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Theoretical and Phenomenological Study of Non-standard Cosmology

Par

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Contents

List of Tables

List of Figures

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Introduction

. Cosmology is the study of the cosmos. From early on, humans were always fascinated by the stars. They noticed constellations that looked like animals and people, so they weaved stories and myths to rationalize what they had seen. This fascination with the stars grew stronger with the passage of time. With each discovery and theory that explains a cosmological phenomenon, more questions are raised.

On one hand, after science was developed, when Isaac Newton presented the now eminently renowned inverse-square gravitational force law, which united the earthly physics of falling apples with the cosmic movement of planets and stars, it was very successful in both explaining and anticipating a variety of phenomena, but on the other hand, Newtonian gravity couldn't explain many other events. In 1907, the prediction of gravitational red-shift was made possible due to Einstein's introducing the equivalence between gravitation and inertia, in 1915, the General Relativity theory was presented in the differential geometry language, based on two key elements: pseudo-Riemannian geometry (Lorentzian geometry) and specific field equations for the Ricci tensor. Depending on these two elements. It succeeded in explaining a famous observation that Newtonian gravity could not, such as the experimental result of Mercury's orbit and the gravitational deflection of light passing near the Sun, as measured in 1919 during a solar eclipse by Arthur Eddington [1].

Actually, physicists describe the universe using two major theories: general relativity and quantum mechanics, which are considered two pillars of modern physics. General relativity describes the gravitational force and the structure of the large-scale universe, whereas quantum mechanics studies phenomena on small scales.

After about a century, the field equations of general relativity are still our best description of how space-time behaves on a macroscopic scale. However, this success hasn't stopped people from proposing alternatives. Even in the early days after General Relativity emerged, Einstein himself attempted to alter the framework of General Relativity by introducing the "Cosmological Constant". We are not claiming that General Relativity is wrong, but these suggestions on how to extend it and incorporate it into a larger, more unified theory (and one of the most important goals of modern physics is to obtain a unified theory) can solve some theoretical and current observational problems. These early changes were sparked mostly by scientific curiosity, such as Weyl's theory of scale independence, Eddington's connection theory, and Kaluza and Klein's theory of higher dimensions. During this period, Dirac suggested an explanation of the Large-Number Coincidence (LNC) problem by suggesting that the Newtonian gravitation constant can change over time [2],the possibility of changing Newton's constant was discussed again in the 1960s by Brans and Dicke. Modifying gravity, on the other hand, inspired Andrei Sakharov, who proposed in 1967 a hypothesis that Einstein–Hilbert action is linked to a change in the action of quantum fluctuations, and he regarded gravitation and the discussion of quantum electrodynamics as analogous [3],[4].

The period from 1960 to 1980 was a Golden Age in experimental gravity. The first successful lunar ranging tests were carried out in 1962, when Massachusetts Institute of Technology's Louis Smullin and Giorgio Fiocco observed laser pulses reflected from the moon's surface. In 1974 Russell Alan Hulse and Joseph Hooton Taylor Jr recognized the source as a pulsar after detecting pulsed radio radiation, and this ended with the

Hulse–Taylor binary pulsar's orbital period decreasing at a pace consistent with the general relativistic prediction of gravitational-wave energy loss, all these results based on alternative general relativity models such us The parameterized post-Newtonian formalism, or PPN formalism [5], [6].

In addition, the existence of dark matter could be explained by modified gravity. It could also be used to characterize the universe's change from deceleration to acceleration, as well as coincidence problem and high-energy physics problems. This work has recently continued to attract many physicists, motivating us to investigate a variety of models with various modifications.

Our work is divided into two parts. Part one is the Standard Model of Cosmology, where the first chapter contains a reminder of Einstein's ordinary four-dimensional theory of gravitation. This perspective allows us to better understand by presenting as a beginning some concepts that could be considered predictions of general relativity, and by introducing the Einstein field equations and their solutions. In the following chapter, we will be interested in extra dimensions to explain the phase of acceleration in the late universe without the concept of dark matter. In the first case, we start with the ordinary The Friedmann–Lemaître–Robertson–Walker FRW metric plus 1 extra dimension. By assuming that the universe is a perfect fluid and is defined by dimensionless numbers w (for four dimensions) and γ (for the fifth dimension), in order to discuss the effect of extra dimension. On the other hand, we discuss it with viscosity fluid where $\overline{p} = p + h(t)H_R$, it has been discussed numerous times that the possibility of viscosity might influence the expansion history of the universe. In the same chapter, we will also discuss another model, Kantowski-Sachs space-time in five dimensions, because it represents the most famous anisotropic model. Finally, we take a look at the simplest class of modified gravity theories, the F(R) modified. We will discuss all these models using dynamical study. In the second part, in order to extend our work on alternative models of general relativity, we will be concerned with introducing quantum information in curved space-time and this part is organized as follows: Chapter three contains the definition of quantum entanglement by giving a basic mathematical overview of quantum entanglement, which covers the fundamentals of entanglement classification and quantification. Similarly, in Chapter 4 we review the essentials of entanglement and inertial observers. In this chapter, we will use de Sitter–Schwarzschild space-time to see the effect of a gravitational field near a massive black hole on the spin entanglement in the case of triplet and singlet states presented by a system of two particles described by wave packets moving in a gravitational field (GF). For a more in-depth discussion, we will elaborate on a general formalism for quantum spin entanglement in curved space-time. This formulation allows us to study different models in curved space-time. The Kerr and the non-commutative Reissner-Nordström models will be considered. The concurrence behavior as well as the spin entanglement of a system of two spin-1/2 particles will be discussed. This model allows us to do a detailed study of this purely quantum phenomenon in different frames of space and geometry, or both at the same time. Then, in Chapter 5 we will draw our conclusion.

Part I

The Standard Model of Cosmology

Chapter 1

General Relativity

1.1 Physics Before General Relativity

Isaac Newton's universal law of gravitation represents a quantum leap in human understanding of the forces that define the universe. Newton published this law in the late 17^{th} century. He asserted that all objects attract one another and that this force of gravity is responsible for holding the planets in their orbits around the sun. He had discovered the force that holds the universe together. This force of gravity acting between two objects is directly proportional to the mass of the objects and inversely proportional to the square of the distance that separates their centers

$$\overrightarrow{F} = G\frac{M_1 M_2}{r^2}\overrightarrow{U}_r \tag{1.1}$$

Newton knew that the force that caused the apple's acceleration (gravity) must be dependent upon the mass of the apple. And since the force acting to cause the apple's downward acceleration also causes the earth's upward acceleration, consider Newton's famous equation

$$\overrightarrow{F} = m\overrightarrow{a} \tag{1.2}$$

where $G = 6.67259 \times 10^{-11} m^3 Kg^{-1} s^{-2}$,the gravitational constant $\overrightarrow{F}$ gravitational force, $\overrightarrow{U}_r = \frac{\overrightarrow{r}}{r}$ where $\overrightarrow{r}$ is the vector radius of $\overrightarrow{M_1 M_2}$ (Distance between the two bodies), and M_1, M_2 masses of the two bodies considered.

Newtonian physics based on the concept of force and action at a distance, Table (1.1) shows a comparison between Newton's ideas and Maxwell

Newton (1686)	Maxwell (1904)
Absolute space, without relation to external things, remains similar and immobile	There isn't absolute space
Absolute,true and mathematical time, of itself,and from its own nature flows equably without regard to anything external	There is no absolute time
Simultaneity $V = \infty$	There is no simultaneity of two events that Produce on different theaters C is a limiting speed

Table 1.1: Comparison between Newton's ideas and Maxwell

1.2 Relativity and absolute space

The special relativity elaborated by Einstein in 1905 was intended to introduce the principle of relativity: All the laws of physics are identical in all Galilean references, including the constancy of the speed of light, this principle will be translated mathematically by the invariance of the equations under the transformations of Lorentz, which allows us to determine the time and space intervals in a transformation from one reference to another.

Relativity refuses the idea of absolute space, whereby special relativity (1905) refuses the absolute space of Michelson, and general relativity refuses the absolute space of Newton. The groundwork for the concept of Absolute Space was laid by Newton in his book (Mathematical Principles for Natural Philosophy):

"Absolute space, in its own nature, without regard to anything external, remains always similar and immovable. Relative space is some movable dimension or measure of the absolute spaces; which our senses determine by its position to bodies: and which is vulgarly taken for immovable space ."

The emergence of general relativity is the result of the failure of all endeavors that attempted to reform Newton's theory of gravity and Einstein's philosophical desire to delete absolute space.

1.3 Equivalence Principle

Albert Einstein was thinking about gravity and the fact that weight disappears during the acceleration of free-fall. When your weight disappears, you do not feel the pull of gravity on your body. You have become weightless. Einstein's suspicion was that gravity had become something else. He believed it had become acceleration. Gravity and acceleration were equivalent, according to eq. (1.2) , where $\overrightarrow{a}$ represents the particle's acceleration, it follows from the law of dynamics that

$$m_I \overrightarrow{a} = m_g \overrightarrow{g} \tag{1.3}$$

where $\overrightarrow{g}$ is the gravitational field and m_I, m_g are the inertial and gravitational masses, respectively, the equality of the last two implies that

$$\frac{m_I}{m_g} = \left\| \frac{\overrightarrow{g}}{\overrightarrow{a}} \right\| = 1 \tag{1.4}$$

Later on, Einstein was so confident of his idea that he declared that an observer in a windowless room would be unable to determine whether his weight was being created by the pull of gravity or the force of acceleration. An observer watching a light beam in a dark motionless room that is not affected by a gravitational field would see the beam move in a straight line across the room; however, if the room began to accelerate, the observer would feel a force similar to gravity and perceive the light beam bending. If gravity and acceleration are equivalent, then an observer in a gravitational field sees the light beam curve, meaning that light that is without mass can be deflected by a gravitational field. Einstein called this idea the equivalence principle. Einstein called this idea the "equivalence principle," proving this principle by observing the universe when the light from a distant star passes close to the sun, but the problem is that they can't see any stars during the daytime. They are lost in the glare of the sun. British astronomer Arthur Eddington (as we mentioned in the introduction) proposed doing this experiment during a total solar eclipse, where the sun is blocked by the moon and the stars become visible.

