^3 p^2 + 192\,i\left(1+\beta\,p^2\right)x)\epsilon^5 \quad \text{(C.20)}$$

C.2.2 The Energy Spectrum E_n

Let's assume that `a-` `a+` are the annihilation and creation operators and their effect on the Fock states as follows

$$\texttt{a-}\,|\phi_n\rangle = \sqrt{n}\,|\phi_{n-1}\rangle \quad \text{(C.21)}$$

$$\sqrt{n}\,\langle\phi_{n-1}| \tag{C.22}$$

$$\sqrt{n+1}\,|\phi_{n+1}\rangle \tag{C.23}$$

$$\sqrt{n+1}\,\langle\phi_{n+1}| \tag{C.24}$$

The commutator

$$[\text{a-},\text{a+}] = 1 \tag{C.25}$$

According to the previous subsection we use the expression of h_d and plug in it the expressions of the coordinates (x, p) written in terms of the annihilation and creation operators : $x = \frac{1}{\sqrt{2}}(\text{a-} + \text{a+})$, $p = \frac{i}{\sqrt{2}}(\text{a+} - \text{a-})$, after we have finished this heavy computation ,we got a long equation

$$h_d = 6\cdot\epsilon^2\cdot\beta\cdot\text{‘a+‘}^3\cdot\text{‘a-‘}^3 - 675\cdot256^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^4\cdot\text{‘a-‘}^{13} - 3\cdot\epsilon^2\cdot\text{‘a-‘}^2\cdot\text{‘a+‘}^6\cdot\beta^2 +$$
$$3\cdot4^{-1}\cdot i\cdot\epsilon\cdot2^{1\cdot2^{-1}}\cdot\beta\cdot\text{‘a+‘}\cdot\text{‘a-‘}^2 + 9\cdot32^{-1}\cdot\epsilon^2\cdot\beta^4\cdot\text{‘a+‘}^9\cdot\text{‘a-‘}^3 + 189\cdot32^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^2\cdot$$
$$\text{‘a-‘}^9 + 27\cdot64^{-1}\cdot\epsilon^4\cdot\beta^6\cdot\text{‘a-‘}^{11}\cdot\text{‘a+‘}^7 + 2925\cdot512^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^6\cdot\text{‘a-‘}^{12}\cdot\text{‘a+‘}^7 + 79\cdot$$
$$32^{-1}\cdot i\cdot\epsilon^3\cdot2^{1\cdot2^{-1}}\cdot\beta^4\cdot\text{‘a+‘}^6\cdot\text{‘a-‘}^7 + 2\cdot\text{‘a+‘}\cdot\text{‘a-‘} + 21\cdot32^{-1}\cdot\epsilon^2\cdot\beta^4\cdot\text{‘a+‘}^6\cdot\text{‘a-‘}^6 - 19\cdot$$
$$16^{-1}\cdot\epsilon^4\cdot\beta^3\cdot\text{‘a+‘}^{12} - 5103\cdot64^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^2\cdot\text{‘a+‘}^4\cdot\text{‘a-‘}^7 + 99\cdot512^{-1}\cdot\epsilon^4\cdot\beta^6\cdot\text{‘a-‘}^{10}\cdot$$
$$\text{‘a+‘}^8 - 15\cdot32^{-1}\cdot\epsilon^4\cdot\beta^4\cdot\text{‘a-‘}^{14} + 30\cdot\epsilon^4\cdot\text{‘a+‘}^7\cdot\text{‘a-‘}^3\cdot\beta^3 - 1\cdot128^{-1}\cdot i\cdot\epsilon^3\cdot2^{1\cdot2^{-1}}\cdot\beta^4\cdot\text{‘a+‘}^{13} +$$
$$675\cdot8^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^2\cdot\text{‘a-‘}^7\cdot\text{‘a+‘}^2 + i\cdot\epsilon^3\cdot2^{1\cdot2^{-1}}\cdot\beta\cdot\text{‘a+‘}^6\cdot\text{‘a-‘} + 1\cdot8^{-1}\cdot\beta^2\cdot\text{‘a-‘}^6 - 3\cdot$$
$$16^{-1}\cdot\epsilon^2\cdot\beta^4\cdot\text{‘a+‘}^7\cdot\text{‘a-‘}^5 + 3\cdot4^{-1}\cdot i\cdot\epsilon^3\cdot2^{1\cdot2^{-1}}\cdot\beta\cdot\text{‘a-‘}^3 - 1\cdot2^{-1}\cdot\beta\cdot\text{‘a+‘}^4 - 135\cdot32^{-1}\cdot\epsilon^4\cdot\beta^4\cdot$$
$$\text{‘a+‘}^3\cdot\text{‘a-‘}^9 - 135\cdot32^{-1}\cdot\epsilon^4\cdot\beta^4\cdot\text{‘a+‘}^9\cdot\text{‘a-‘}^3 + 7\cdot4^{-1}\cdot i\cdot\epsilon^3\cdot2^{1\cdot2^{-1}}\cdot\text{‘a-‘}^2\cdot\text{‘a+‘}^3 - 3\cdot8^{-1}\cdot i\cdot$$
$$\epsilon^3\cdot2^{1\cdot2^{-1}}\cdot\beta^3\cdot\text{‘a-‘}^9 + 6543\cdot32^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^3\cdot\text{‘a+‘}^7\cdot\text{‘a-‘}^4 + 9\cdot8^{-1}\cdot i\cdot\epsilon^3\cdot2^{1\cdot2^{-1}}\cdot\beta^3\cdot\text{‘a-‘}^8\cdot$$
$$\text{‘a+‘} - 45\cdot8^{-1}\cdot\epsilon^4\cdot\text{‘a-‘}^2\cdot\text{‘a+‘}^8\cdot\beta^3 + 3753\cdot32^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta\cdot\text{‘a+‘}^6\cdot\text{‘a-‘} - 3\cdot64^{-1}\cdot$$
$$\epsilon^4\cdot\beta^6\cdot\text{‘a+‘}^{15}\cdot\text{‘a-‘}^3 + 441\cdot16^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\text{‘a-‘}^5 - 15\cdot256^{-1}\cdot i\cdot\epsilon^3\cdot2^{1\cdot2^{-1}}\cdot\beta^5\cdot\text{‘a-‘}^{12}\cdot$$
$$\text{‘a+‘}^3 - 3267\cdot32^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^2\cdot\text{‘a-‘}^8\cdot\text{‘a+‘} + 5\cdot2^{-1}\cdot i\cdot\epsilon^3\cdot2^{1\cdot2^{-1}}\cdot\beta^2\cdot\text{‘a-‘}^6\cdot\text{‘a+‘}^3 + 15\cdot$$
$$8^{-1}\cdot\epsilon^4\cdot\text{‘a-‘}^{10}\cdot\beta^3 + 9\cdot8^{-1}\cdot\epsilon^4\cdot\text{‘a-‘}^8\cdot\beta^2 - 1\cdot256^{-1}\cdot i\cdot\epsilon^3\cdot2^{1\cdot2^{-1}}\cdot\beta^5\cdot\text{‘a-‘}^{15} + 157\cdot8^{-1}\cdot\epsilon^4\cdot$$
$$\text{‘a-‘}^4 + 675\cdot2048^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^6\cdot\text{‘a+‘}^{17}\cdot\text{‘a-‘}^2 - 2925\cdot512^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^6\cdot\text{‘a+‘}^{12}\cdot$$
$$\text{‘a-‘}^7 + 147\cdot16^{-1}\cdot i\cdot\epsilon^3\cdot2^{1\cdot2^{-1}}\cdot\beta^2\cdot\text{‘a-‘}^5\cdot\text{‘a+‘}^4 + 579\cdot512^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^4\cdot\text{‘a+‘}^{15} - 69\cdot$$
$$16^{-1}\cdot i\cdot\epsilon^3\cdot2^{1\cdot2^{-1}}\cdot\beta^2\cdot\text{‘a-‘}^4\cdot\text{‘a+‘}^5 - 675\cdot512^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^5\cdot\text{‘a-‘}^{16}\cdot\text{‘a+‘} + 57\cdot32^{-1}\cdot i\cdot$$
$$\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta\cdot\text{‘a+‘}^9 - 45\cdot2048^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^6\cdot\text{‘a-‘}^{19} - 48\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta\cdot\text{‘a+‘}^3 + 3375\cdot$$
$$256^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^4\cdot\text{‘a-‘}^9\cdot\text{‘a+‘}^4 + 45\cdot128^{-1}\cdot\epsilon^4\cdot\beta^5\cdot\text{‘a+‘}^{13}\cdot\text{‘a-‘}^3 + 75\cdot64^{-1}\cdot\epsilon^4\cdot\beta^4\cdot$$
$$\text{‘a-‘}^{12}\cdot\text{‘a+‘}^2 + 513\cdot32^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\text{‘a-‘}^3\cdot\text{‘a+‘}^4 - 15\cdot4^{-1}\cdot\epsilon^2\cdot\beta\cdot\text{‘a+‘}^2\cdot\text{‘a-‘}^4 + 33\cdot$$
$$64^{-1}\cdot i\cdot\epsilon^3\cdot2^{1\cdot2^{-1}}\cdot\beta^3\cdot\text{‘a+‘}^9\cdot\text{‘a-‘}^2 - 5391\cdot32^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^3\cdot\text{‘a-‘}^6\cdot\text{‘a+‘}^5 - 9\cdot128^{-1}\cdot\epsilon^4\cdot$$
$$\beta^6\cdot\text{‘a+‘}^{13}\cdot\text{‘a-‘}^5 - 4725\cdot512^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^5\cdot\text{‘a-‘}^{11}\cdot\text{‘a+‘}^4 - 135\cdot512^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot$$
$$\beta^5\cdot\text{‘a+‘}^{17} - 5391\cdot32^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^3\cdot\text{‘a+‘}^6\cdot\text{‘a-‘}^5 + 24\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta\cdot\text{‘a+‘}^2\cdot\text{‘a-‘}^7 - 45\cdot$$
$$64^{-1}\cdot\epsilon^4\cdot\beta^4\cdot\text{‘a+‘}^{10}\cdot\text{‘a-‘}^2 - 9\cdot2^{-1}\cdot\epsilon^4\cdot\text{‘a-‘}\cdot\text{‘a+‘}^7\cdot\beta^2 + 243\cdot4^{-1}\cdot\epsilon^4\cdot\text{‘a-‘}^2\cdot\text{‘a+‘}^6\cdot\beta^2 - 7\cdot$$
$$4^{-1}\cdot i\cdot\epsilon^3\cdot2^{1\cdot2^{-1}}\cdot\beta\cdot\text{‘a-‘}^7 - 29\cdot32^{-1}\cdot i\cdot\epsilon^3\cdot2^{1\cdot2^{-1}}\cdot\beta^2\cdot\text{‘a+‘}^9 - 67\cdot64^{-1}\cdot i\cdot\epsilon^3\cdot2^{1\cdot2^{-1}}\cdot\beta^3\cdot\text{‘a+‘}^{10}\cdot$$
$$\text{‘a-‘} + 9\cdot4^{-1}\cdot i\cdot\epsilon^3\cdot2^{1\cdot2^{-1}}\cdot\beta^3\cdot\text{‘a+‘}^4\cdot\text{‘a-‘}^5 + 9\cdot4^{-1}\cdot i\cdot\epsilon^3\cdot2^{1\cdot2^{-1}}\cdot\beta^3\cdot\text{‘a+‘}^5\cdot\text{‘a-‘}^4 - 120\cdot i\cdot\epsilon^5\cdot$$
$$2^{1\cdot2^{-1}}\cdot\text{‘a+‘}^3 + 135\cdot128^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^5\cdot\text{‘a-‘}^{15}\cdot\text{‘a+‘}^2 - 5391\cdot64^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^3\cdot\text{‘a+‘}^9\cdot$$
$$\text{‘a-‘}^2 - 3\cdot2^{-1}\cdot\epsilon^2\cdot\text{‘a+‘}^7\cdot\text{‘a-‘}^3\cdot\beta^3 + 30\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^2\cdot\text{‘a-‘}^5\cdot\text{‘a+‘}^2 + 13\cdot4^{-1}\cdot i\cdot\epsilon^3\cdot2^{1\cdot2^{-1}}\cdot$$

$\beta^{2}\cdot\text{‘a+‘}^{7}\cdot\text{‘a−‘}^{2}+315\cdot 2048^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{6}\cdot\text{‘a+‘}^{16}\cdot\text{‘a−‘}^{3}+675\cdot 64^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{5}\cdot$
$\text{‘a+‘}^{13}\cdot\text{‘a−‘}^{4}-6075\cdot 512^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{5}\cdot\text{‘a+‘}^{8}\cdot\text{‘a−‘}^{7}+9\cdot 2^{-1}\cdot\epsilon^{2}\cdot\text{‘a−‘}\cdot\text{‘a+‘}^{3}+15\cdot 4^{-1}\cdot$
$\epsilon^{2}\cdot\text{‘a−‘}^{2}\cdot\text{‘a+‘}^{2}+99\cdot 2^{-1}\cdot\epsilon^{4}\cdot\text{‘a−‘}^{5}\cdot\beta^{2}\cdot\text{‘a+‘}^{3}-23\cdot 64^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{4}\cdot\text{‘a+‘}^{11}\cdot\text{‘a−‘}^{2}+$
$135\cdot 512^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{5}\cdot\text{‘a−‘}^{15}-5391\cdot 64^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{3}\cdot\text{‘a−‘}^{9}\cdot\text{‘a+‘}^{2}-13\cdot 2^{-1}\cdot i\cdot\epsilon^{3}\cdot$
$2^{1\cdot 2^{-1}}\cdot\beta^{2}\cdot\text{‘a+‘}^{6}\cdot\text{‘a−‘}^{3}-513\cdot 32^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\text{‘a−‘}^{4}\cdot\text{‘a+‘}^{3}+18\cdot\epsilon^{4}-2\cdot\epsilon^{2}+135\cdot 128^{-1}\cdot$
$\epsilon^{4}\cdot\beta^{5}\cdot\text{‘a−‘}^{13}\cdot\text{‘a+‘}^{3}-3375\cdot 64^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{4}\cdot\text{‘a+‘}^{6}\cdot\text{‘a−‘}^{7}-27\cdot 1024^{-1}\cdot\epsilon^{4}\cdot\beta^{6}\cdot\text{‘a−‘}^{16}\cdot$
$\text{‘a+‘}^{2}-3\cdot 64^{-1}\cdot\epsilon^{4}\cdot\beta^{6}\cdot\text{‘a−‘}^{15}\cdot\text{‘a+‘}^{3}-75\cdot 8^{-1}\cdot\epsilon^{4}\cdot\text{‘a−‘}\cdot\text{‘a+‘}^{9}\cdot\beta^{3}-18\cdot\epsilon^{4}\cdot\text{‘a+‘}^{5}\cdot\text{‘a−‘}^{3}\cdot$
$\beta^{2}+77\cdot 64^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{3}\cdot\text{‘a−‘}^{10}\cdot\text{‘a+‘}-3\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{3}\cdot\text{‘a−‘}^{6}\cdot\text{‘a+‘}^{3}+9\cdot 8^{-1}\cdot i\cdot\epsilon^{3}\cdot$
$2^{1\cdot 2^{-1}}\cdot\beta^{3}\cdot\text{‘a+‘}^{8}\cdot\text{‘a−‘}+765\cdot 128^{-1}\cdot\epsilon^{4}\cdot\beta^{4}\cdot\text{‘a+‘}^{8}\cdot\text{‘a−‘}^{4}-6159\cdot 64^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{3}\cdot\text{‘a+‘}^{9}\cdot$
$\text{‘a−‘}^{4}+981\cdot 16^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta\cdot\text{‘a+‘}^{5}\cdot\text{‘a−‘}^{4}+45\cdot 2048^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{6}\cdot\text{‘a+‘}^{19}+150\cdot$
$i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{2}\cdot\text{‘a−‘}^{4}\cdot\text{‘a+‘}^{3}+45\cdot 128^{-1}\cdot\epsilon^{4}\cdot\beta^{5}\cdot\text{‘a−‘}^{14}\cdot\text{‘a+‘}^{2}-45\cdot 128^{-1}\cdot\epsilon^{4}\cdot\beta^{5}\cdot\text{‘a+‘}^{14}\cdot$
$\text{‘a−‘}^{2}-3\cdot 16^{-1}\cdot\epsilon^{2}\cdot\text{‘a+‘}^{10}\cdot\beta^{3}-3\cdot 16^{-1}\cdot\epsilon^{2}\cdot\text{‘a−‘}^{10}\cdot\beta^{3}+8775\cdot 512^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{5}\cdot\text{‘a+‘}^{9}\cdot$
$\text{‘a−‘}^{6}+135\cdot 32^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{5}\cdot\text{‘a+‘}^{12}\cdot\text{‘a−‘}^{5}+\beta\cdot\text{‘a+‘}^{2}\cdot\text{‘a−‘}^{2}-3\cdot 4^{-1}\cdot\epsilon^{2}\cdot\beta\cdot\text{‘a−‘}^{6}+$
$90\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{2}\cdot\text{‘a+‘}^{6}\cdot\text{‘a−‘}+5\cdot 256^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{5}\cdot\text{‘a−‘}^{14}\cdot\text{‘a+‘}+1\cdot 8^{-1}\cdot\beta^{2}\cdot\text{‘a+‘}^{6}-$
$1\cdot 2^{-1}\cdot\beta\cdot\text{‘a−‘}^{4}+675\cdot 128^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{5}\cdot\text{‘a−‘}^{14}\cdot\text{‘a+‘}^{3}-1\cdot 8^{-1}\cdot\beta^{2}\cdot\text{‘a+‘}^{4}\cdot\text{‘a−‘}^{2}+1\cdot$
$2^{-1}\cdot\beta^{2}\cdot\text{‘a+‘}^{3}\cdot\text{‘a−‘}^{3}+675\cdot 512^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{5}\cdot\text{‘a+‘}^{16}\cdot\text{‘a−‘}-2103\cdot 512^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot$
$\beta^{4}\cdot\text{‘a−‘}^{11}\cdot\text{‘a+‘}^{4}-675\cdot 64^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{5}\cdot\text{‘a−‘}^{13}\cdot\text{‘a+‘}^{4}+15\cdot 128^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{4}\cdot$
$\text{‘a+‘}^{12}\cdot\text{‘a−‘}+93\cdot 16^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{2}\cdot\text{‘a−‘}^{4}\cdot\text{‘a+‘}^{3}+48\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta\cdot\text{‘a+‘}\cdot\text{‘a−‘}^{2}+$
$27\cdot 64^{-1}\cdot\epsilon^{4}\cdot\beta^{6}\cdot\text{‘a+‘}^{11}\cdot\text{‘a−‘}^{7}-135\cdot 512^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{5}\cdot\text{‘a+‘}^{10}\cdot\text{‘a−‘}^{5}+35\cdot 256^{-1}\cdot i\cdot$
$\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{5}\cdot\text{‘a+‘}^{11}\cdot\text{‘a−‘}^{4}+9\cdot 8^{-1}\cdot\epsilon^{2}\cdot\text{‘a−‘}^{4}+149\cdot 128^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{4}\cdot\text{‘a+‘}^{9}\cdot\text{‘a−‘}^{4}+$
$11\cdot 32^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{3}\cdot\text{‘a−‘}^{7}\cdot\text{‘a+‘}^{4}+675\cdot 8^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{2}\cdot\text{‘a+‘}^{7}\cdot\text{‘a−‘}^{2}+3\cdot 64^{-1}\cdot\epsilon^{2}\cdot$
$\beta^{4}\cdot\text{‘a+‘}^{10}\cdot\text{‘a−‘}^{2}+19\cdot 16^{-1}\cdot\epsilon^{4}\cdot\beta^{3}\cdot\text{‘a−‘}^{12}-133\cdot 32^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{3}\cdot\text{‘a+‘}^{7}\cdot\text{‘a−‘}^{4}+3\cdot$
$4^{-1}\cdot i\cdot\epsilon\cdot 2^{1\cdot 2^{-1}}\cdot\beta\cdot\text{‘a+‘}\cdot\text{‘a−‘}^{4}-675\cdot 256^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{4}\cdot\text{‘a+‘}^{13}+3\cdot 2^{-1}\cdot\epsilon^{2}\cdot\beta\cdot\text{‘a−‘}^{4}-$
$111\cdot 32^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{3}\cdot\text{‘a+‘}^{6}\cdot\text{‘a−‘}^{5}-99\cdot 16^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\text{‘a−‘}\cdot\text{‘a+‘}^{4}+9\cdot 32^{-1}\cdot\epsilon^{2}\cdot$
$\beta^{4}\cdot\text{‘a+‘}^{3}\cdot\text{‘a−‘}^{9}-17\cdot 32^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{2}\cdot\text{‘a−‘}^{8}\cdot\text{‘a+‘}-3\cdot 8^{-1}\cdot\epsilon^{2}\cdot\text{‘a−‘}^{4}\cdot\beta^{3}\cdot\text{‘a+‘}^{6}-3\cdot$
$16^{-1}\cdot\epsilon^{2}\cdot\beta^{4}\cdot\text{‘a+‘}^{5}\cdot\text{‘a−‘}^{7}+45\cdot 256^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{5}\cdot\text{‘a+‘}^{8}\cdot\text{‘a−‘}^{7}-65\cdot 256^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot$
$\beta^{5}\cdot\text{‘a+‘}^{6}\cdot\text{‘a−‘}^{9}-945\cdot 512^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{6}\cdot\text{‘a−‘}^{14}\cdot\text{‘a+‘}^{5}+1035\cdot 512^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{6}\cdot$
$\text{‘a+‘}^{13}\cdot\text{‘a−‘}^{6}+9\cdot 16^{-1}\cdot i\cdot\epsilon\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{2}\cdot\text{‘a+‘}^{6}\cdot\text{‘a−‘}+45\cdot 256^{-1}\cdot\epsilon^{4}\cdot\beta^{6}\cdot\text{‘a−‘}^{14}\cdot\text{‘a+‘}^{4}-23\cdot$
$4^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{2}\cdot\text{‘a−‘}^{7}\cdot\text{‘a+‘}^{2}+387\cdot 64^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{3}\cdot\text{‘a−‘}^{11}+3\cdot 8^{-1}\cdot\epsilon^{2}\cdot\text{‘a−‘}\cdot\text{‘a+‘}^{9}\cdot$
$\beta^{3}-1035\cdot 512^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{6}\cdot\text{‘a−‘}^{13}\cdot\text{‘a+‘}^{6}-3\cdot 1024^{-1}\cdot\epsilon^{4}\cdot\beta^{6}\cdot\text{‘a−‘}^{18}+15\cdot 64^{-1}\cdot\epsilon^{4}\cdot$
$\beta^{4}\cdot\text{‘a+‘}^{14}-3\cdot 1024^{-1}\cdot\epsilon^{4}\cdot\beta^{6}\cdot\text{‘a+‘}^{18}+48\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta\cdot\text{‘a+‘}^{2}\cdot\text{‘a−‘}-3\cdot 16^{-1}\cdot i\cdot\epsilon\cdot 2^{1\cdot 2^{-1}}\cdot$
$\beta^{2}\cdot\text{‘a+‘}^{5}\cdot\text{‘a−‘}^{2}-9\cdot 128^{-1}\cdot\epsilon^{4}\cdot\beta^{6}\cdot\text{‘a−‘}^{13}\cdot\text{‘a+‘}^{5}-165\cdot 256^{-1}\cdot\epsilon^{4}\cdot\beta^{6}\cdot\text{‘a−‘}^{9}\cdot\text{‘a+‘}^{9}+9\cdot$
$512^{-1}\cdot\epsilon^{4}\cdot\beta^{6}\cdot\text{‘a−‘}^{17}\cdot\text{‘a+‘}-33\cdot 64^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{3}\cdot\text{‘a−‘}^{11}+15\cdot 64^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{3}\cdot$
$\text{‘a+‘}^{11}+99\cdot 128^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{2}\cdot\text{‘a+‘}^{11}+9\cdot 128^{-1}\cdot\epsilon^{4}\cdot\beta^{5}\cdot\text{‘a+‘}^{15}\cdot\text{‘a−‘}+45\cdot 32^{-1}\cdot\epsilon^{4}\cdot$
$\beta^{4}\cdot\text{‘a−‘}^{13}\cdot\text{‘a+‘}-161\cdot 8^{-1}\cdot\epsilon^{4}\cdot\beta^{3}\cdot\text{‘a+‘}^{9}\cdot\text{‘a−‘}^{3}+233\cdot 4^{-1}\cdot\epsilon^{4}\cdot\beta\cdot\text{‘a+‘}^{2}\cdot\text{‘a−‘}^{4}+70\cdot\epsilon^{4}\cdot$
$\beta\cdot\text{‘a+‘}^{5}\cdot\text{‘a−‘}+38\cdot\epsilon^{4}\cdot\beta\cdot\text{‘a+‘}\cdot\text{‘a−‘}^{5}+16875\cdot 256^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{4}\cdot\text{‘a−‘}^{8}\cdot\text{‘a+‘}^{5}-45\cdot$
$128^{-1}\cdot\epsilon^{4}\cdot\beta^{5}\cdot\text{‘a−‘}^{15}\cdot\text{‘a+‘}+675\cdot 128^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{4}\cdot\text{‘a−‘}^{11}\cdot\text{‘a+‘}^{2}-99\cdot 128^{-1}\cdot i\cdot\epsilon^{5}\cdot$
$2^{1\cdot 2^{-1}}\cdot\beta^{2}\cdot\text{‘a−‘}^{11}+43\cdot 32^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{2}\cdot\text{‘a−‘}^{9}+99\cdot 512^{-1}\cdot\epsilon^{4}\cdot\beta^{6}\cdot\text{‘a−‘}^{8}\cdot\text{‘a+‘}^{10}+11\cdot$
$128^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{4}\cdot\text{‘a−‘}^{13}-108\cdot\epsilon^{4}\cdot\beta\cdot\text{‘a+‘}^{3}\cdot\text{‘a−‘}^{3}+9\cdot 2^{-1}\cdot\epsilon^{4}\cdot\beta^{3}\cdot\text{‘a+‘}^{10}\cdot\text{‘a−‘}^{2}+15\cdot$
$4^{-1}\cdot\epsilon^{4}\cdot\beta^{4}\cdot\text{‘a+‘}^{11}\cdot\text{‘a−‘}^{3}+15\cdot 32^{-1}\cdot\epsilon^{4}\cdot\beta^{4}\cdot\text{‘a+‘}^{9}\cdot\text{‘a−‘}^{5}+693\cdot 32^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\text{‘a+‘}^{5}\cdot$