1.4 Space-Time Metrics

1.4.1 Flat Space

We define the space-time coordinates of 2 events

$$x^\mu = (x^0, \overrightarrow{x}) \quad \text{and} \quad y^\mu = (y^0, \overrightarrow{y}) \tag{1.5}$$

The exponent 0 designates the coordinate of time and x^μ, y^μ are quadri-vectors (this will not be possible in General Relativity where coordinates are not vectors). We define the scalar product of these two quadri-vectors by

$$x.y = x^0 y^0 - \overrightarrow{x}\,\overrightarrow{y} = x^0 y^0 - x^i y^i \tag{1.6}$$

in space-time, the separation or interval S, defines the distance between two events

$$\Delta S^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 \tag{1.7}$$

where t is the coordinate time, or real time. xyz are the space coordinates. We can define two regions related to our own position according to Minkowski space-time, one time-like and one space-like, where these two regions are separated by the light-cone, defined by the incoming and outgoing light rays. A separation between two events in space-time is time-like if inside the cone and space-like if outside. At the intersection, the separation is light-like since this is the trajectory of a photon. Matter cannot travel along space-like trajectories. Light and massless particles travel along the light-cone.

The trajectory of a particle in space-time is called a world-line. The trajectory of a particle in free fall in space-time is called a geodesic. Remember that the value of this parameter is independent of the observer and invariant under coordinate transformations. From Minkowski

$$\eta_{\mu\nu} = diagonal(1, -1, -1, -1) \tag{1.8}$$

we can rewrite eq. (1.6)

$$x.y = \eta_{\mu\nu} x^\mu y^\nu = x_\nu y^\nu \qquad \text{where} \qquad x_\mu = \eta_{\mu\nu} x^\nu \tag{1.9}$$

for an infinitesimal vector dx^μ, eq. (1.7) becomes

$$dS^2 = (dx^0)^2 - d\overrightarrow{x}^2 = dt^2 - d\overrightarrow{x}^2 = \eta_{\mu\nu}dx^\mu dx^\nu \tag{1.10}$$

1.4.2 Curved Space-Time

If we have a free fall and we carry out its coordinate transformation from an inertial system to an arbitrary coordinate system it can be shown that the time derivative of the velocity along a world line contains a term, the affine connection, that acts as an acceleration if the metric is not flat. Einstein gravity can thus be regarded a property of curved space-time. The tensor that describes the curvature of space-time is the Riemann-Christoffel (or simply the Riemann) tensor $R_{\alpha\beta\gamma\delta}$, corresponding to the Gaussian curvature.

A- Manifold

A manifold is one of the most fundamental concepts in mathematics and physics. *A* manifold is a space which is locally similar to Euclidean space in that it can be covered by coordinate patches (*A* "coordinate patch" on a manifold M is an open subset $U \subseteq M$ together with a map $\phi : U \to \mathbf{R}^n$ that is a homeomorphism between U and its image $\phi(U)$. We can also say that a manifold is a topological space where every point can be contained in some coordinate patch), and we say that a real C^r n-dimensional manifold M is a set M together with a C^r Atlas $\{U_\alpha, \varphi_\alpha\}$,i.e. a collection of charts $(U_\alpha, \varphi_\alpha)$, where the U_α are subsets of M (in topology, one describes a manifold using an atlas. Atlas Set of all cards) and the φ_α are one-one maps of the corresponding U_α to open sets in $\mathbf{R}^n$ [8]

B- Differentiable Manifold

In all of the following we will assume that the applications are infinitely derivable in their domains of definitions.

C- Curves Coordinates

Any point of a manifold can be delimited by coordinated curves, which play the role of the axes in Euclidean geometry. $\overrightarrow{dM} = \overrightarrow{MM'}$ is a manifold tangent vector at M (a tangent vector is an infinitesimal displacement at a specific point on a manifold) The set of tangent vectors at a point P forms a vector space called the tangent space at P

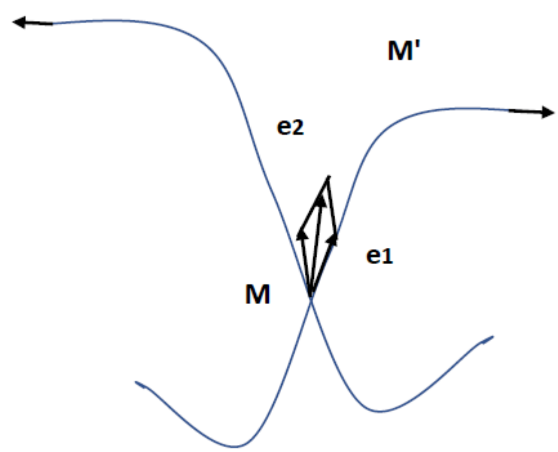


Figure 1-1: Curves Coordinates

$$\overrightarrow{MM'} = dx^1 \overrightarrow{e}_1 + dx^2 \overrightarrow{e}_2 \tag{1.11}$$

In general we have:

$$\overrightarrow{dM} = dx^\mu \overrightarrow{e}_\mu(M) \tag{1.12}$$

where $\overrightarrow{dM}$ is a basic vector

D- Metric Tensor

In the mathematical field of differential geometry, a metric tensor is a type of function that takes as input a pair of tangent vectors V and U at a point on a surface or the distance between two points. These are additional concepts, and a new entity is needed to define them. This entity is called the metric or the metric tensor. A general space endowed with a metric is called a Riemannian space. Let us begin with the simple case R^2 by considering two vectors, V and U ,their Cartesian components can be written as follows

$$V = (v_x, v_y) = (v^1, v^2); U = (u_x, u_y) = (u^1, u^2) \tag{1.13}$$

as is well known

$$V.V = v_x^2 + v_y^2 \tag{1.14}$$

this is a scalar quantity. Similarly $U.U$ and $V.U$ are scalars. To accomplish this, we introduce a new geometric object, the length of which is known as the 'scalar product' of two vectors. We write, in some coordinate system:

$$V.V = g_{ik}v^i v^k = g_{11}v^1 v^1 + (g_{12} + g_{21})v^1 v^2 + g_{22}v^2 v^2$$

where in this coordinate system g_{ik} are the metric tensor components. In addition, the metric tensor is used to express the distance ds between the points (x^1, x^2) and $(x^1 + dx^1, x^2 + dx^2)$

$$ds^2 = d\overrightarrow{M}^2 = dx^\mu dx^\nu \overrightarrow{e}_\mu \overrightarrow{e}_\nu = g_{\mu\nu} dx^\mu dx^\nu \tag{1.15}$$

1.4.3 Tensor

Mathematically, space-time is represented by a four-dimensional differentiable manifold M [7], this point will be characterized by its coordinates $M(x^0, x^1, x^2, x^3)$,we have transformation equations, which specify one coordinate system in terms of the other

$$x^{\mu'} = \phi^{\mu}(x^0, x^1, x^2, x^3) \tag{1.16}$$

where ϕ^{μ} have continuous partial derivatives and single valued [8], by deriving the eq. (1.16)

$$dx^{\mu'} = \sum_{\nu=0}^{3} \frac{\partial \phi^{\mu}}{\partial x^{\nu}} dx^{\nu} \tag{1.17}$$

$$dx^{\mu'}(x) = \frac{\partial x^{\mu'}(x)}{\partial x^{\nu}} dx^{\nu} = a^{\mu'}_{\nu} dx^{\nu} \tag{1.18}$$

and as dM is invariant, we can write

$$\overrightarrow{dM} = dx^{\mu} \overrightarrow{e}_{\mu}(M) = dx^{\nu'} \overrightarrow{e}_{\nu'}(M) \tag{1.19}$$

One can also deduce the transformations of the unit vectors

$$\begin{aligned} dx^{\mu} \overrightarrow{e}_{\mu}(M) &= dx^{\lambda'} \overrightarrow{e}_{\lambda'}(M) \\ &= a^{\lambda'}_{\nu} dx^{\nu} \overrightarrow{e}_{\lambda'}(M) \end{aligned} \tag{1.20}$$

inverse of the matrix $a^{\mu'}_{\nu}$

$$a^{\nu}_{\mu'} = \frac{\partial x^{\nu}(x)}{\partial x^{\mu'}} \tag{1.21}$$

which allows us to write

$$a^{\nu}_{\mu'} a^{\lambda'}_{\nu} = \delta^{\lambda'}_{\mu'} \qquad \text{where } \delta^{\lambda'}_{\mu'} = \delta^{\lambda}_{\mu} = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{if } \lambda \neq \mu \end{cases} \tag{1.22}$$

$\delta^{\lambda'}_{\mu'}$ is the Kronecker symbol, we also have

$$\begin{aligned} g_{\mu'\nu'} &= \vec{e}_{\mu'} \vec{e}_{\nu'} \\ &= a^{\lambda}_{\mu'} a^{\rho}_{\nu'} g_{\lambda\rho} \end{aligned} \tag{1.23}$$