$\text{‘a−‘}^{2}+5\cdot4^{-1}\cdot i\cdot\epsilon^{3}\cdot2^{1\cdot2^{-1}}\cdot\beta\cdot\text{‘a+‘}^{7}+3753\cdot32^{-1}\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\beta\cdot\text{‘a+‘}\cdot\text{‘a−‘}^{6}-315\cdot128^{-1}\cdot$
$\epsilon^{4}\cdot\beta^{5}\cdot\text{‘a−‘}^{12}\cdot\text{‘a+‘}^{4}-675\cdot64^{-1}\cdot\epsilon^{4}\cdot\beta^{4}\cdot\text{‘a+‘}^{10}\cdot\text{‘a−‘}^{4}-9\cdot128^{-1}\cdot\epsilon^{4}\cdot\beta^{5}\cdot\text{‘a−‘}^{11}\cdot\text{‘a+‘}^{5}-$
$197\cdot4^{-1}\cdot\epsilon^{4}\cdot\beta\cdot\text{‘a+‘}^{4}\cdot\text{‘a−‘}^{2}-4725\cdot128^{-1}\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\beta^{4}\cdot\text{‘a−‘}^{10}\cdot\text{‘a+‘}^{3}+9\cdot512^{-1}\cdot\epsilon^{4}\cdot\beta^{6}\cdot$
$\text{‘a+‘}^{17}\cdot\text{‘a−‘}-65\cdot256^{-1}\cdot i\cdot\epsilon^{3}\cdot2^{1\cdot2^{-1}}\cdot\beta^{5}\cdot\text{‘a+‘}^{9}\cdot\text{‘a−‘}^{6}+1\cdot256^{-1}\cdot i\cdot\epsilon^{3}\cdot2^{1\cdot2^{-1}}\cdot\beta^{5}\cdot\text{‘a+‘}^{10}\cdot$
$\text{‘a−‘}^{5}-231\cdot4^{-1}\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\beta\cdot\text{‘a+‘}^{3}\cdot\text{‘a−‘}^{6}+7407\cdot64^{-1}\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\beta^{2}\cdot\text{‘a+‘}^{6}\cdot\text{‘a−‘}^{5}-$
$1755\cdot1024^{-1}\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\beta^{6}\cdot\text{‘a−‘}^{11}\cdot\text{‘a+‘}^{8}-7407\cdot64^{-1}\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\beta^{2}\cdot\text{‘a+‘}^{5}\cdot\text{‘a−‘}^{6}+$
$3\cdot4^{-1}\cdot\epsilon^{2}\cdot\text{‘a−‘}\cdot\text{‘a+‘}^{7}\cdot\beta^{2}+120\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\beta\cdot\text{‘a+‘}\cdot\text{‘a−‘}^{4}+9\cdot128^{-1}\cdot\epsilon^{4}\cdot\beta^{5}\cdot\text{‘a−‘}^{16}-3\cdot$
$32^{-1}\cdot\epsilon^{2}\cdot\beta^{4}\cdot\text{‘a+‘}^{11}\cdot\text{‘a−‘}-87\cdot256^{-1}\cdot\epsilon^{4}\cdot\beta^{6}\cdot\text{‘a−‘}^{12}\cdot\text{‘a+‘}^{6}+315\cdot2048^{-1}\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\beta^{6}\cdot$
$\text{‘a−‘}^{18}\cdot\text{‘a+‘}+17931\cdot512^{-1}\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\beta^{4}\cdot\text{‘a+‘}^{9}\cdot\text{‘a−‘}^{6}-315\cdot128^{-1}\cdot\epsilon^{4}\cdot\beta^{5}\cdot\text{‘a+‘}^{11}\cdot$
$\text{‘a−‘}^{5}+465\cdot32^{-1}\cdot\epsilon^{4}\cdot\beta^{4}\cdot\text{‘a−‘}^{9}\cdot\text{‘a+‘}^{5}-45\cdot128^{-1}\cdot\epsilon^{4}\cdot\beta^{4}\cdot\text{‘a+‘}^{12}-81\cdot8^{-1}\cdot\epsilon^{4}\cdot\text{‘a+‘}^{8}\cdot\beta^{2}+$
$2103\cdot512^{-1}\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\beta^{4}\cdot\text{‘a+‘}^{11}\cdot\text{‘a−‘}^{4}+945\cdot512^{-1}\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\beta^{6}\cdot\text{‘a+‘}^{14}\cdot\text{‘a−‘}^{5}-$
$675\cdot128^{-1}\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\beta^{5}\cdot\text{‘a+‘}^{14}\cdot\text{‘a−‘}^{3}+125\cdot64^{-1}\cdot\epsilon^{4}\cdot\beta^{2}\cdot\text{‘a+‘}^{10}+1\cdot2^{-1}\cdot\epsilon^{4}\cdot\beta\cdot\text{‘a+‘}^{8}-$
$42\cdot\epsilon^{4}\cdot\text{‘a+‘}^{2}+165\cdot8^{-1}\cdot\epsilon^{4}\cdot\text{‘a+‘}^{4}-87\cdot8^{-1}\cdot i\cdot\epsilon^{3}\cdot2^{1\cdot2^{-1}}\cdot\beta\cdot\text{‘a+‘}\cdot\text{‘a−‘}^{4}+15\cdot4^{-1}\cdot\epsilon^{4}\cdot\text{‘a+‘}^{10}\cdot$
$\beta^{3}-12147\cdot512^{-1}\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\beta^{4}\cdot\text{‘a−‘}^{12}\cdot\text{‘a+‘}^{3}-1\cdot256^{-1}\cdot i\cdot\epsilon^{3}\cdot2^{1\cdot2^{-1}}\cdot\beta^{5}\cdot\text{‘a+‘}^{15}-45\cdot$
$128^{-1}\cdot\epsilon^{4}\cdot\beta^{4}\cdot\text{‘a−‘}^{12}+9\cdot4^{-1}\cdot i\cdot\epsilon^{3}\cdot2^{1\cdot2^{-1}}\cdot\text{‘a−‘}^{3}\cdot\text{‘a+‘}^{2}-1\cdot8^{-1}\cdot\beta^{2}\cdot\text{‘a+‘}^{2}\cdot\text{‘a−‘}^{4}-1\cdot4^{-1}\cdot$
$\beta^{2}\cdot\text{‘a+‘}^{5}\cdot\text{‘a−‘}-1\cdot4^{-1}\cdot\beta^{2}\cdot\text{‘a−‘}^{5}\cdot\text{‘a+‘}+3375\cdot256^{-1}\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\beta^{4}\cdot\text{‘a+‘}^{9}\cdot\text{‘a−‘}^{4}+$
$11\cdot4^{-1}\cdot\epsilon^{4}\cdot\beta\cdot\text{‘a+‘}^{6}+13\cdot2^{-1}\cdot\epsilon^{4}\cdot\beta\cdot\text{‘a+‘}^{4}-693\cdot32^{-1}\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\text{‘a−‘}^{5}\cdot\text{‘a+‘}^{2}+31\cdot8^{-1}\cdot$
$i\cdot\epsilon^{3}\cdot2^{1\cdot2^{-1}}\cdot\text{‘a+‘}\cdot\text{‘a−‘}^{4}-5\cdot256^{-1}\cdot i\cdot\epsilon^{3}\cdot2^{1\cdot2^{-1}}\cdot\beta^{5}\cdot\text{‘a−‘}^{13}\cdot\text{‘a+‘}^{2}-255\cdot64^{-1}\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot$
$\beta^{3}\cdot\text{‘a+‘}^{12}\cdot\text{‘a−‘}-4617\cdot16^{-1}\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\beta^{2}\cdot\text{‘a−‘}^{4}\cdot\text{‘a+‘}^{5}-4617\cdot16^{-1}\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\beta^{2}\cdot$
$\text{‘a−‘}^{5}\cdot\text{‘a+‘}^{4}+597\cdot32^{-1}\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\beta\cdot\text{‘a−‘}^{8}\cdot\text{‘a+‘}+5859\cdot32^{-1}\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\beta\cdot\text{‘a+‘}^{5}\cdot$
$\text{‘a−‘}^{2}-240\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\beta\cdot\text{‘a+‘}^{3}\cdot\text{‘a−‘}^{2}-3\cdot8^{-1}\cdot i\cdot\epsilon^{3}\cdot2^{1\cdot2^{-1}}\cdot\beta^{3}\cdot\text{‘a+‘}^{9}-120\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot$
$\text{‘a−‘}\cdot\text{‘a+‘}^{2}-24\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\beta\cdot\text{‘a+‘}^{7}\cdot\text{‘a−‘}^{2}+75\cdot4^{-1}\cdot\epsilon^{4}\cdot\text{‘a−‘}^{6}\cdot\beta^{3}\cdot\text{‘a+‘}^{4}-15\cdot8^{-1}\cdot$
$\epsilon^{4}\cdot\text{‘a−‘}^{9}\cdot\beta^{3}\cdot\text{‘a+‘}+2403\cdot8^{-1}\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\beta^{2}\cdot\text{‘a−‘}^{6}\cdot\text{‘a+‘}^{3}-4725\cdot128^{-1}\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot$
$\beta^{4}\cdot\text{‘a+‘}^{10}\cdot\text{‘a−‘}^{3}+809\cdot64^{-1}\cdot\epsilon^{4}\cdot\beta^{2}\cdot\text{‘a+‘}^{2}\cdot\text{‘a−‘}^{8}+1353\cdot512^{-1}\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\beta^{4}\cdot\text{‘a−‘}^{14}\cdot$
$\text{‘a+‘}-15\cdot8^{-1}\cdot\epsilon^{2}\cdot\text{‘a+‘}^{4}+6435\cdot1024^{-1}\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\beta^{6}\cdot\text{‘a−‘}^{9}\cdot\text{‘a+‘}^{10}+81\cdot32^{-1}\cdot\epsilon^{4}\cdot$
$\beta^{2}\cdot\text{‘a+‘}^{9}\cdot\text{‘a−‘}-27\cdot1024^{-1}\cdot\epsilon^{4}\cdot\beta^{6}\cdot\text{‘a+‘}^{16}\cdot\text{‘a−‘}^{2}+4239\cdot64^{-1}\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\beta^{3}\cdot\text{‘a+‘}^{8}\cdot$
$\text{‘a−‘}^{5}+2025\cdot256^{-1}\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\beta^{4}\cdot\text{‘a−‘}^{12}\cdot\text{‘a+‘}-405\cdot128^{-1}\cdot\epsilon^{4}\cdot\beta^{5}\cdot\text{‘a+‘}^{7}\cdot\text{‘a−‘}^{9}+9\cdot$
$4^{-1}\cdot\epsilon^{2}\cdot\text{‘a−‘}^{5}\cdot\beta^{3}\cdot\text{‘a+‘}^{5}+9\cdot16^{-1}\cdot\epsilon^{2}\cdot\text{‘a−‘}^{2}\cdot\text{‘a+‘}^{8}\cdot\beta^{3}-1251\cdot8^{-1}\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\text{‘a−‘}^{2}\cdot$
$\text{‘a+‘}^{3}+3\cdot2^{-1}\cdot i\cdot\epsilon\cdot2^{1\cdot2^{-1}}\cdot\beta\cdot\text{‘a+‘}^{2}\cdot\text{‘a−‘}^{3}-3\cdot4^{-1}\cdot i\cdot\epsilon^{3}\cdot2^{1\cdot2^{-1}}\cdot\beta\cdot\text{‘a+‘}^{2}\cdot\text{‘a−‘}-5\cdot256^{-1}\cdot i\cdot$
$\epsilon^{3}\cdot2^{1\cdot2^{-1}}\cdot\beta^{5}\cdot\text{‘a+‘}^{13}\cdot\text{‘a−‘}^{2}-517\cdot16^{-1}\cdot\epsilon^{4}\cdot\beta^{2}\cdot\text{‘a+‘}^{5}\cdot\text{‘a−‘}^{5}-90\cdot\epsilon^{4}\cdot\text{‘a−‘}^{4}\cdot\beta^{2}\cdot\text{‘a+‘}^{4}+$
$3375\cdot128^{-1}\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\beta^{5}\cdot\text{‘a+‘}^{6}\cdot\text{‘a−‘}^{11}-1485\cdot128^{-1}\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\beta^{5}\cdot\text{‘a+‘}^{7}\cdot\text{‘a−‘}^{10}-$
$3\cdot2^{-1}\cdot\epsilon^{2}\cdot\beta\cdot\text{‘a+‘}^{4}+3\cdot128^{-1}\cdot\epsilon^{2}\cdot\beta^{4}\cdot\text{‘a−‘}^{12}+235\cdot8^{-1}\cdot\epsilon^{4}\cdot\beta^{2}\cdot\text{‘a+‘}^{3}\cdot\text{‘a−‘}^{7}+57\cdot4^{-1}\cdot i\cdot\epsilon^{3}\cdot$
$2^{1\cdot2^{-1}}\cdot\beta\cdot\text{‘a+‘}^{3}\cdot\text{‘a−‘}^{2}-135\cdot128^{-1}\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\beta^{5}\cdot\text{‘a+‘}^{15}\cdot\text{‘a−‘}^{2}-1053\cdot32^{-1}\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot$
$\beta\cdot\text{‘a−‘}^{7}-1495\cdot64^{-1}\cdot\epsilon^{4}\cdot\beta^{2}\cdot\text{‘a+‘}^{8}\cdot\text{‘a−‘}^{2}+35\cdot256^{-1}\cdot i\cdot\epsilon^{3}\cdot2^{1\cdot2^{-1}}\cdot\beta^{5}\cdot\text{‘a−‘}^{11}\cdot\text{‘a+‘}^{4}+$
$2799\cdot128^{-1}\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\beta^{2}\cdot\text{‘a+‘}^{2}\cdot\text{‘a−‘}^{9}+3\cdot2^{-1}\cdot i\cdot\epsilon\cdot2^{1\cdot2^{-1}}\cdot\text{‘a−‘}-9\cdot2^{-1}\cdot i\cdot\epsilon^{3}\cdot2^{1\cdot2^{-1}}\cdot$
$\text{‘a−‘}+3\cdot8^{-1}\cdot i\cdot\epsilon^{3}\cdot2^{1\cdot2^{-1}}\cdot\text{‘a+‘}^{5}+1\cdot8^{-1}\cdot i\cdot\epsilon^{3}\cdot2^{1\cdot2^{-1}}\cdot\text{‘a−‘}^{5}-1\cdot2^{-1}\cdot i\cdot\epsilon^{3}\cdot2^{1\cdot2^{-1}}\cdot\text{‘a−‘}^{3}-i\cdot$
$\epsilon^{3}\cdot2^{1\cdot2^{-1}}\cdot\text{‘a+‘}^{3}+3\cdot2^{-1}\cdot i\cdot\epsilon^{3}\cdot2^{1\cdot2^{-1}}\cdot\text{‘a+‘}+441\cdot16^{-1}\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\text{‘a+‘}^{5}+9711\cdot128^{-1}\cdot$
$i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\beta^{2}\cdot\text{‘a+‘}^{3}\cdot\text{‘a−‘}^{8}-120\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\beta\cdot\text{‘a+‘}^{4}\cdot\text{‘a−‘}-3\cdot2^{-1}\cdot\epsilon^{2}\cdot\text{‘a+‘}\cdot\text{‘a−‘}^{3}-$
$9\cdot\epsilon^{2}\cdot\text{‘a+‘}\cdot\text{‘a−‘}+3\cdot4^{-1}\cdot\epsilon^{2}\cdot\beta\cdot\text{‘a+‘}^{6}-13\cdot4^{-1}\cdot\epsilon^{4}\cdot\beta^{2}\cdot\text{‘a+‘}^{4}\cdot\text{‘a−‘}^{2}-3\cdot16^{-1}\cdot i\cdot\epsilon\cdot2^{1\cdot2^{-1}}\cdot$
$\beta^{2}\cdot\text{‘a+‘}^{7}-1053\cdot32^{-1}\cdot i\cdot\epsilon^{5}\cdot2^{1\cdot2^{-1}}\cdot\beta\cdot\text{‘a+‘}^{7}-87\cdot256^{-1}\cdot\epsilon^{4}\cdot\beta^{6}\cdot\text{‘a+‘}^{12}\cdot\text{‘a−‘}^{6}-15\cdot2^{-1}\cdot$

$\epsilon^4\cdot\beta^4\cdot\text{‘a−‘}^{11}\cdot\text{‘a+‘}^3+27\cdot8^{-1}\cdot\epsilon^4\cdot\beta^3\cdot\text{‘a+‘}^{11}\cdot\text{‘a−‘}-3\cdot4^{-1}\cdot i\cdot\epsilon\cdot2^{1\cdot2^{-1}}\cdot\text{‘a−‘}\cdot\text{‘a+‘}^2-29\cdot 8^{-1}\cdot\epsilon^4\cdot\text{‘a−‘}^4\cdot\text{‘a+‘}^2-23\cdot2^{-1}\cdot\epsilon^4\cdot\text{‘a−‘}^3\cdot\text{‘a+‘}^3-51\cdot128^{-1}\cdot\epsilon^2\cdot\beta^4\cdot\text{‘a+‘}^4\cdot\text{‘a−‘}^8-315\cdot 2048^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^6\cdot\text{‘a−‘}^{16}\cdot\text{‘a+‘}^3+13\cdot4^{-1}\cdot\epsilon^4\cdot\beta^2\cdot\text{‘a−‘}^6-9\cdot8^{-1}\cdot\epsilon^4\cdot\beta\cdot\text{‘a−‘}^8-39\cdot4^{-1}\cdot \epsilon^4\cdot\text{‘a−‘}^5\cdot\text{‘a+‘}+45\cdot16^{-1}\cdot\epsilon^4\cdot\beta^4\cdot\text{‘a−‘}^{10}\cdot\text{‘a+‘}^4+3\cdot64^{-1}\cdot\epsilon^2\cdot\beta^4\cdot\text{‘a+‘}^2\cdot\text{‘a−‘}^{10}-45\cdot4^{-1}\cdot \epsilon^4\cdot\text{‘a−‘}^8\cdot\beta^3\cdot\text{‘a+‘}^2+15\cdot\epsilon^4\cdot\text{‘a−‘}^7\cdot\beta^3\cdot\text{‘a+‘}^3+2403\cdot8^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^2\cdot\text{‘a+‘}^6\cdot\text{‘a−‘}^3+ 6543\cdot32^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^3\cdot\text{‘a−‘}^7\cdot\text{‘a+‘}^4+225\cdot8^{-1}\cdot\epsilon^4\cdot\beta\cdot\text{‘a+‘}^6\cdot\text{‘a−‘}^2+163\cdot8^{-1}\cdot\epsilon^4\cdot\beta\cdot \text{‘a+‘}^5\cdot\text{‘a−‘}^3-15\cdot256^{-1}\cdot i\cdot\epsilon^3\cdot2^{1\cdot2^{-1}}\cdot\beta^5\cdot\text{‘a+‘}^{12}\cdot\text{‘a−‘}^3-111\cdot16^{-1}\cdot i\cdot\epsilon^3\cdot2^{1\cdot2^{-1}}\cdot\beta^2\cdot\text{‘a−‘}^5\cdot \text{‘a+‘}^2+3\cdot2^{-1}\cdot\epsilon^2\cdot\text{‘a−‘}^2+15\cdot2^{-1}\cdot\epsilon^2\cdot\text{‘a+‘}^2+45\cdot256^{-1}\cdot\epsilon^4\cdot\beta^6\cdot\text{‘a+‘}^{14}\cdot\text{‘a−‘}^4+387\cdot64^{-1}\cdot i\cdot \epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^3\cdot\text{‘a+‘}^{11}-27\cdot8^{-1}\cdot i\cdot\epsilon^3\cdot2^{1\cdot2^{-1}}\cdot\beta\cdot\text{‘a+‘}^5+15\cdot8^{-1}\cdot\epsilon^4\cdot\beta^4\cdot\text{‘a+‘}^{12}\cdot\text{‘a−‘}^2-23\cdot8^{-1}\cdot \epsilon^4\cdot\text{‘a+‘}^6+19\cdot8^{-1}\cdot\epsilon^4\cdot\text{‘a−‘}^6+36\cdot\epsilon^4\cdot\text{‘a−‘}^2-855\cdot512^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^6\cdot\text{‘a+‘}^{15}\cdot\text{‘a−‘}^4+ 5859\cdot32^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta\cdot\text{‘a+‘}^2\cdot\text{‘a−‘}^5+61\cdot4^{-1}\cdot\epsilon^4\cdot\text{‘a+‘}^5\cdot\text{‘a−‘}-39879\cdot512^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot \beta^4\cdot\text{‘a+‘}^7\cdot\text{‘a−‘}^8+33717\cdot512^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^4\cdot\text{‘a−‘}^{10}\cdot\text{‘a+‘}^5-135\cdot32^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^5\cdot \text{‘a−‘}^{12}\cdot\text{‘a+‘}^5+12147\cdot512^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^4\cdot\text{‘a+‘}^{12}\cdot\text{‘a−‘}^3-17931\cdot512^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot \beta^4\cdot\text{‘a+‘}^6\cdot\text{‘a−‘}^9+33\cdot16^{-1}\cdot i\cdot\epsilon^3\cdot2^{1\cdot2^{-1}}\cdot\beta^2\cdot\text{‘a−‘}^7+39879\cdot512^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^4\cdot\text{‘a+‘}^8\cdot \text{‘a−‘}^7+13\cdot4^{-1}\cdot\epsilon^4\cdot\beta^2\cdot\text{‘a+‘}^6-65\cdot2^{-1}\cdot\epsilon^4\cdot\beta\cdot\text{‘a−‘}^4+135\cdot128^{-1}\cdot\epsilon^4\cdot\beta^5\cdot\text{‘a+‘}^{12}\cdot\text{‘a−‘}^4- 35\cdot8^{-1}\cdot i\cdot\epsilon^3\cdot2^{1\cdot2^{-1}}\cdot\text{‘a−‘}\cdot\text{‘a+‘}^4-33717\cdot512^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^4\cdot\text{‘a+‘}^{10}\cdot\text{‘a−‘}^5-67\cdot64^{-1}\cdot \epsilon^4\cdot\beta^2\cdot\text{‘a−‘}^{10}+121\cdot8^{-1}\cdot\epsilon^4\cdot\beta\cdot\text{‘a−‘}^7\cdot\text{‘a+‘}-315\cdot2048^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^6\cdot\text{‘a+‘}^{18}\cdot\text{‘a−‘}- 129\cdot64^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^3\cdot\text{‘a+‘}^{13}+2025\cdot512^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^5\cdot\text{‘a+‘}^{12}\cdot\text{‘a−‘}^3-9\cdot16^{-1}\cdot i\cdot \epsilon\cdot2^{1\cdot2^{-1}}\cdot\beta^2\cdot\text{‘a−‘}^6\cdot\text{‘a+‘}-3\cdot4^{-1}\cdot\epsilon^2\cdot\text{‘a−‘}^5\cdot\beta^2\cdot\text{‘a+‘}^3+675\cdot512^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^5\cdot\text{‘a−‘}^{13}\cdot \text{‘a+‘}^2-149\cdot8^{-1}\cdot\epsilon^4\cdot\beta\cdot\text{‘a+‘}^7\cdot\text{‘a−‘}-1353\cdot512^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^4\cdot\text{‘a+‘}^{14}\cdot\text{‘a−‘}+15\cdot16^{-1}\cdot i\cdot\epsilon\cdot2^{1\cdot2^{-1}}\cdot\beta^2\cdot\text{‘a−‘}^4\cdot\text{‘a+‘}^3+765\cdot64^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^3\cdot\text{‘a+‘}^{10}\cdot\text{‘a−‘}-3\cdot2^{-1}\cdot i\cdot\epsilon\cdot2^{1\cdot2^{-1}}\cdot\beta\cdot \text{‘a+‘}^3\cdot\text{‘a−‘}^2-6075\cdot512^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^5\cdot\text{‘a+‘}^7\cdot\text{‘a−‘}^8+3\cdot8^{-1}\cdot\epsilon^2\cdot\text{‘a−‘}^9\cdot\beta^3\cdot\text{‘a+‘}+9\cdot 16^{-1}\cdot\epsilon^2\cdot\text{‘a−‘}^8\cdot\beta^3\cdot\text{‘a+‘}^2-3\cdot2^{-1}\cdot\epsilon^2\cdot\text{‘a−‘}^7\cdot\beta^3\cdot\text{‘a+‘}^3-15\cdot16^{-1}\cdot i\cdot\epsilon\cdot2^{1\cdot2^{-1}}\cdot\beta^2\cdot\text{‘a+‘}^4\cdot \text{‘a−‘}^3-3\cdot4^{-1}\cdot i\cdot\epsilon\cdot2^{1\cdot2^{-1}}\cdot\beta\cdot\text{‘a+‘}^4\cdot\text{‘a−‘}+8775\cdot512^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^5\cdot\text{‘a+‘}^6\cdot\text{‘a−‘}^9- 3375\cdot128^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^5\cdot\text{‘a+‘}^{11}\cdot\text{‘a−‘}^6-3\cdot\epsilon^2\cdot\text{‘a−‘}^6\cdot\beta^2\cdot\text{‘a+‘}^2-3\cdot8^{-1}\cdot\epsilon^2\cdot\text{‘a−‘}^6\cdot\beta^3\cdot \text{‘a+‘}^4+135\cdot512^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^5\cdot\text{‘a−‘}^{17}+129\cdot64^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^3\cdot\text{‘a−‘}^{13}-3\cdot32^{-1}\cdot \epsilon^2\cdot\beta^4\cdot\text{‘a+‘}\cdot\text{‘a−‘}^{11}+2025\cdot512^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^5\cdot\text{‘a−‘}^{12}\cdot\text{‘a+‘}^3+675\cdot512^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot \beta^5\cdot\text{‘a+‘}^{13}\cdot\text{‘a−‘}^2-675\cdot512^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^5\cdot\text{‘a+‘}^{14}\cdot\text{‘a−‘}+16875\cdot256^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot \beta^4\cdot\text{‘a+‘}^8\cdot\text{‘a−‘}^5-5\cdot8^{-1}\cdot\epsilon^4\cdot\beta\cdot\text{‘a+‘}^2\cdot\text{‘a−‘}^6+5103\cdot64^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^2\cdot\text{‘a+‘}^7\cdot\text{‘a−‘}^4+ 3\cdot16^{-1}\cdot i\cdot\epsilon\cdot2^{1\cdot2^{-1}}\cdot\beta^2\cdot\text{‘a−‘}^5\cdot\text{‘a+‘}^2-9\cdot128^{-1}\cdot\epsilon^4\cdot\beta^5\cdot\text{‘a+‘}^{10}\cdot\text{‘a−‘}^6-9711\cdot128^{-1}\cdot i\cdot\epsilon^5\cdot 2^{1\cdot2^{-1}}\cdot\beta^2\cdot\text{‘a+‘}^8\cdot\text{‘a−‘}^3-129\cdot32^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\text{‘a−‘}^6\cdot\text{‘a+‘}+3\cdot\epsilon^2\cdot\beta\cdot\text{‘a+‘}^3\cdot\text{‘a−‘}-2799\cdot 128^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^2\cdot\text{‘a+‘}^9\cdot\text{‘a−‘}^2+585\cdot128^{-1}\cdot\epsilon^4\cdot\beta^5\cdot\text{‘a+‘}^9\cdot\text{‘a−‘}^7+21\cdot4^{-1}\cdot\epsilon^2\cdot\text{‘a−‘}^4\cdot \beta^2\cdot\text{‘a+‘}^4+129\cdot32^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\text{‘a+‘}^6\cdot\text{‘a−‘}+585\cdot128^{-1}\cdot\epsilon^4\cdot\beta^5\cdot\text{‘a+‘}^6\cdot\text{‘a−‘}^{10}-405\cdot 128^{-1}\cdot\epsilon^4\cdot\beta^5\cdot\text{‘a+‘}^8\cdot\text{‘a−‘}^8+3\cdot4^{-1}\cdot\epsilon^2\cdot\text{‘a−‘}^7\cdot\beta^2\cdot\text{‘a+‘}-4239\cdot64^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^3\cdot\text{‘a−‘}^8\cdot \text{‘a+‘}^5-45\cdot32^{-1}\cdot\epsilon^4\cdot\beta^4\cdot\text{‘a+‘}^{13}\cdot\text{‘a−‘}+7425\cdot256^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^5\cdot\text{‘a+‘}^9\cdot\text{‘a−‘}^8-579\cdot 512^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\beta^4\cdot\text{‘a−‘}^{15}+3\cdot16^{-1}\cdot i\cdot\epsilon\cdot2^{1\cdot2^{-1}}\cdot\beta^2\cdot\text{‘a−‘}^7-3\cdot4^{-1}\cdot i\cdot\epsilon\cdot2^{1\cdot2^{-1}}\cdot\beta\cdot\text{‘a−‘}^3-3\cdot 4^{-1}\cdot i\cdot\epsilon\cdot2^{1\cdot2^{-1}}\cdot\beta\cdot\text{‘a+‘}^3+3\cdot4^{-1}\cdot i\cdot\epsilon\cdot2^{1\cdot2^{-1}}\cdot\beta\cdot\text{‘a+‘}^5+33\cdot2^{-1}\cdot i\cdot\epsilon^3\cdot2^{1\cdot2^{-1}}\cdot\text{‘a−‘}\cdot\text{‘a+‘}^2+9\cdot i\cdot \epsilon^3\cdot2^{1\cdot2^{-1}}\cdot\text{‘a+‘}\cdot\text{‘a−‘}^2+39\cdot16^{-1}\cdot i\cdot\epsilon^3\cdot2^{1\cdot2^{-1}}\cdot\beta^2\cdot\text{‘a+‘}^7-15\cdot8^{-1}\cdot i\cdot\epsilon^3\cdot2^{1\cdot2^{-1}}\cdot\beta\cdot\text{‘a−‘}^5+96\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\text{‘a+‘}+96\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\text{‘a−‘}-51\cdot32^{-1}\cdot i\cdot\epsilon^5\cdot2^{1\cdot2^{-1}}\cdot\text{‘a+‘}^7-13\cdot2^{-1}\cdot i\cdot\epsilon^3\cdot2^{1\cdot2^{-1}}\cdot\beta\cdot \text{‘a+‘}^5\cdot\text{‘a−‘}^2-7\cdot4^{-1}\cdot i\cdot\epsilon^3\cdot2^{1\cdot2^{-1}}\cdot\beta\cdot\text{‘a+‘}^4\cdot\text{‘a−‘}^3+29\cdot4^{-1}\cdot i\cdot\epsilon^3\cdot2^{1\cdot2^{-1}}\cdot\beta\cdot\text{‘a+‘}^3\cdot\text{‘a−‘}^4+5\cdot$