1.4.4 A Some Basic Rules of Tensor Calculus

We are now defining covariant and contravariant vectors or tensors as being represented with lower and upper indices, respectively, in an orthogonal system of coordinates with perpendicular axes, according to the previous equation

$$A_{\mu'} = \frac{\partial x^{\nu}}{\partial x^{\mu'}} A_{\nu}, \qquad B^{\nu'} = \frac{\partial x^{\nu'}}{\partial x^{\mu}} B^{\mu} \tag{1.24}$$

a rank tensor (k, l)

$$T^{\mu'_1 \ldots\ldots \mu'_k}_{\nu'_1 \ldots\ldots \nu'_l} = \frac{\partial x^{\mu'_1}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\mu'_k}}{\partial x^{\mu_k}} \frac{\partial x^{\nu_1}}{\partial x^{\nu'_1}} \cdots \frac{\partial x^{\nu_l}}{\partial x^{\nu'_l}} T^{\mu_1 \ldots\ldots \mu_k}_{\nu_1 \ldots\ldots \nu_l} \tag{1.25}$$

we can show that the sum T of two tensors T_1 and T_2 defined by

$$T^{\mu_1 \ldots\ldots \mu_k}_{\nu_1 \ldots\ldots \nu_l} = (T_1)^{\mu_1 \ldots\ldots \mu_k}_{\nu_1 \ldots\ldots \nu_l} + (T_2)^{\mu_1 \ldots\ldots \mu_k}_{\nu_1 \ldots\ldots \nu_l} \tag{1.26}$$

and

$$(\alpha T)^{\mu_1 \ldots\ldots \mu_k}_{\nu_1 \ldots\ldots \nu_l} = \alpha T^{\mu_1 \ldots\ldots \mu_k}_{\nu_1 \ldots\ldots \nu_l} \tag{1.27}$$

In the other hand

$$g = \det g_{\mu\nu} = |G| \tag{1.28}$$

the transformation of a matrix is given by

$$G' = aGa^{+} \tag{1.29}$$

where $a^{+} = (a^{T})^{*}$. In the same way we can write

$$g_{\mu'\nu'} = a^{\lambda}_{\mu'} g_{\lambda\rho} a^{\rho}_{\nu'} \tag{1.30}$$

so we have

$$|G'| = a^{2}\,|G| \tag{1.31}$$

$$g' = a^{2} g \tag{1.32}$$

where

$$a = \det a^{\lambda}_{\mu'} = \det \frac{\partial x^{\lambda}}{\partial x_{\mu'}} \text{ (Jacobian)} \tag{1.33}$$

We define the antisymmetric tensor

$$\varepsilon_{\mu\nu\rho\sigma} = \frac{1}{\sqrt{g}}\eta_{\mu\nu\rho\sigma}$$

$\eta_{\mu\nu\rho\sigma}$ is totally antisymmetric, where $\eta_{0123} = 1$, in Minkowski space

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} ; g = -1 \tag{1.34}$$

in spherical coordinates we have

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix} ; g = -r^4 \sin^2\theta \tag{1.35}$$

the event in spherical coordinates is given by

$$ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 \tag{1.36}$$

Covariant Derivative of Tensor

Consider a scalar field $\varphi(x)$, a contravariant vector A^μ, and a covariant vector A_μ. Let us see how the quantities $\frac{\partial\varphi}{\partial x^\nu}$, $\frac{\partial A^\mu}{\partial x^\nu}$ and $\frac{\partial A_\mu}{\partial x^\nu}$ are transformed under a general coordinate transformation $x^\mu \to x^{\mu'}$, and we get

$$d\varphi(x) = \sum_\mu \frac{\partial\varphi}{\partial x^\mu} dx^\mu = \partial_\mu \varphi dx^\mu \tag{1.37}$$

and

$$\frac{\partial\varphi}{\partial x^\mu} \to \frac{\partial\varphi}{\partial x^{\mu'}} = \frac{\partial x^\nu}{\partial x^{\mu'}} \frac{\partial\varphi}{\partial x^\nu} \tag{1.38}$$

which shows that $\frac{\partial\varphi}{\partial x^\nu}$ is a covariant vector, and we know that

$$A^\mu(x) \to A^{\mu'}(x) = a^{\mu'}_\lambda A^\lambda(x) \tag{1.39}$$

$$\partial_\mu A^{\mu'}(x) = a^{\mu'}_\lambda \partial_\mu A^\lambda + \partial_\mu a^{\mu'}_\lambda A^\lambda \tag{1.40}$$

In the other hand we have:

$$\overrightarrow{A}(x) = A^{\mu}(x)\overrightarrow{e}_{\mu} \tag{1.41}$$

this quantity is invariant (because it is independent of the system)

$$\begin{aligned} d\overrightarrow{A}(x) &= DA^{\mu}(x)\overrightarrow{e}_{\mu}(x) \\ &= dA^{\mu}(x)\overrightarrow{e}_{\mu}(x) + A^{\mu}(x)d\overrightarrow{e}_{\mu}(x) \end{aligned} \tag{1.42}$$

where $DA^{\mu}(x)$ covariant derivative. We put:

$$d\overrightarrow{e}_{\mu}(x) = \Gamma^{\lambda}_{\mu\nu}(x)dx^{\nu}\overrightarrow{e}_{\lambda}(x) \tag{1.43}$$

the connection $\Gamma^{\lambda}_{\mu\nu}(x)$ is defined in terms of the metric tensor $g_{\mu\nu}$. From eq. (1.42) we obtain

$$D_{\nu}A^{\mu}(x) = \partial_{\nu}A^{\mu}(x) + A^{\lambda}(x)\Gamma^{\mu}_{\lambda\nu}(x) \tag{1.44}$$

similarly, we can find the product covariant derivative:

$$D_{\nu}\left(A_{\mu}B_{\lambda}\right) = \left(D_{\nu}A_{\mu}\right)B_{\lambda} + A_{\mu}\left(D_{\nu}B_{\lambda}\right) \tag{1.45}$$

$$= \partial_{\nu}A_{\mu}B_{\lambda} - \Gamma^{\rho}_{\nu\mu}A_{\rho}B_{\lambda} + A_{\mu}\partial_{\nu}B_{\lambda} - A_{\mu}\Gamma^{\rho}_{\nu\lambda}B_{\rho} \tag{1.46}$$

this allows us to write for two covariants tensors:

$$D_{\nu}\left(T_{\mu\lambda}\right) = \partial_{\nu}T_{\mu\lambda} - \Gamma^{\rho}_{\nu\mu}T_{\rho\lambda} - \Gamma^{\rho}_{\nu\lambda}T_{\mu\rho} \tag{1.47}$$

then we can do this generalization

$$D_\nu\left(T^{\mu\lambda}_{\rho}\right) = \partial_\nu T^{\mu\lambda}_{\rho} + \Gamma^{\mu}_{\sigma\nu} T^{\sigma\lambda}_{\rho} + \Gamma^{\lambda}_{\sigma\nu} T^{\mu\sigma}_{\rho} - \Gamma^{\sigma}_{\rho\nu} T^{\mu\lambda}_{\sigma} \tag{1.48}$$

and for mertic tensor:

$$D_\nu\left(g_{\mu\lambda}\right) = \partial_\nu g_{\mu\lambda} - \Gamma^{\rho}_{\nu\mu} g_{\rho\lambda} - \Gamma^{\rho}_{\nu\lambda} g_{\mu\rho} = 0 \tag{1.49}$$

where (see annex):

$$\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu} \tag{1.50}$$

$$\Gamma^{\lambda}_{\mu\nu} = g^{\lambda\rho}\Gamma_{\mu\nu,\lambda} = g^{\lambda\rho}\frac{1}{2}\left[\partial_\nu g_{\mu\rho} + \partial_\mu g_{\nu\rho} - \partial_\rho g_{\mu\nu}\right] \tag{1.51}$$

This formula is one of the most important in this subject. It is known by different names, like the Christoffel connection, the Levi-Civita connection, and sometimes the Riemannian connection. The associated connection coefficients are sometimes called Christoffel symbols and are written as $\left\{ \begin{array}{c} \sigma \\ \mu\nu \end{array} \right\}$.

1.5 Principle of General Relativity

1.5.1 Postulate

All physical laws are the same in all curvilinear coordinate systems (their laws are in the form of tensor equations). In the absence of gravitational fields, its laws are reduced to those of special relativity.