$2^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta\cdot\text{‘a+‘}^{2}\cdot\text{‘a-‘}^{5}-2\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta\cdot\text{‘a+‘}\cdot\text{‘a-‘}^{6}-71\cdot 128^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{4}\cdot$
$\text{‘a-‘}^{8}\cdot\text{‘a+‘}^{5}-199\cdot 128^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{4}\cdot\text{‘a-‘}^{9}\cdot\text{‘a+‘}^{4}+65\cdot 64^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{4}\cdot\text{‘a-‘}^{10}\cdot$
$\text{‘a+‘}^{3}+13\cdot 64^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{4}\cdot\text{‘a-‘}^{11}\cdot\text{‘a+‘}^{2}-45\cdot 128^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{4}\cdot\text{‘a-‘}^{12}\cdot\text{‘a+‘}-$
$45\cdot 2^{-1}\cdot\epsilon^{4}\cdot\beta^{3}\cdot\text{‘a-‘}^{6}\cdot\text{‘a+‘}^{6}-129\cdot 4^{-1}\cdot\epsilon^{4}\cdot\beta^{3}\cdot\text{‘a-‘}^{7}\cdot\text{‘a+‘}^{5}+447\cdot 16^{-1}\cdot\epsilon^{4}\cdot\beta^{3}\cdot\text{‘a-‘}^{8}\cdot$
$\text{‘a+‘}^{4}+71\cdot 8^{-1}\cdot\epsilon^{4}\cdot\beta^{3}\cdot\text{‘a-‘}^{9}\cdot\text{‘a+‘}^{3}+3\cdot 8^{-1}\cdot\epsilon^{4}\cdot\beta^{3}\cdot\text{‘a-‘}^{11}\cdot\text{‘a+‘}-12\cdot\epsilon^{4}\cdot\beta^{3}\cdot\text{‘a-‘}^{10}\cdot$
$\text{‘a+‘}^{2}+585\cdot 32^{-1}\cdot\epsilon^{4}\cdot\beta^{4}\cdot\text{‘a+‘}^{8}\cdot\text{‘a-‘}^{6}-855\cdot 64^{-1}\cdot\epsilon^{4}\cdot\beta^{4}\cdot\text{‘a+‘}^{6}\cdot\text{‘a-‘}^{8}-45\cdot 4^{-1}\cdot\epsilon^{4}\cdot\beta^{4}\cdot$
$\text{‘a+‘}^{7}\cdot\text{‘a-‘}^{7}-675\cdot 2048^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{6}\cdot\text{‘a-‘}^{17}\cdot\text{‘a+‘}^{2}+3\cdot 4^{-1}\cdot i\cdot\epsilon\cdot 2^{1\cdot 2^{-1}}\cdot\text{‘a-‘}^{3}+$
$3\cdot 2^{-1}\cdot i\cdot\epsilon\cdot 2^{1\cdot 2^{-1}}\cdot\text{‘a+‘}-3657\cdot 512^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{4}\cdot\text{‘a+‘}^{13}\cdot\text{‘a-‘}^{2}+675\cdot 128^{-1}\cdot i\cdot\epsilon^{5}\cdot$
$2^{1\cdot 2^{-1}}\cdot\beta^{4}\cdot\text{‘a+‘}^{11}\cdot\text{‘a-‘}^{2}-105\cdot 16^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{2}\cdot\text{‘a+‘}^{5}\cdot\text{‘a-‘}^{2}+159\cdot 4^{-1}\cdot\epsilon^{4}\cdot\beta^{3}\cdot\text{‘a+‘}^{7}\cdot$
$\text{‘a-‘}^{5}+1755\cdot 1024^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{6}\cdot\text{‘a-‘}^{8}\cdot\text{‘a+‘}^{11}-255\cdot 32^{-1}\cdot\epsilon^{4}\cdot\beta^{2}\cdot\text{‘a-‘}^{9}\cdot\text{‘a+‘}+120\cdot$
$i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta\cdot\text{‘a+‘}^{5}+30\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{2}\cdot\text{‘a-‘}^{7}+45\cdot 16^{-1}\cdot\epsilon^{4}\cdot\beta^{4}\cdot\text{‘a+‘}^{5}\cdot\text{‘a-‘}^{7}-315\cdot 32^{-1}\cdot$
$\epsilon^{4}\cdot\beta^{4}\cdot\text{‘a+‘}^{6}\cdot\text{‘a-‘}^{6}+1935\cdot 64^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{3}\cdot\text{‘a-‘}^{8}\cdot\text{‘a+‘}^{3}+1645\cdot 32^{-1}\cdot\epsilon^{4}\cdot\beta^{2}\cdot\text{‘a+‘}^{6}\cdot$
$\text{‘a-‘}^{4}+2403\cdot 128^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{2}\cdot\text{‘a+‘}^{10}\cdot\text{‘a-‘}+120\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\text{‘a-‘}^{3}-179\cdot 128^{-1}\cdot$
$i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{4}\cdot\text{‘a+‘}^{8}\cdot\text{‘a-‘}^{5}-7425\cdot 256^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{5}\cdot\text{‘a+‘}^{8}\cdot\text{‘a-‘}^{9}+195\cdot 32^{-1}\cdot i\cdot\epsilon^{5}\cdot$
$2^{1\cdot 2^{-1}}\cdot\beta^{3}\cdot\text{‘a-‘}^{10}\cdot\text{‘a+‘}^{3}+2025\cdot 256^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{4}\cdot\text{‘a+‘}^{12}\cdot\text{‘a-‘}-963\cdot 32^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot$
$\beta^{3}\cdot\text{‘a-‘}^{11}\cdot\text{‘a+‘}^{2}-135\cdot 512^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{5}\cdot\text{‘a-‘}^{10}\cdot\text{‘a+‘}^{5}-4725\cdot 512^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{5}\cdot$
$\text{‘a+‘}^{11}\cdot\text{‘a-‘}^{4}-51\cdot 128^{-1}\cdot\epsilon^{2}\cdot\beta^{4}\cdot\text{‘a+‘}^{8}\cdot\text{‘a-‘}^{4}+225\cdot 32^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{3}\cdot\text{‘a-‘}^{6}\cdot\text{‘a+‘}^{5}+$
$5\cdot 256^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{5}\cdot\text{‘a+‘}^{14}\cdot\text{‘a-‘}-3\cdot 4^{-1}\cdot i\cdot\epsilon\cdot 2^{1\cdot 2^{-1}}\cdot\text{‘a+‘}^{3}+55\cdot 32^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{2}\cdot$
$\text{‘a+‘}^{8}\cdot\text{‘a-‘}-57\cdot 32^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta\cdot\text{‘a-‘}^{9}+5\cdot 64^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{4}\cdot\text{‘a+‘}^{10}\cdot\text{‘a-‘}^{3}-309\cdot$
$64^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{3}\cdot\text{‘a-‘}^{8}\cdot\text{‘a+‘}^{3}+81\cdot 64^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{3}\cdot\text{‘a-‘}^{9}\cdot\text{‘a+‘}^{2}-21\cdot 4^{-1}\cdot\epsilon^{4}\cdot$
$\text{‘a-‘}^{2}\cdot\text{‘a+‘}^{2}-63\cdot 2^{-1}\cdot\epsilon^{4}\cdot\text{‘a+‘}\cdot\text{‘a-‘}^{3}+58\cdot\epsilon^{4}\cdot\text{‘a+‘}\cdot\text{‘a-‘}-675\cdot 512^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{5}\cdot$
$\text{‘a-‘}^{14}\cdot\text{‘a+‘}-3\cdot\epsilon^{2}\cdot\beta\cdot\text{‘a+‘}\cdot\text{‘a-‘}^{3}+1485\cdot 128^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{5}\cdot\text{‘a+‘}^{10}\cdot\text{‘a-‘}^{7}+15\cdot 4^{-1}\cdot$
$\epsilon^{2}\cdot\beta\cdot\text{‘a+‘}^{4}\cdot\text{‘a-‘}^{2}-6\cdot\epsilon^{2}\cdot\beta\cdot\text{‘a+‘}^{5}\cdot\text{‘a-‘}-1989\cdot 16^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{3}\cdot\text{‘a-‘}^{7}\cdot\text{‘a+‘}^{6}+231\cdot$
$4^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta\cdot\text{‘a+‘}^{6}\cdot\text{‘a-‘}^{3}+255\cdot 64^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{3}\cdot\text{‘a-‘}^{12}\cdot\text{‘a+‘}+765\cdot 64^{-1}\cdot i\cdot\epsilon^{5}\cdot$
$2^{1\cdot 2^{-1}}\cdot\beta^{3}\cdot\text{‘a-‘}^{10}\cdot\text{‘a+‘}+33\cdot 16^{-1}\cdot\epsilon^{4}\cdot\beta^{3}\cdot\text{‘a+‘}^{8}\cdot\text{‘a-‘}^{4}-195\cdot 32^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{3}\cdot\text{‘a+‘}^{10}\cdot$
$\text{‘a-‘}^{3}-1331\cdot 32^{-1}\cdot\epsilon^{4}\cdot\beta^{2}\cdot\text{‘a+‘}^{4}\cdot\text{‘a-‘}^{6}-8559\cdot 32^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta\cdot\text{‘a+‘}^{3}\cdot\text{‘a-‘}^{4}+1935\cdot$
$64^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{3}\cdot\text{‘a+‘}^{8}\cdot\text{‘a-‘}^{3}+1989\cdot 16^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{3}\cdot\text{‘a-‘}^{6}\cdot\text{‘a+‘}^{7}+6159\cdot 64^{-1}\cdot i\cdot$
$\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{3}\cdot\text{‘a-‘}^{9}\cdot\text{‘a+‘}^{4}+1\cdot 256^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{5}\cdot\text{‘a-‘}^{10}\cdot\text{‘a+‘}^{5}-981\cdot 16^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot$
$\beta\cdot\text{‘a+‘}^{4}\cdot\text{‘a-‘}^{5}-48\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta\cdot\text{‘a-‘}^{3}-90\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{2}\cdot\text{‘a-‘}^{6}\cdot\text{‘a+‘}+67\cdot 8^{-1}\cdot\epsilon^{4}\cdot\beta^{2}\cdot$
$\text{‘a+‘}^{7}\cdot\text{‘a-‘}^{3}+123\cdot 16^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{2}\cdot\text{‘a+‘}^{4}\cdot\text{‘a-‘}^{3}-47\cdot 4^{-1}\cdot\epsilon^{4}\cdot\beta\cdot\text{‘a-‘}^{6}-187\cdot 2^{-1}\cdot\epsilon^{4}\cdot$
$\text{‘a-‘}\cdot\text{‘a+‘}^{3}+3\cdot 128^{-1}\cdot\epsilon^{2}\cdot\beta^{4}\cdot\text{‘a+‘}^{12}+3\cdot 8^{-1}\cdot\epsilon^{2}\cdot\text{‘a+‘}^{8}\cdot\beta^{2}+3657\cdot 512^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{4}\cdot$
$\text{‘a-‘}^{13}\cdot\text{‘a+‘}^{2}-15\cdot 8^{-1}\cdot\epsilon^{4}\cdot\text{‘a-‘}^{2}\cdot\text{‘a+‘}^{4}+45\cdot 32^{-1}\cdot\epsilon^{4}\cdot\beta^{4}\cdot\text{‘a+‘}^{11}\cdot\text{‘a-‘}+963\cdot 32^{-1}\cdot i\cdot\epsilon^{5}\cdot$
$2^{1\cdot 2^{-1}}\cdot\beta^{3}\cdot\text{‘a+‘}^{11}\cdot\text{‘a-‘}^{2}-30\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{2}\cdot\text{‘a+‘}^{5}\cdot\text{‘a-‘}^{2}-75\cdot 8^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta\cdot\text{‘a+‘}^{4}\cdot$
$\text{‘a-‘}-3375\cdot 64^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{4}\cdot\text{‘a+‘}^{7}\cdot\text{‘a-‘}^{6}+26\cdot\epsilon^{4}\cdot\beta\cdot\text{‘a+‘}^{2}\cdot\text{‘a-‘}^{2}+45\cdot 32^{-1}\cdot\epsilon^{4}\cdot$
$\beta^{4}\cdot\text{‘a+‘}\cdot\text{‘a-‘}^{11}-13\cdot 2^{-1}\cdot\epsilon^{4}\cdot\beta^{2}\cdot\text{‘a+‘}^{5}\cdot\text{‘a-‘}-13\cdot 2^{-1}\cdot\epsilon^{4}\cdot\beta^{2}\cdot\text{‘a-‘}^{5}\cdot\text{‘a+‘}+120\cdot i\cdot\epsilon^{5}\cdot$
$2^{1\cdot 2^{-1}}\cdot\text{‘a+‘}\cdot\text{‘a-‘}^{2}+219\cdot 64^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{3}\cdot\text{‘a+‘}^{8}\cdot\text{‘a-‘}^{3}-99\cdot 16^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\text{‘a+‘}\cdot$
$\text{‘a-‘}^{4}+45\cdot 256^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{5}\cdot\text{‘a+‘}^{7}\cdot\text{‘a-‘}^{8}+3\cdot 4^{-1}\cdot i\cdot\epsilon\cdot 2^{1\cdot 2^{-1}}\cdot\beta\cdot\text{‘a+‘}^{2}\cdot\text{‘a-‘}-1251\cdot$
$8^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\text{‘a-‘}^{3}\cdot\text{‘a+‘}^{2}-6435\cdot 1024^{-1}\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{6}\cdot\text{‘a-‘}^{10}\cdot\text{‘a+‘}^{9}+855\cdot 512^{-1}\cdot i\cdot$
$\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{6}\cdot\text{‘a-‘}^{15}\cdot\text{‘a+‘}^{4}-3\cdot 4^{-1}\cdot\epsilon^{2}\cdot\text{‘a+‘}^{5}\cdot\text{‘a-‘}^{3}\cdot\beta^{2}+3\cdot 4^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta\cdot\text{‘a+‘}^{3}-30\cdot$
$i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{2}\cdot\text{‘a+‘}^{7}-120\cdot i\cdot\epsilon^{5}\cdot 2^{1\cdot 2^{-1}}\cdot\beta\cdot\text{‘a-‘}^{5}-29\cdot 32^{-1}\cdot i\cdot\epsilon^{3}\cdot 2^{1\cdot 2^{-1}}\cdot\beta^{4}\cdot\text{‘a+‘}^{7}\cdot\text{‘a-‘}^{6}+$

$$51\cdot 32^{-1}\cdot i\cdot \epsilon^5\cdot 2^{1\cdot 2^{-1}}\cdot\ \text{‘a−‘}^7-45\cdot 64^{-1}\cdot \epsilon^4\cdot \beta^4\cdot \text{‘a+‘}^2\cdot \text{‘a−‘}^{10}+45\cdot 16^{-1}\cdot \epsilon^4\cdot \beta^4\cdot \text{‘a+‘}^{\ 7}\cdot$$
$$\text{‘a−‘}^5+765\cdot 128^{-1}\cdot \epsilon^4\cdot \beta^4\cdot \text{‘a+‘}^4\cdot \text{‘a−‘}^8-8559\cdot 32^{-1}\cdot i\cdot \epsilon^5\cdot 2^{1\cdot 2^{-1}}\cdot \beta\cdot \text{‘a+‘}^4\cdot \text{‘a−‘}^3-27\cdot \epsilon^4\cdot$$
$$\text{‘a−‘}^7\cdot \beta^2\cdot \text{‘a+‘}+153\cdot 4^{-1}\cdot \epsilon^4\cdot \text{‘a−‘}^6\cdot \beta^2\cdot \text{‘a+‘}^2+3\cdot 8^{-1}\cdot \epsilon^2\cdot \text{‘a−‘}^8\cdot \beta^2-150\cdot i\cdot \epsilon^5\cdot 2^{1\cdot 2^{-1}}\cdot$$
$$\beta^2\cdot \text{‘a+‘}^4\cdot \text{‘a−‘}^3+135\cdot 512^{-1}\cdot i\cdot \epsilon^5\cdot 2^{1\cdot 2^{-1}}\cdot \beta^5\cdot \text{‘a+‘}^{15}-45\cdot 16^{-1}\cdot i\cdot \epsilon^3\cdot 2^{1\cdot 2^{-1}}\cdot \beta^2\cdot \text{‘a+‘}^6\cdot$$
$$\text{‘a−‘}-27\cdot 16^{-1}\cdot i\cdot \epsilon^3\cdot 2^{1\cdot 2^{-1}}\cdot \beta^2\cdot \text{‘a−‘}^6\cdot \text{‘a+‘}-597\cdot 32^{-1}\cdot i\cdot \epsilon^5\cdot 2^{1\cdot 2^{-1}}\cdot \beta\cdot \text{‘a+‘}^8\cdot \text{‘a−‘}-3267\cdot$$
$$32^{-1}\cdot i\cdot \epsilon^5\cdot 2^{1\cdot 2^{-1}}\cdot \beta^2\cdot \text{‘a+‘}^8\cdot \text{‘a−‘}+45\cdot 4^{-1}\cdot i\cdot \epsilon^3\cdot 2^{1\cdot 2^{-1}}\cdot \beta\cdot \text{‘a+‘}^2\cdot \text{‘a−‘}^3-3\cdot i\cdot \epsilon^3\cdot 2^{1\cdot 2^{-1}}\cdot \beta^3\cdot$$
$$\text{‘a+‘}^6\cdot \text{‘a−‘}^3+13\cdot \epsilon^4\cdot \beta^2\cdot \text{‘a+‘}^3\cdot \text{‘a−‘}^3-13\cdot 4^{-1}\cdot \epsilon^4\cdot \beta^2\cdot \text{‘a+‘}^2\cdot \text{‘a−‘}^{\ 4}-3\cdot 4^{-1}\cdot i\cdot \epsilon\cdot 2^{1\cdot 2^{-1}}\cdot \beta\cdot$$
$$\text{‘a−‘}^5+240\cdot i\cdot \epsilon^5\cdot 2^{1\cdot 2^{-1}}\cdot \beta\cdot \text{‘a+‘}^2\cdot \text{‘a−‘}^3-39\cdot \epsilon^4\cdot \beta\cdot \text{‘a+‘}^3\cdot \text{‘a−‘}+39\cdot \epsilon^4\cdot \beta\cdot \text{‘a+‘}\cdot \text{‘a−‘}^3+$$
$$3\cdot 4^{-1}\cdot i\cdot \epsilon\cdot 2^{1\cdot 2^{-1}}\cdot \text{‘a+‘}\cdot \text{‘a−‘}^2-2403\cdot 128^{-1}\cdot i\cdot \epsilon^5\cdot 2^{1\cdot 2^{-1}}\cdot \beta^2\cdot \text{‘a−‘}^{10}\cdot \text{‘a+‘}\ -3\cdot 4^{-1}\cdot i\cdot \epsilon^3\cdot$$
$$2^{1\cdot 2^{-1}}\cdot \beta\cdot \text{‘a+‘}\cdot \text{‘a−‘}^2+189\cdot 32^{-1}\cdot i\cdot \epsilon^5\cdot 2^{1\cdot 2^{-1}}\cdot \beta^2\cdot \text{‘a+‘}^9-215\cdot 8^{-1}\cdot \epsilon^4\cdot \beta\cdot \text{‘a+‘}^4\cdot \text{‘a−‘}^4-$$
$$135\cdot 8^{-1}\cdot \epsilon^4\cdot \beta\cdot \text{‘a+‘}^3\cdot \text{‘a−‘}^5-15\cdot 2^{-1}\cdot \epsilon^4\cdot \text{‘a−‘}^4\cdot \beta^3\cdot \text{‘a+‘}^6-135\cdot 4^{-1}\cdot \epsilon^4\cdot \text{‘a−‘}^5\cdot \beta^3\cdot \text{‘a+‘}^5$$