1.5.2 Generalized Law of Inertia

In special relativity we have

$$\frac{dU^\mu}{ds} = 0 \tag{1.52}$$

as we know that

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = c^2dt^2 - d\overrightarrow{r}^2 \tag{1.53}$$

and

$$\frac{dU^\mu}{ds} = 0 \Leftrightarrow \frac{d\overrightarrow{U}}{ds} = 0 \quad \underset{\text{GR}}{\Leftrightarrow} \quad \frac{DU^\mu}{ds} = 0 \tag{1.54}$$

this is called the Law of generalized inertia. And the eq. (1.54) can be written as follows

$$\frac{DU^\mu}{ds} = 0 \to dx^\nu \frac{D_\nu U^\mu}{ds} = \frac{dx^\nu}{ds} D_\nu U^\mu = U^\nu D_\nu U^\mu \tag{1.55}$$

Assuming that $ds^2 > 0$, we have

$$\frac{DU^\mu}{ds} = 0 \to U^\nu D_\nu U^\mu = 0$$

$$\to U^\nu \partial_\nu U^\mu + U^\nu \Gamma^\mu_{\lambda\nu}(x) U^\lambda$$

$$\Rightarrow \frac{dU^\mu}{ds} = -\Gamma^\mu_{\lambda\nu}(x) U^\lambda U^\nu \tag{1.56}$$

This is the equation of geodesic motion, it is the shortest path between two given points in a curved space, and that can be represented by the following expression

$$\frac{d^2x^\mu}{ds} + \Gamma^\mu_{\lambda\nu}(x) \frac{dx^\lambda}{ds} \frac{dx^\nu}{ds} = 0 \tag{1.57}$$

1.5.3 Limitations

Consider the case of a weak static field (such as, to a good approximation, that of the Sun) and a particle moving slowly in it ($v << c$). With $x^0 = ct, x^1 = x, x^2 = y, x^3 = z$, an inertial frame is one in which the metric tensor is

$$g_{\mu\nu} = \eta_{\mu\nu} = diag(-1, 1, 1, 1) \tag{1.58}$$

so that $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$. A weak field is one for which

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{1.59}$$

by assuming that

$$|h_{\mu\nu}| << 1 \tag{1.60}$$

each element of $g_{\mu\nu}$ is close to its inertial value. Non-relativistic motion, on the other hand, implies that

$$\tau \approx t, \frac{dx^0}{d\tau} \approx c, \frac{dx^i}{d\tau} \approx v^i << c \tag{1.61}$$

so the geodesic equation becomes

$$\frac{d^2x^i}{dt^2} = -c^2\Gamma^i_{00} \tag{1.62}$$

then the right hand side represents the "gravitational force", which gives the particle its acceleration. Where

$$\begin{aligned} U^0 &= \tfrac{1}{\sqrt{1-\beta^2}} = 1 \\ U^i &= \tfrac{v^i}{\sqrt{1-\beta^2}} \simeq v^i \end{aligned} \tag{1.63}$$

$$a^i = \frac{dv^i}{dt} \simeq -\Gamma^i_{00}U^0U^0 - 2\Gamma^i_{0j}U^0U^j - \Gamma^i_{jk}U^jU^k = \frac{1}{2}\eta^{ij}\partial_j h_{00} \tag{1.64}$$

In the end we can write

$$\frac{d\overrightarrow{v}}{dt} = \frac{c^2}{2}\overrightarrow{\nabla}h_{00} \tag{1.65}$$

this is to be compared with Newton's equation

$$\frac{d\overrightarrow{v}}{dt} = g = -\overrightarrow{\nabla}\phi \tag{1.66}$$

where ϕ is the gravitational potential. Comparison of eq. (1.65) and eq. (1.66) gives

$$h_{00} = -\frac{2\phi}{c^2} \tag{1.67}$$

and we have $g_{00} = \eta_{00} + h_{00} = -1 + h_{00} = -(1 + \frac{2\phi}{c^2})$. We have found one component of the metric tensor $g_{\mu\nu}$, Actually, this is all we can find by comparing Einstein's theory with Newton's. At a distance r from a gravitating body of mass M, we have $\phi = -\frac{MG}{r}$, is the Newtonian potential so

$$g_{00} = -(1 - 2\frac{GM}{rc^2}) \tag{1.68}$$

we have a potential of the following form:

$$g_{ij} = \delta_{ij}\left(1 - 2\frac{GM}{r}\right) \tag{1.69}$$

Example: Table (1.2) shows the value of $2\frac{GM}{R}$ on the different surfaces.

$2\frac{GM}{R}$	on the surface of
10^{-39}	proton
10^{-9}	Earth
10^{-6}	Sun
10^{-4}	white dwarf
1	Black hole

Table 1.2: The $2\dfrac{GM}{R}$ value of the various objects

1.5.4 Curvature Tensor

Riemann Tensor

$R^{\nu}_{\rho\mu\lambda}$: Riemann tensor, defined by (see Annex)

$$R^{\nu}_{\rho\mu\lambda} = \left[\partial_{\lambda}\Gamma^{\nu}_{\rho\mu} - \partial_{\mu}\Gamma^{\nu}_{\rho\lambda} + \Gamma^{\nu}_{\sigma\lambda}\Gamma^{\sigma}_{\rho\mu} - \Gamma^{\nu}_{\sigma\mu}\Gamma^{\sigma}_{\rho\lambda}\right] \tag{1.70}$$

eq. (1.70) is a mixed tensor, calculated in terms of the metric thanks to Christoffel symbol Γ, $R^{\nu}_{\rho\mu\lambda}$ is called Riemann curvature tensor or Riemann–Christoffel tensor, all information on the curvature of a manifold is contained in this tensor. Here are some algebraic properties:

1- $R^{\nu}_{\rho\mu\lambda} = -R^{\nu}_{\rho\lambda\mu}$

2- $R^{\mu}_{\rho\nu\lambda} + R^{\mu}_{\nu\lambda\rho} + R^{\mu}_{\lambda\rho\nu} = 0$

3- $R_{\mu\nu\rho\lambda} = -R_{\nu\mu\rho\lambda}$ $(R_{\sigma\rho\mu\lambda} = g_{\nu\sigma}R^{\nu}_{\rho\mu\lambda})$

4- $R_{\mu\nu\rho\lambda} = R_{\rho\lambda\mu\nu}$

Of these 4 properties, it remains $\frac{n^2\left(n^2-1\right)}{12} \rightarrow \frac{16(16-1)}{12} = 20$

Ricci Tensor

$$R_{\mu\nu} = R^{\lambda}_{\mu\nu\lambda} = g^{\lambda\rho} R_{\rho\mu\nu\lambda} \tag{1.71}$$

it is a symmetric tensor, which gives $\frac{n(n+1)}{2}$ components. On the other hand, we have the Bianchi identity

$$D_{\mu} R^{\rho}_{\sigma\nu\lambda} + D_{\nu} R^{\rho}_{\sigma\lambda\mu} + D_{\lambda} R^{\rho}_{\sigma\nu\mu} = 0 \tag{1.72}$$

and (see Annex)

$$D_{\nu} \left(R^{\nu}{}_{\mu} - \frac{1}{2} R \delta^{\nu}_{\mu} \right) = 0 \tag{1.73}$$

The Bianchi identities are interesting in their own right, but they are introduced here only because of their usefulness in the context of Einstein's equations for the gravitational field. They are identities involving the covariant derivatives of the Riemann

$$\begin{gathered} S_{\mu\nu} = \left(R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} \right) \\ D_{\nu} S^{\nu}_{\mu} = 0 \end{gathered} \qquad S_{\mu\nu} : \text{Einstein tensor} \tag{1.74}$$

Einstein Equations :

a) Energy–Momentum Tensor

In the case of a continuous distribution of the matter and without pressure

$$T^{\mu\nu} = \mu(x) U^{\mu} U^{\nu} \tag{1.75}$$

where μ is mass density, $T^{\mu\nu}$ Tension tensor is a symmetric tensor, and all these properties are for the perfect fluid. Case of a perfect fluid with pressure $T^{\mu\nu} = (\mu + p) U^{\mu} U^{\nu} - p g^{\mu\nu}$, and for non-relativistic case

$$T^{00} = \mu + p - p = \mu$$
$$U^i = 0, U^0 = 1 \rightarrow \quad T^{0i} = T^{i0} = 0$$
$$T^{ij} = p\delta^{ij} \qquad \text{because} \ - pg^{ij} = +p\delta^{ijij}$$

b- Equations:

$S_{\mu\nu}$ is a symmetric tensor in function of 2^{sd} derivatives of $g_{\mu\nu}$, and it is satisfied

$$S_{\mu\nu} = \kappa T_{\mu\nu} \tag{1.76}$$

we can write

$$S_{\mu\nu} = aR_{\mu\nu} + bRg_{\mu\nu} + cg_{\mu\nu} \tag{1.77}$$

and according to continuity condition $\Rightarrow$

$$S_{\mu\nu} = \left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \lambda g_{\mu\nu}\right] \tag{1.78}$$

so the first Einstein equation is

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{1.79}$$

Chapter 2

Extra-Dimensions

2.1 Introduction

In the early twentieth century, the mathematician and physicist Theodor Kaluza and the theoretical physicist Oskar Klein advanced the first idea about the space–time could have more than four dimensions. By publishing a paper [9], Kaluza extended Einstein's theory of general relativity from four to five dimensions, which is still the most well-known description of gravitation. Klein supposed in 1926 that an extra fourth spatial dimension is wrapped up into a circle with a very small radius, the extra-dimension bends around on itself, where it can be proven that 5D space–time can be divided into Einstein's four-dimensional gravitational theory and Maxwell's electromagnetism theory. As a result, a Kaluza–Klein theory (KK theory) is a scientific model that attempts to unify electromagnetism with Einstein's gravity.

There has been a growing believe that extra-dimensions may potentially play a key role in resolving a number of other pressing phenomenological issues, such as, the cosmological constant's smallness [10], hierarchy problems [11], accelerated expansion of the universe.

Observational cosmology indicates that our universe is experiencing a large scale accelerated expansion. This was first observed from the high redshift supernova Ia [18, 20], [22], and later confirmed by cross-checks with cosmic microwave background radiation [12], [21]. The expansion rate was explained in the cosmological standard model by the addition of dark energy, which has a negative pressure. However, the nature of dark energy as well as dark matter is yet unknown as long as the solution is not yet obtained in the context of Standard General Relativity. This leads to suggesting many different models, such as quintessence [23], [26]. Bianchi type-I cosmological models with time dependent deceleration parameter [24] , Kantowski-Sachs model with viscous fluid in presence of cosmological term Λ [25] and F(R) model [27], Recently, a new model has been proposed that unifies dark energy and dark matter into a single phenomenon, a fluid that possesses "negative mass" [91], and other works [28, 31].