by definition

$$E_n = \langle \phi, n|\, h_d \,|\phi, n\rangle \tag{C.26}$$

by substituting the expression of h_d in (C.26) and using a Maple compilation, then we get the following expression of energy E_n

$$\begin{aligned}
E_n = {} & 1024^{-1}[-\Big(\Big((65520n^7 + 660n^9 + 2970n^8 + 1472940n^5 + 215460n^6 \\
& + 3150630n^4 + 6832350 \cdot n + 8764740 \cdot n^2 + 7873080 \cdot n^3 + 2055375)\beta^6 \\
& + \Big(3240n^8 + 3936600 + 10500480n^3 + 1249920n^5 + 255360n^6 + 11983680n \\
& + 14579040n^2 + 25920n^7 + 4337760n^4)\beta^5 + \Big(80640n^6 + 2296800n^4 \\
& + 624960n^5 + 7753056n + 6019968n^3 + 9227232n^2 + 11520n^7 + 2659536)\beta^4 \\
& + \Big(102144n^5 + 1436160n^3 + 2079360n + 731712 + 23040n^6 + 654720n^4\Big)\beta^3 \\
& + \Big(535392 \cdot n + 399488 \cdot n^3 + 33088 \cdot n^5 + 90400n^4 + 179760 + 544160n^2)\beta^2 \\
& + \Big(83840 + 226304n + 27520n^4 + 117248n^3 + 196608n^2\Big)\beta + 2944 + 17664n^2 \\
& + 11776 \cdot n + 11776 \cdot n^3)\epsilon^4) + \Big(\Big(2016 \cdot n^5 + 672 \cdot n^6 + 46224 \cdot n + 31680 \cdot n^3 \\
& + 61728n^2 + 17520n^4 + 16200)\beta^4 + \Big(2304n^5 + 14400 + 30720n^3 + 40320n^2 \\
& + 5760n^4 + 41856n)\beta^3 + \Big(29952n + 35328n^2 + 12288 + 5376n^4 + 10752n^3\Big)\beta^2 \\
& + \Big(5376 + 13824 \cdot n + 13824 \cdot n^2 + 6144n^3\Big)\beta + 1408 + 3840n + 3840n^2)\epsilon^2] \\
& + 1024^{-1}[1024n + 512n^3 + 768n^2 + 384)\beta^2 + \Big(512 + 1024n^2 + 1024n\Big)\beta] \\
& + 1 + 2 \cdot n
\end{aligned} \tag{C.27}$$

for $(\beta, \epsilon) << 1$

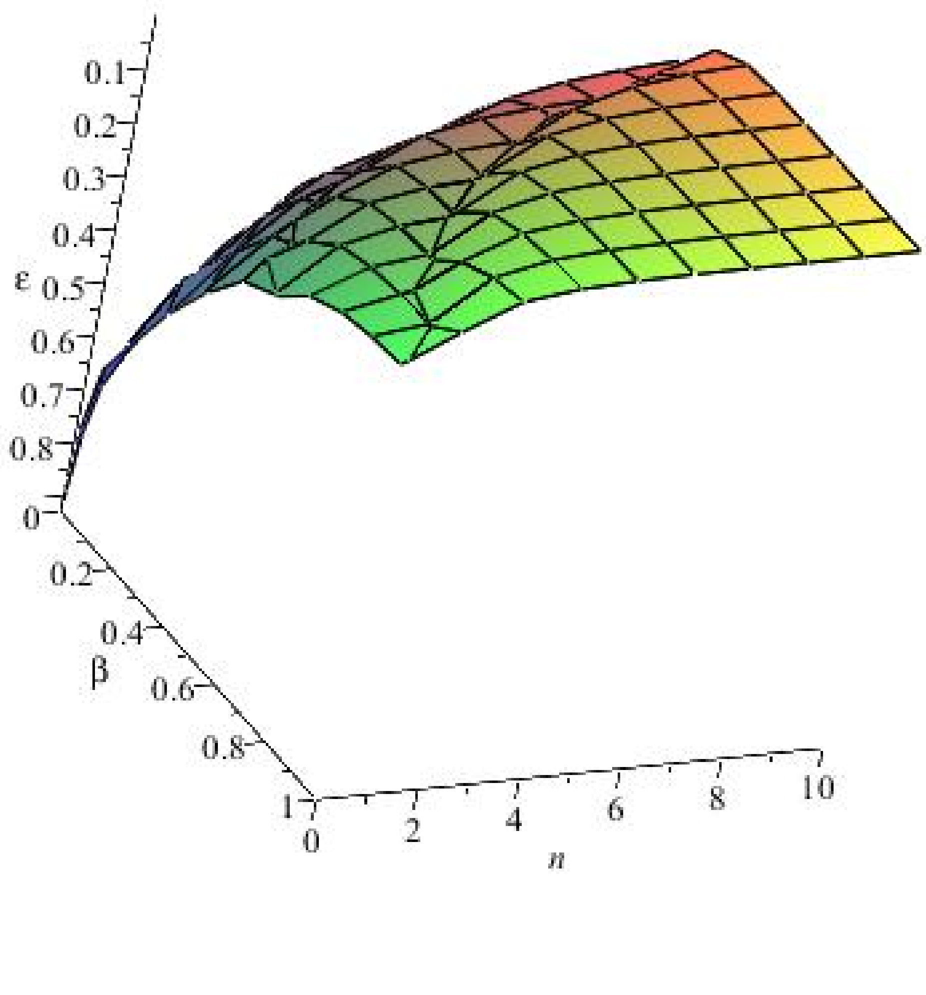
$$E_n = 1 + 2\,n + \left(n^2 + 1/2 + n\right)\beta + \left(3/8 + 1/2\,n^3 + 3/4\,n^2 + n\right)\beta^2$$
$$+ \left(\frac{15}{4}\,n + \frac{15}{4}\,n^2 + \frac{11}{8}\right)\epsilon^2 + \left(\frac{21}{4} + \frac{27}{2}\,n + \frac{27}{2}\,n^2 + 6\,n^3\right)\epsilon^2\beta + o(\beta^3, \epsilon^3) \quad \text{(C.28)}$$

Particular Cases

1. $\epsilon \neq 0$ and $\beta \neq 0$

$$
\begin{aligned}
E_n = {} & 1 + 2n + \left(2^{-1} + n^2 + n\right)\beta + \left(n + 2^{-1}n^3 + 3\cdot 4^{-1}n^2 + 3\cdot 8^{-1}\right)\beta^2 \\
& + (11\cdot 8^{-1} + 15\cdot 4^{-1}n + 15\cdot 4^{-1}n^2)\epsilon^2 \\
& + \left(21\cdot 4^{-1} + 27\cdot 2^{-1}n + 27\cdot 2^{-1}\cdot n^2 + 6n^3\right)\epsilon^2\cdot\beta
\end{aligned}
\tag{C.29}
$$

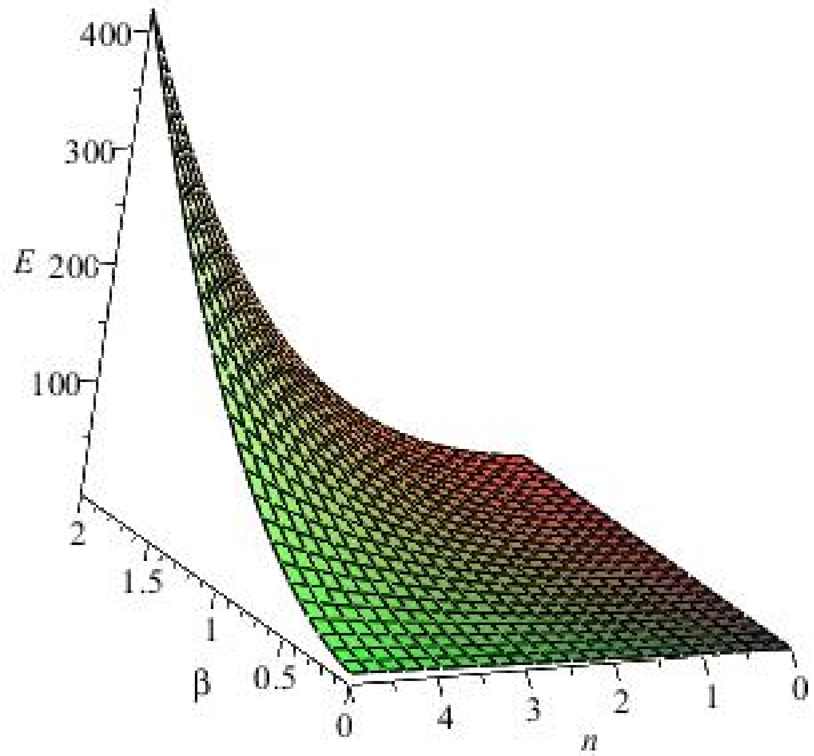
Figure C.1: Deformed Cubic An-harmonic Oscillator



2. $\epsilon = 0$ and $\beta \neq 0$

$$E_n = 1+2n+\left(2^{-1}+n^2+n\right)\beta+\left(n+2^{-1}n^3+3\cdot 4^{-1}n^2+3\cdot 8^{-1}\right)\beta^2 \quad \text{(C.30)}$$

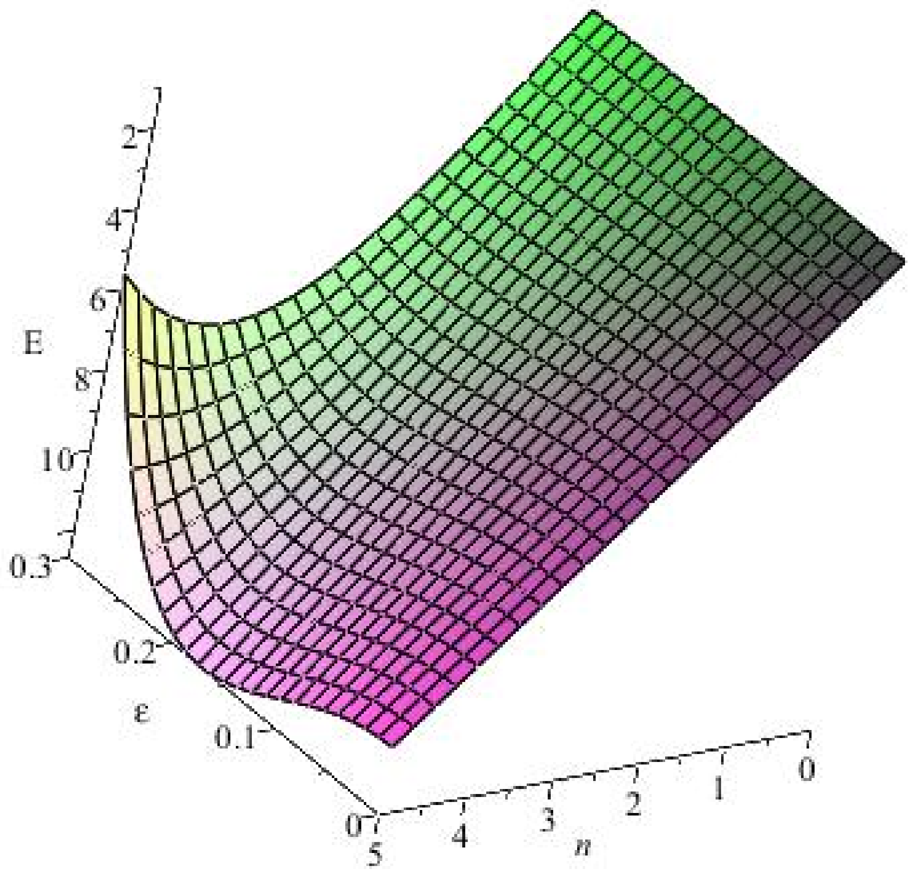
Figure C.2: Deformed Harmonic Oscillator



3. $\epsilon \neq 0$ and $\beta = 0$

$$\begin{aligned} E_n = &\left(-23 \cdot 8^{-1} - 69 \cdot 4^{-1} n^2 - 23 \cdot 2^{-1} n - 23 \cdot 2^{-1} n^3\right) \epsilon^4 \\ &+ \left(11 \cdot 8^{-1} + 15 \cdot 4^{-1} \cdot n + 15 \cdot 4^{-1} \cdot n^2\right) \epsilon^2 + 1 + 2 \cdot n \end{aligned} \tag{C.31}$$

Figure C.3: Cubic An-harmonic Oscillator



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Abstract

Quantum gravity is a physical theory designed to unify general relativity and quantum mechanics, which are inconsistent and incompatible and each one has a different scale from the other, the first is macroscopic while the second is microscopic, thus it would be difficult to make them integrated into a single theory. There are numerous attempts to unify them using three separate roads: super string theory, loop-quantum gravity and black hole approach, and since these three approaches expect a minimal length on either the position x or the momentum p coordinates, or both of them, this necessitates a deformation on the phase space, with a slight deformation coefficient. Certainly; such deformation affects Quantum Mechanics in particular Heisenberg algebra by producing a modifications in the commutation relations. In such circumstances, the resulted Hamiltonians could be non-Hermitian which means they do not respect Dirac Hermiticity condition defined in a Hermitian-inner product, which is necessary and sufficient condition to obtain a real and positive energy spectrum.In fact, it is contrary to what was found; because there are Hamiltonian operators do not respect this condition.In this context it can be categorized them into three classes, PT-Hermitian, pseudo-Hermitian and Quasi-Hermitan hamiltonian, where all they have the property of Hermiticity but in non standard inner product. First, two examples of PT-symmetric Hamiltonians have been studied, Shifted(displace)-harmonic oscillator $p^2 + x^2 + i\epsilon x$ and Cubic-anharmonic oscillator $p^2 + x^2 + i\epsilon x^3$, then we have deformed the position operator x, where the Hamiltonians H became pseudo-hermitian or more precisely quasi-Hermitian characterized by the similarity transformation which is written in terms of a positive-definite, Hermitian and invertible operator η. We found that the energy spectrum is real and positive and related to a deformation coefficient β is given for small values,and we studied the effect of time evolution on both the space-time and non-Hermiticity of PT-symmetric spin Hamiltonian H_{2*2} with assumption that there is no correlation in the momentum. It turns out that it is possible to generate a bipartite spin quantum entanglement quantified in the Von Newman entropy, as well as the possibility of magnified the amount of quantum entanglement and becomes maximal or reduced (decoherence) depending on the various parameters and physical quantities of the system (Hermiticity and metric) Finally we also expect that quantum entanglement depends strongly on the metric space, especially near the horizon (Schwarzchild's radius). This result can be generalized to multisystems (Bell, Werner,...etc).

pass words: Quantum gravity, Hermiticity, PT-symmetry,Pseudo-Hermiticity Quasi-Hermiticity,minimal length,Quantum Entanglement

Résumé

La gravité quantique est une théorie physique conçue à unifier la relativité générale et la mécanique quantique, qui sont inconsistantes et incompatibles et chacune possède une échelle différente de l'autre.Il y a de nombreuse tentatives pour les unifier en utilisant trois voies distictes : la théorie des super cordes, la théorie des boucles quantique et l'approche du trou noir, et étant donné que ces approches atteignent une longueur minimale de coordonnées de la position x ou l'impulsion p, ou les deux, autrement, ce qui nécessité une déformation dans l'espace des phases, avec un coefficient de déformation un peu petit. Certainement ; cette déformation affecte l'algèbre de Heisenberg en produisant quelques modifications qui apparaissent dans en particulier les relations de commutation.Dans ce cas, les Hamiltoniens obtenus pourraient être non-Hermitiques, cela signifie qu'ils ne respectent pas la condition d'Herméticité de Dirac définie dans un produit scalaire Hermitique, cette condition étant considérée comme une condition nécessaire et suffisante pour obtenir un spectre d'énergie réel et positif.En fait, c'est contraire à ce qui a été trouvé, car il y a des opérateurs Hamiltoniens qui ne respectent pas cette condition. Dans ce contexte, nous les avons classés en trois classes, PT-hermitien, pseudo-hermitien et Hamiltonien quasi-Hermitain, où ils ont tous la propriété d'Herméticité mais dans un produit intérieur non-standard. En premier lieu on a étudié deux exemples d'hamiltoniens PT-symmetrique, un oscillateur harmonique déplacé $p^2+x^2+i\epsilon x$ et un oscillateur cubeibique anharmonique $p^2+x^2+i\epsilon x^3$, En suite, on a déformé l'opérateur de position x, où l' Hamiltonien H est devenu pseudo-hermitiens ou plus précisément quasi-hermitien caractérisé par la transformation de similarité qui s'écrive en fonction d'un opérateur η ; positif défini, hermitien et inversible. Nous avons trouvé que le spectre d'énergie est réel et positif et lié à un coefficient de déformation β est donné pour de petites valeurs En deuxième lieu, nous avons étudié l'effet de l'évolution du temps sur l'espace-temps et la non-Herméticité du Hamiltonien de spin PT-symétrique H_{2*2} en supposant qu'il n'y a pas de corrélation dans l'impulsion. Il s'avère qu'il est possible de générer une intrication quantique de spin bipartite quantifiée dans l'entropie de Von Newman. Ainsi que la possibilité d'augmenter la quantité d'intrication quantique et de devenir maximal ou réduit (décohérence) en fonction de différents paramètres, et nous prédisons également que l'intrication quantique dépende fortement de l'espace métrique,en particulier près de l'horizon (rayon de Schwarzchild). Ce résultat peut être généralisé à multi-systèmes (Bell, Werner,...etc).

mots de passe : La gravité quantique, herméticité, PT-symétrie,quasi-Herméticité, Pseudo-Herméticité,Longueur milimale,Intrication quantique

ملخص

مؤثرات هاملتون اللاهرميتية والمشوهة

الجاذبية الكمية هي نظرية فيزيائية تهدف إلى جمع كل من النسبية العامة وميكانيكا الكم، اللذان يتميزان بكونهما غير منسجمان و يختلفان من حيث المستوى، فالأولى ماكروسكوبية و الثاني ميكروسكوبي، و بالتالي من الصعوبة بما كان دمجهما في نظرية واحدة. هناك العديد من المحاولات لتوحيديهما من خلال ثلاثة مقاربات : النظرية الخيطية الفائقة ، الحلقات الكمية و مقاربة الثقب الأسود، وحيث أن هذه المقاربات تصل الى نتيجة وهي أقصر طول في إحداثيات الموضع x أو الزخم p أو فيهما معا،والذي يتطلب تشوه في الفضاء الطور بمعامل تشوه يكون صغيرا، هذا التشوه بالتأكيد يوثر على جبر هايزنبرغ لميكانيكا الكم، حيث تطرأ عليه بعض التعديلات، في هذه الحالة يمكن الحصول على مؤثرات هاميلتون لا تحقق المسلمة الهرميتية لديراك والمعرفة بواسطة جداء سلمي هرميتي والتي كانت تعتبر شرط لازم وكاف للحصول على طيف طاقة حقيقي و موجب. حيث وجد انه هناك مؤثرات هاملتون لا تحقق هذا الشرط . منذ 1998 عمل بندر على هذا الفئة و قد أرجع السبب أنها تحقق التماثل PT في مقال قدمه مع بوتشر، وقد سميت مؤثرات هاملتون ذات التماثل PT ولكن للحصول على طيف طاقة حقيقي كله موجب اشترط بندر و دوري وآخرون على أن يملك المؤثر H تماثل PT- دقيق. لكن منذ 2001 عمل مصطفازاديه على مؤثرات هاملتون غير هرميتية لها إمكانية امتلاك طيف طاقة حقيقي و موجب وبين أنها فئة أكثر عموما و تشمل أيضا فئة تماثل PT وسماها مؤثرات هرميتية-زائفة والتي تتميز بأنها تحقق شرط الهرميتية لكن في جداء سلمي يكون معدل بمؤثر يسمى المتري η . من خلال دراستنا لنوعين من مؤثرات هاملتون للمذبذب التوافقي النازح $p^2 + x^2 + i\epsilon x$ و للمذبذب اللاتوافقي المكعب $p^2 + x^2 + i\epsilon x^3$ قمنا بتشويه مؤثر الموضع x حيث طرأت بعد ذلك تغيرات على مؤثر هاملتون H وأصبح زائف-هرميتي أو بشكل أدق شبه-هرميتي أي يحقق علاقة التحويل المشابه عن طريق مؤثر موجب-معرف ، هرميتي و قابل للانعكاس و هو المتري η ، و قد وجدنا أن طيف الطاقة حقيقي و موجب ويتعلق بمعامل التشوه β الذي يعطى من أجل قيم صغيرة. و أخيرا درسنا تأثير تطور الزمن على كل من الفضاء الزمكاني و لاهرميتية هاملتون لمؤثر H_{2*2} عند افتراض انه ليس هناك ارتباط في العزم. وقد تبين أنه يمكن توليد ترابط كمي لنظام ثنائي للعزم الذاتي $Spin$ في أنتروبي فون نيومان وكذا إمكانية تضخيم مقدار الترابط الكمي الذي يتم إنشاؤه ويصبح أكبر أو أقل (فك الارتباط) اعتمادا على مختلف المقادير والكميات الفيزيائية للنظام (الهرميتية والمتري) توصلنا أيضا الترابط الكمي يتعلق أساسا بمتري الفضاء الزمكاني بالخصوص بجوار الأفق (نصف قطر شوارتشيلد) ويمكن تعميم هذه النتيجة على أنظمة متعددة(حالات بيل، ويرنر...) و ندرس تأثيرارتباط العزم على ظاهرة الترابط الكمومي.

كلمات مفتاحية : الجاذبية الكمومية،الهرميتية، اللاهرمتية ، تماثل-PT ، تماثل زائف-هرميتي ، تماثل شبه-هرمتيي ، ترابط كمومي.