Motivated by the works discussed above, in this chapter, we will focus on the possibility that extra-dimensions could provide a solution to the universe's accelerated expansion without mentioning the concept of dark energy. where we will discuss three models in five dimensions. The Friedmann Robertson-Walker model with perfect fluid and non-perfect fluid, Kantowski-Sachs space-time and F (R) gravity, by using dynamical study.

2.2 Friedmann-Robertson-Walker Model in Five Dimensions

2.2.1 Introduction

Robertson-Walker (RW) metric describes a homogeneous (all places look the same), isotropic (all directions look the same), expanding or contracting (observations of distant galaxies and quasars have shown the universe to be approximately homogeneous and isotropic on spatial scales larger than a few hundred million light years). Because of homogeneity and isotropy, the curvature of space-time must be the same everywhere and in every direction, allowing the metric to be given in the form

$$ds^2 = -c^2dt + R^2(t)(\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2) \tag{2.1}$$

where $R(t)$ is scale factor, k is a constant that represents the space's curvature.

For flat non-empty space $k = 0$. In this case, space is flat but expanding or contracting. The RW metric can be reduced to a form that is similar to the Minkowski metric. In this universe, parallel lines stay parallel, and a triangle is still 180. It holds the Pythagorean theorem (see the figure) as the $3D$ spatial hypersurface still has euclidean geometry, but the $4D$ space-time is still curved, so there is gravity (matter and radiation) as $4D$ space is curved, but $3D$ space is flat. To the best of our knowledge, this is the universe we live in now. From WMAP (Wilkinson Microwave Anisotropy Probe) $K = 0$ to approximately 0.4% (https://map.gsfc.nasa.gov).

For closed space $k = 1$**.** In this case, space has a positive curvature. In this universe, traveling on a straight line would eventually travel back to points visited before.

For open space $k = -1$. In this case, space has a negative curvature. In this universe, a triangle would be less than 180 degrees.

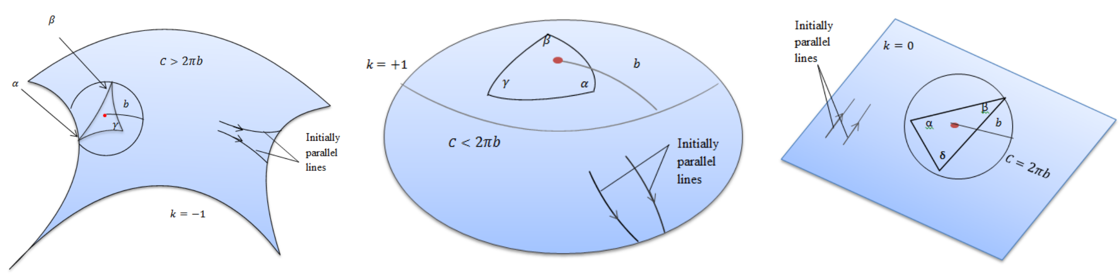


Figure 2-1: Shape of the Universe

2.2.2 RW Metric in 5 Dimensions

We choose the metric in five dimensions that takes this form

$$ds^2 = dt^2 - R^2(t)[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)] - A^2(t)dy^2 \tag{2.2}$$

where $A(t)$ is a scale factor of the extra-dimension, y is the fifth coordinate.

2.2.3 General Form of Einstein Equations in 5 Dimensions

The Einstein field equations in five dimensions of the form:

$$G_A^B = \overline{k} T_A^B \tag{2.3}$$

where A and B are indices which run over all space-time dimensions, in this chapter, we set the higher dimensional coupling constant equal to one ($\overline{k} = 1$). The higher dimensional stress-energy tensor will be transformed into

$$T_A^B = diag[\rho(t), -p(t), -p(t), -p(t), -p_5(t)] \tag{2.4}$$

where $p_5(t)$ is the pressure in the extra dimension. This stress-energy describes a homogenous, isotropic perfect fluid in five dimensions.

2.2.4 Friedmann-Robertson-Walker (FRW) Equations

By adopting the metric above and the perfect fluid stress-energy tensor, the 5D FRW field equations are of the form

$$\rho = 3\frac{\dot{R}^2}{R^2} + 3\frac{k}{R^2} + 3\frac{\dot{R}\dot{A}}{RA} \tag{2.5}$$

$$p = -[2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2} + \frac{\ddot{A}}{A} + 2\frac{\dot{R}\dot{A}}{RA}] \tag{2.6}$$

$$p_5 = -3(\frac{\ddot{R}}{R} + \frac{k}{R^2} + \frac{\dot{R}^2}{R^2}) \tag{2.7}$$

and the 5D conservation equation

$$0 = \dot{\rho} + 3\frac{\dot{R}}{R}(\rho + p) + \frac{\dot{A}}{A}(\rho + p_5) \tag{2.8}$$

where a dot denotes a time derivative, you can easily show that the conservation equation is in fact satisfied when eqs. (2.6) and (2.7) are employed. We have three unique equations and five unknowns. This is analogous to Standard $4D$, FRW cosmology, where one usually adopts an equation state relating to the pressure to proceed. Here, we adopt two equations of state of the form

$$p = w\rho \tag{2.9}$$

$$p_5 = \gamma\rho \tag{2.10}$$

where w and γ can in general be time-dependent but, in this case, are constants. We consider a flat space-time $k = 0$, with a constant speed of expansion in the extra dimension $\ddot{A} = 0$. Then, we have the following equation

$$p = -[2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{\ddot{A}}{A} + 2\frac{\dot{R}\dot{A}}{RA}] \tag{2.11}$$

By using the eqs. (2.9) and (2.11) we find the following Ricatti non-linear equation

$$\dot{H} + \frac{3}{2}(1+w)H^2 + (\frac{2+3w}{2})H\frac{\dot{A}}{A} = 0 \tag{2.12}$$

if $\ddot{A} = 0 \Leftrightarrow A = c_1 t + c_0$, then

$$\dot{H} + \frac{3}{2}(1+w)H^2 + (\frac{2+3w}{2})(\frac{c_1}{c_1 t + c_0})H = 0 \tag{2.13}$$

where c_1 and c_0 are the integration constants, and $H = \frac{\dot{R}}{R}$ is the Hubble parameter. The solution of the form (the solution of Bernoulli)

$$H(t) = \frac{(2+3w)H_0}{-(t-t_0)3H_0(1+w) + (2+3w)}$$

we assumed that $c_0 = 0$ and $c_1 = 1$. For the current universe, the Hubble constant is H_0. When we put $H_0 t_0 = \tau = 1$, $\frac{t}{t_0} = \widehat{t}$ and $\widehat{H} = \frac{H}{H_0}$, we get

$$\widehat{H} = \frac{2+3w}{-3(\widehat{t}-1)(1+w) + (2+3w)} \tag{2.14}$$

The variation of $\widehat{H}$ as a function of $\widehat{t}$ is shown in Fig (2-2) for $w = 0$, $w = \frac{1}{3}$.

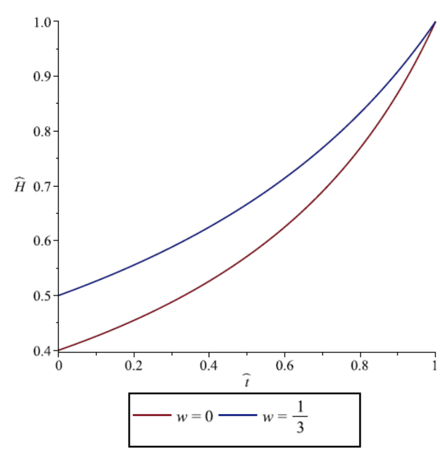


Figure 2-2: Variation as a $\hat{H}$ in function of $\widehat{t}$

· Notice that the expansion rate increase with time but when $w = 0$, the cosmological fluid is dominated by matter stays less than the case of $w = \frac{1}{3}$, where the cosmological fluid is dominated by radiation. And for $w = -1 \Leftrightarrow \hat{H} = 1 \Rightarrow H = H_0$, the expansion rate stays constant which is explained by dark energy. On the other hand we have

$$\frac{dH}{dt} = -H^2(1+q) \tag{2.15}$$

where q the deceleration parameter $q = \frac{\frac{dH}{dt}+H^2}{-H^2}$,notice that $H > 0$, and $\frac{dH}{dt} > 0$, then $q < 0$, we can say that the universe is in accelerated expansion. Table (2.1) shows the behavior of H and q with varying w

w	$w > \frac{-2}{3}$	$\frac{-5+3\hat{t}}{3\hat{t}+6} < w < \frac{-2}{3}$
H	$+$	$-$
q	$-$	$+$
The nature of movement	accelerated expansion	decelerated contraction

Table 2.1: H and q behavior , with varying w

2.2.5 Dynamical Study

Dynamical systems theory is an area of mathematics used to describe the behavior of complex dynamical systems. We call a dynamic system any system that develops with time or whose development depends on time. We will write the Friedmann equations in terms of the Hubble parameters H_R and H_A, according to eqs. (2.5), (2.6) and (2.7), with the relations $p = w\rho$, $p_5 = \gamma\rho$, we find:

$$\dot{H}_A = (\frac{2\gamma+1}{3} - w)\rho - H_A^2 - 3H_AH_R \tag{2.16}$$

and

$$\dot{H}_R = -(1+\gamma)\frac{\rho}{3} - H_R^2 + H_AH_R \tag{2.17}$$

according the conservation equation

$$\dot{\rho} = -[(3H_R(1+w) + H_A(1+\gamma)]\rho \tag{2.18}$$

the quantities H_R and H_A are the Hubble parameters of the $4D$ universe and $1D$ extra-dimensional space, respectively. If we take as dynamical variables ρ, H_R and H_A,the analysis leads to the following cases:

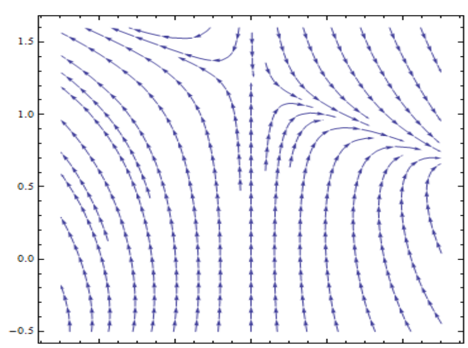
1) if $\gamma = -1$ and $w = -1$, one has the following critical points:

for $\rho = 0$, $H_R = 0$, $H_A = 0$, which corresponds to a flat and static space for both $4D$ and $1D$ extra-dimensional space.

for $H_R = 0, H_A = 1.101$, such that $\rho = 3\frac{H_A^2}{2} = 1.819$, it corresponds to static space for the $4D$ universe and an accelerated $1D$ extra-dimensional space. The matrix of stability is

$$M_1 \approx \begin{pmatrix} 1.101 & 0 \\ -3.304 & -2.202 \end{pmatrix} \tag{2.19}$$

Fig (2-9) displays the phase portrait (see annex) for critical point $\{(H_R, H_A) = (0, 1.101)\}$, we have a "saddle node point" (By using Mathematica program):



· $H_R = 0.545, H_A = 0.545$, such that $\rho = 6H_A^2 = 1.787$, it corresponds to a flat space

Figure 2-3: Phase portrait for (H_R, H_A)

and an accelerated for 4D universe and 1D extra dimensional space and we have the matrix of stability is:

$$M_2 \approx \begin{pmatrix} -0.545 & 0.545 \\ -1.637 & -2.729 \end{pmatrix} \tag{2.20}$$

Fig (2-4) displays the phase portrait for critical point $\{(H_R, H_A) = (0.545, 0.545)\}$ which is "stable nodal sink".

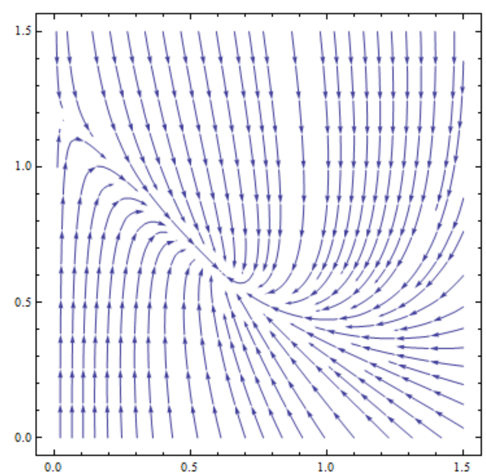


Figure 2-4: Phase portrait for (H_R, H_A)

$\cdot$ $H_R = 0, H_A = -0.63$, such that $\rho = \frac{3}{2}H_A^2 = 0.5954$, it corresponds to static space for 4D universe and an contracted 1D extra dimensional space and we have the matrix of stability is:

$$M_3 \approx \begin{pmatrix} -0.63 & 0 \\ 1.89 & 1.26 \end{pmatrix}$$

FIG (2-5) displays the phase portrait for critical point $\{(H_R, H_A) = (0, -0.63)\}$ which is "saddle node point ".

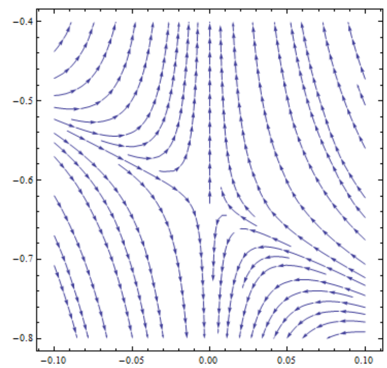


Figure 2-5: Phase portrait for (H_R, H_A)

· $H_R = -0.317, H_A = -0.317$, such that $\rho = 6H_A^2 = 0.606$, it corresponds to a contracted space both 4D and 1D universe and we have the matrix of stability is:

$$M_4 \approx \begin{pmatrix} 0.317 & -0.317 \\ 0.953 & 1.589 \end{pmatrix}$$

FIG (2-6) displays the phase portrait for critical point $\{(H_R, H_A) = (-0.317, -0.317)\}$ which is "unstable nodal source ".

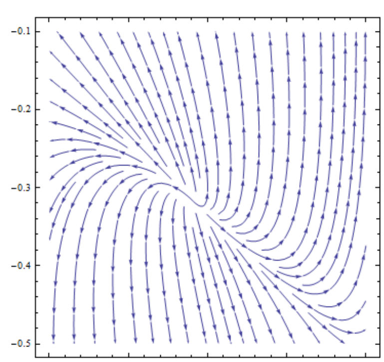


Figure 2-6: Phase portrait for (H_R, H_A)

2.2.6 Friedmann Equation with Shear Viscosity

Introduction

Based upon experience in fluid mechanisms, scientists expected that the viscosity concept would be important in cosmology. In the early universe, viscosity may arise due to various processes such as the decoupling of neutrinos during the radiation reaction [32, 33]. And viscosity mechanisms in cosmology can explain the anomalously high entropy of the present universe [15, 16].

The first suggestion in viscosity theory was made by Eckart [13] and it was developed by [14], where they studied the effect of energy dissipation, occurring during the motion of a fluid, on that motion itself. From these results, S. Floerchinger, N.Tetradis, A.Wiedemann [17] proposed a paper showing the shear and bulk viscosity could account for the acceleration of the cosmological expansion.

In this part, we tried to see the effect of shear viscosity on acceleration expansion with the FRW model in five dimensions.

FRW Equations

We consider $\overline{p} = p + h(t)H_R$, equations of Friedmann become

$$\rho = 3\frac{\dot{R}^2}{R^2} + 3\frac{k}{R^2} + \frac{\dot{R}\dot{A}}{RA} \tag{2.21}$$

$$p = -[2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2} + \frac{\ddot{A}}{A} + 2\frac{\dot{R}\dot{A}}{RA}] - h(t)H_R \tag{2.22}$$

$$p_5 = -3(\frac{\ddot{R}}{R} + \frac{k}{R^2} + \frac{\dot{R}^2}{R^2}) \tag{2.23}$$

we consider a space-time flat $(k = 0)$ and $h(t) = \alpha H_R$ with a constant speed of expansion in extra dimension $(\ddot{A} = 0)$, and we find the following Ricatti non-linear equation using the equations $p = w\rho$, $p_5 = \gamma\rho$ and eq. (2.22)

$$\dot{H} + \frac{3}{2}(\alpha + w)H^2 + (\frac{2+3w}{2})H\frac{\dot{A}}{A} = 0 \tag{2.24}$$

if $\ddot{A} = 0 \Leftrightarrow A = ct + c_0$, then

$$\dot{H} + \frac{3}{2}(\alpha + w)H^2 + (\frac{2+3w}{2})(\frac{c_1}{c_1 t + c_0})H = 0 \tag{2.25}$$

where $c_1$and c_0 are the integration constants and $H = \frac{\dot{R}}{R}$ is the Hubble parameter. The form of the equation is (solution of Bernoulli)

$$H(t) = \frac{(2+3w)H_0}{-(t-t_0)3H_0(\alpha + w) + (2+3w)} \tag{2.26}$$

where H_0 denotes the Hubble constant for the current universe. If $H_0 t_0 = \tau = 1$, $\frac{t}{t_0} = \widehat{t}$, $\hat{H} = \frac{H}{H_0}$, then

$$\hat{H} = \frac{2+3w}{-3(\widehat{t} - 1)(1 + \frac{\alpha}{2} + w) + (2+3w)} \tag{2.27}$$

we attempt to obtain α in the function of w, t . From eq. (2.26), we find

$$\alpha = \frac{(\hat{t} - 1)3(1+w) - (2+3w)}{-\frac{3}{2}(\hat{t} - 1)} \tag{2.28}$$

then we obtain the following equation

$$\hat{H} = \frac{2+3w}{-9(\widehat{t}-1)(1+w)+2(2+3w)} \tag{2.29}$$

Fig (2-7) shows the variation of $\hat{H}$ as a function of $\widehat{t}$, notice that when $w = 0$, the cosmological fluid is dominated by matter, notice that the expansion rate increase with time but it stays less than the case of $w = \frac{1}{3}$, where the cosmological fluid is dominated by radiation. We see the same behavior as in the previous graph but it stays less (see fig (2-2))

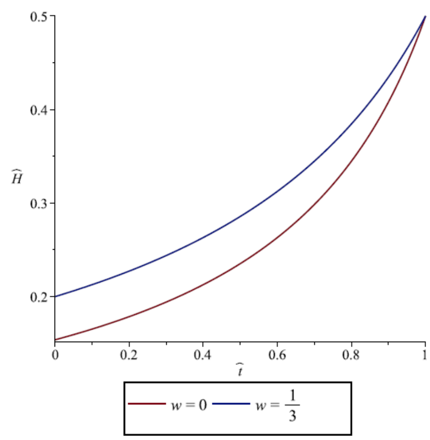


Figure 2-7: Variation of $\hat{H}$ as a function of $\widehat{t}$

In the case where $w = -1 \Leftrightarrow \hat{H} = \frac{1}{2} \Rightarrow 2\hat{H} = H_0$, the expansion rate stays constant which is explained by dark energy. From eq. (2.15), notice that $H > 0$, and $\frac{dH}{dt} > 0$, notice that $H > 0$ and $\frac{dH}{dt} > 0$, then $q < 0$, we can say that the universe is experiencing accelerated expansion.

2.2.7 Dynamical Study

The same way leads us to find these dynamical equations (by considering $h(t) = \alpha H_R$)

$$\dot{H}_A = (\frac{2\gamma+1}{3} - w)\rho - H_A^2 - 3H_A H_R - \alpha H_R \tag{2.30}$$

$$\dot{H}_R = -(1+\gamma)\frac{\rho}{3} - H_R^2 + H_A H_R \tag{2.31}$$

$$\dot{\rho} = -[(3H_R(1+w) + H_A(1+\gamma)]\rho - 3\alpha H_R^2 \tag{2.32}$$

Finally, we obtain these critical points.

$$H_R = -0.5\gamma^2 \frac{\alpha}{2+\gamma^2-3\gamma w}, H_A = 0.5\frac{\alpha\gamma(2+\gamma)}{2+\gamma^2-3\gamma w}, \rho = -\frac{1.5\alpha^2\gamma^3}{(2+\gamma^2-3\gamma w)^2} \tag{2.33}$$

and

$$H_R = 0, H_A = 0, \rho = 0 \tag{2.34}$$

for this point, we have a static space in both 4D and 1D extra-dimensional space.

Discussion In the case eq. (2.33), where critical points are defined if

$$2+\gamma^2-3\gamma w \neq 0 \tag{2.35}$$

the region($w < 0$) gives us value negative of pressure (dark energy). For accelerated expansion in 4 dimensions, for the first point, it must

$$\begin{cases} \alpha < 0 \\ 2+\gamma^2-3\gamma w > 0 \end{cases} \tag{2.36}$$

for positive values of density it must $\gamma < 0$, by using the eq. (2.35), we obtain Fig (2-8) which displays the allowed values of w and γ.

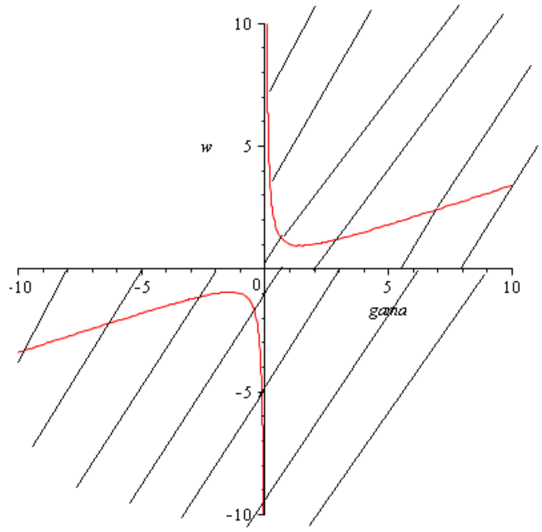


Figure 2-8: w as a function of γ

For $\gamma = -1, \alpha = -1$, For defined eigenvalues, it must be $w > -1$, we obtain this phase portrait (Nodal Sink) see Fig (2-9).

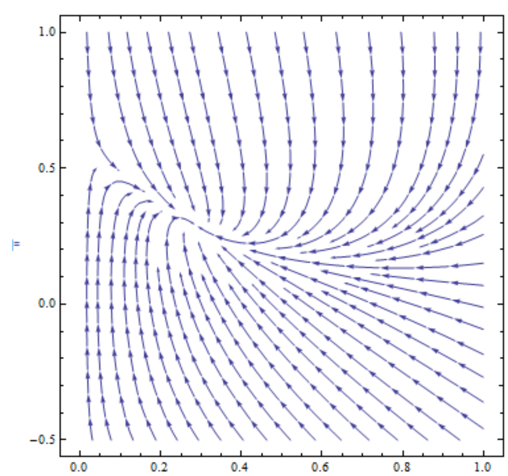


Figure 2-9: Phase portrait

2.3 Dynamical Study of Extra-Dimensional Kantowski-Sachs Space-Time

Introduction

The Kantowski–Sachs (KS) space-time model is the most widely used anisotropic model, and it was proposed by Ronald Kantowski and Rainer Kurt Sachs [34]. The KS model provided a solution to the Einstein field equations for dust space-times, and after advances in particle physics applied to the early universe, the KS model returned to scientific interest. A set of articles developed for the new KS models has appeared [35], [37], where anisotropic fluid developed by Gergely (1999), exotic fluid in (2002), and on the other hand, the study of Mendez and Henriquez [36] that treats an inflationary era for the KS model. K.Adhav et al [38], they found an exact solution of the Einstein field equations for dark energy in the KS metric with anisotropic fluid. V. B. Raut et al [39] investigated the KS cosmological model in the F(R) theory of gravity with an anisotropic fluid. Driven by these works, we will discuss the KS space-time with cosmological constant in 5 dimensions, where the KS metric takes the following form

$$ds^2 = dt^2 - A^2(t)dr^2 - B^2(t)(d\theta^2 + \sin^2\theta d\varphi^2) - C^2(t)dy^2 \tag{2.37}$$

Friedmann equation

$$\begin{aligned}
\frac{1}{B^2}+\frac{\dot{B}^2}{B^2}+2\frac{\dot{C}\dot{B}}{CB}+2\frac{\dot{B}\dot{A}}{BA}+\frac{\dot{C}\dot{A}}{CA}+\Lambda &= \rho \qquad (2.38)\\
\frac{1}{B^2}+\frac{\dot{B}^2}{B^2}+2\frac{\dot{C}\dot{B}}{CB}+2\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}+\Lambda &= -p\\
\frac{\dot{B}\dot{A}}{BA}+\frac{\dot{C}\dot{A}}{CA}+\frac{\dot{C}\dot{B}}{CB}+\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}+\frac{\ddot{A}}{A}+\Lambda &= -p\\
\frac{1}{B^2}+\frac{\dot{B}^2}{B^2}+2\frac{\dot{B}\dot{A}}{BA}+2\frac{\ddot{B}}{B}+\frac{\ddot{A}}{A}+\Lambda &= -p_5
\end{aligned}$$

where Λ is cosmological constant.

2.3.1 Dynamical Study

Direct simplifications give the following autonomous non-linear differential equations

$$\begin{aligned}
\dot{\rho} &= -[(H_A+2H_B)(\rho+p)+H_C(\rho+p_5)] \qquad (2.39)\\
\dot{H}_A &= -H_A^2+\frac{1}{3}\rho-\frac{1}{3}p_5-2H_BH_A-H_CH_A-\frac{2}{3}\Lambda\\
\dot{H}_B &= -H_B^2-\frac{2}{3}\rho-\frac{1}{3}p_5+\frac{1}{3}H_CH_B+\frac{1}{3}H_BH_A+H_CH_A+\frac{1}{3}\Lambda\\
\dot{H}_C &= -H_C^2+\frac{1}{3}\rho-p+\frac{2}{3}p_5-2H_CH_B-H_CH_A+\frac{2}{3}\Lambda
\end{aligned}$$

where $H_A=\frac{\dot{A}}{A}$, $H_B=\frac{\dot{B}}{B}$, $H_C=\frac{\dot{C}}{C}$. For $\gamma=-1$, we find these critical points

$$\{\rho=\Lambda, \quad H_A=0, \quad H_B=0, \quad H_C=-0.57735\sqrt{\Lambda-3\Lambda w}\} \qquad (2.40)$$

$$\{\rho=\Lambda, \quad H_A=0, \quad H_B=0, \quad H_C=0.57735\sqrt{\Lambda-3\Lambda w}\} \qquad (2.41)$$

$$\begin{aligned}\{\rho &= \frac{9\Lambda}{7+6w}, \quad H_A = -\frac{1.0328\sqrt{\Lambda - 3\Lambda w}}{\sqrt{7+6w}}, \\ H_B &= \frac{0.516398\sqrt{\Lambda - 3\Lambda w}}{\sqrt{7+6w}}, \quad H_C = -\frac{1.29099\sqrt{-\Lambda(-1+3w)}}{\sqrt{7+6w}}\}\end{aligned} \tag{2.42}$$

$$\begin{aligned}\{\rho &= \frac{9\Lambda}{7+6w}, \quad H_A = \frac{1.0328\sqrt{\Lambda - 3\Lambda w}}{\sqrt{7+6w}}, \\ H_B &= -\frac{0.516398\sqrt{-\Lambda(-1+3w)}}{\sqrt{7+6w}}, \quad H_C = \frac{1.29099\sqrt{\Lambda - 3\Lambda w}}{\sqrt{7+6w}}\}\end{aligned} \tag{2.43}$$

Discussion

For critical points define it must be $\Lambda > 0$ and

1- For first point (2.40) , It corresponds to a static universe in $4D$ and an accelerated contraction in $5D$ (extra dimension).

2- For third (2.42) and fourth (2.43) points: $-\frac{7}{6} < w \leq \frac{1}{3}$; the values between $-\frac{7}{6} < w < 0$, it is negligible because it gives us negative pressure, so we take $0 \leq w \leq \frac{1}{3}$

- If $0 \leq w < \frac{1}{3}$

For third point (2.42), it corresponds to a decelerated contraction in 4D and in 5D (extra dimension).

For fourth point (2.43), it corresponds to a 4D and 5D (extra dimension) accelerated expansion.

- If $w = \frac{1}{3}$, for all the critical points, we have a static universe in 4D and 5D (extra dimension).

For fourth point (2.43), we take $w = \frac{1}{3}, \Lambda = 1$,the Fig (2-10) displays the (H_A, H_B, H_c) phase portrait.

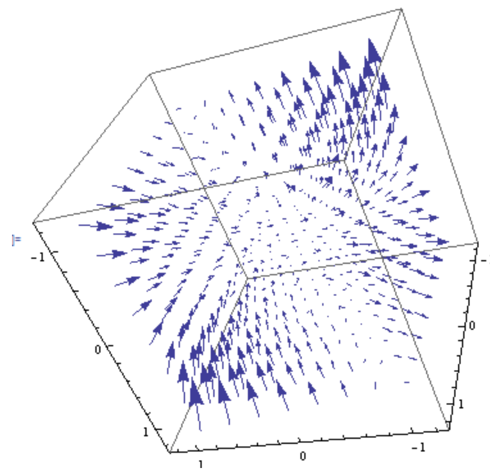


Figure 2-10: Phase portrait for (H_A, H_B, H_c)

For the point (2.43), Fig (2-11) displays the (H_A, H_B, H_c) phase portrait for $w = 0, \Lambda = 1$.

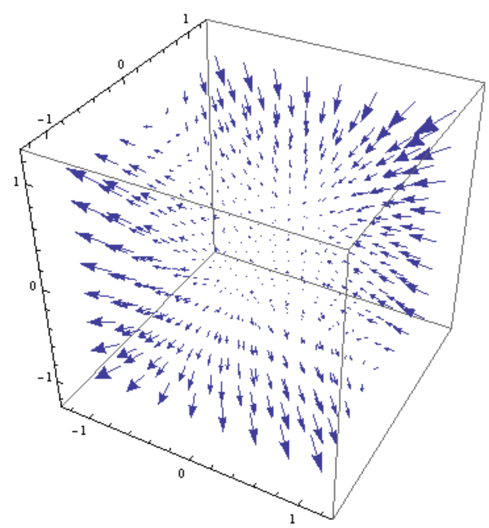


Figure 2-11: Phase portrait for (H_A, H_B, H_c)

2.4 F(R) Gravity

Introduction

Despite the fact that General Relativity (GR) has passed all observational tests so far, there have been observational suggestions that GR may need to be modified. As we pointed out before, this led to the three famous models:

1- A cosmological constant Λ

2- Dark energy

3- Modified gravity

In fact, a cosmological constant is propelling the cosmic acceleration and will eventually dominate the universe. However, the biggest problem with the cosmological constant is answering the question, why is it so small? Why is it not zero? If we accept that the cosmological constant is nonzero, how does its low value explain the current acceleration of the universe? As a result, most cosmologists reject this explanation and believe that a different explanation for cosmic acceleration must be found.

The second proposal postulates the existence of a dark energy fluid with negative pressure. Many dark energy models have been studied, none of which is totally convincing or can be demonstrated to be the correct one (sometimes cosmological constant plays the role of dark energy).

A third possibility was to dispense entirely with the mysterious dark energy and modify gravity. This third possibility was motivated by the question of what would happen if the fundamental action were different. This approach was the correct one in explaining the precession of Mercury's perihelion, not due to an unseen mass but to Einstein's modification of Newtonian gravity.

One approach in this direction is to employ what is known as modified gravity, so-called F (R). This is the simplest class of modified gravity theories. The first papers on this appeared in 1969-1970 [40]. Typically, two approaches are used to investigate F(R) theory: the metric and the Palatini approach [41]. We will not here discuss modified gravities related to the Palatini formulation. We focus on the F(R) theory of gravity using a metric approach.

Einstein gravity is modified by replacing the Ricci curvature scalar R by an arbitrary curvature function $f(R)$, where R is the curvature scalar. A natural modification is to add terms to the action that are proportional to R^n, if $n > 0$, these terms lead to modifications of the standard cosmology at early times that lead to de Sitter behavior (Starobinskii inflation [42]). For $n < 0$, such corrections become important in the late universe. In this case, it may be possible to avoid invoking a cosmological constant to explain cosmic acceleration. This explanation naturally leads to the unification of earlier and later cosmological epochs as the manifestation of different roles of gravitational terms relevant at small and large curvature, as occurs in the model with negative and positive powers of curvature[43] .

These models have raised much recent interest, so a large number of papers are proposed [44, 54]. The cosmological impact of F(R) models on acceleration will be discussed in this study. We begin by introducing a class of models with five dimensions without a cosmological constant. We start with action in F (R) gravity:

$$S = \int d^4x\sqrt{-g}[\frac{f(R)}{2k^2}] \tag{2.44}$$

the equations of motion are

$$-\frac{1}{2}f(R)g_{\mu\nu} + R_{\mu\nu}f'(R) - \nabla_\mu\nabla_\nu f'(R) + g_{\mu\nu}\Box f'(R) = k^2 T_{\mu\nu} \tag{2.45}$$

where $T_{\mu\nu}$ is the energy-momentum tensor. In this case, we chose

$$f(R) = f_0 R^n \tag{2.46}$$

f_0 and n are constant. his model has been used before, by Abdalla et al [55] and others.

The case of Einstein's gravity corresponds to $f_0 = 1, n = 1$. The Einstein field equations for the metric are given by

$$f_0 n R^{n-1}(3\dot{H}_R - \dot{H}_A - 3H_R^2 - H_A^2) + \frac{1}{2} f_0 R^n + (3H_R + H_A) f_0 n(n-1)\dot{R} R^{n-1} = k^2 \rho \quad (2.47)$$

$$\begin{aligned} & f_0 n R^{n-1}\alpha_1 - \frac{1}{2} f_0 R^n - \alpha_2 f_0 n(n-1)\dot{R} R^{n-1} \\ & -f_0 n(n-1)(n-2)\dot{R}^2 R^{n-3} - n f_0 (n-1)\ddot{R} R^{n-2} \\ = \; & k^2 p \end{aligned} \quad (2.48)$$

$$\begin{aligned} & f_0 n R^{n-1}\alpha_3 - \frac{1}{2} f_0 R^n - \alpha_4 f_0 n(n-1)\dot{R} R^{n-1} \\ & -f_0 n(n-1)(n-2)\dot{R}^2 R^{n-3} - n f_0 (n-1)\ddot{R} R^{n-2} \\ = \; & k^2 p_5 \end{aligned} \quad (2.49)$$

where

$$\begin{aligned} \alpha_1 &= (3H_R^2 + H_R + H_R H_A) \\ \alpha_2 &= (3H_R + H_A) \\ \alpha_3 &= (H_A^2 + H_A + 3H_R H_A) \\ \alpha_4 &= (3H_R + H_A) \end{aligned}$$

we take these dynamical variables $(H_R, H_A, x = \frac{\dot{R}}{R}, R)$ in order to obtain these dynamical equations

$$\dot{H}_R = -\frac{3}{2}H_R^2 + \frac{1}{2}H_R H_A + \frac{(n-2)}{4nf_0}(w-\gamma-1)R + \frac{1}{8n}R + \frac{3}{4}(H_A+H_R)(n-1)x \quad (2.50)$$

$$\dot{H}_A = -H_A^2 + \frac{3}{2}H_R^2 - \frac{3}{2}H_R H_A - \frac{(n-2)}{4nf_0}(3w-3\gamma+1)R + \frac{1}{8n}R + \frac{3}{4}(H_A+H_R)(n-1)x \quad (2.51)$$

$$\begin{aligned} \dot{x} \ &= \ -(n-1)x^2 + \frac{3}{2(n-1)}H_R^2 + \frac{3}{2(n-1)}H_R H_A \quad (2.52) \\ &+ \frac{(n-2)}{4nf_0(n-1)}(-3w-\gamma-1)R - \frac{3}{8n}R + \frac{1}{4}(9H_R - H_A)x \end{aligned}$$

$$\dot{R} = -\frac{3}{n}(1+w)RH_R - \frac{(1+\gamma)}{n}RH_A \quad (2.53)$$

where $p = w\rho$, $p_5 = \gamma\rho$, we found these critical points

$$\{H_A = 3H_R, x = -\frac{2H_R}{-1+n}, R = 0\} \quad (2.54)$$

$$\{H_A = H_R, x = \frac{0.666H_R}{-1+n}, R = 0\} \quad (2.55)$$

and

$$\begin{aligned}
\{H_A &= -\frac{3(1+w)H_R}{1+\gamma}, \\
x &= \frac{0.5H_R(G+Q)}{(1+\gamma)(2-3\gamma+\gamma^2)(-2+\gamma-3w)}, \\
R &= -\frac{3H_R^2 f_0 n(4+\gamma+3w)}{(1+\gamma)^2(-2+n)}\}
\end{aligned} \tag{2.56}$$

where

$$\begin{aligned}
G &= -4\gamma - 4\gamma^2 + 2n\gamma + 2n\gamma^2 + 4f_0n^2 \\
Q &= \gamma f_0 n^2 - 12w + 6nw + 3f_0n^2w - 12w^2 + 6nw^2
\end{aligned}$$

2.4.1 Discussion

critical points must be defined

1) for points (2.54) and (2.55) $\Rightarrow n \neq 1$. If $H_A > 0 \Rightarrow H_R > 0, \forall w$, which corresponds to an accelerated expansion of both the 4D universe and 1D extra-dimensional space. If $H_A < 0 \Rightarrow H_R < 0, \forall w$, which is a decelerated contraction of both the 4D universe and the 1D extra-dimensional space.

2) For third point (2.56) $\Rightarrow \gamma \neq -1, n \neq 1, -2+\gamma-3w \neq 0, 2-3n+n^2 \neq 0 \Rightarrow n \neq 1$ and $n \neq 2$, if $\gamma < -1$ and $w > 0$, we have an accelerated expansion in the 4D universe and a decelerated contraction in the 1D extra dimension. The Fig (2-12) displays the allowed values of w and γ.

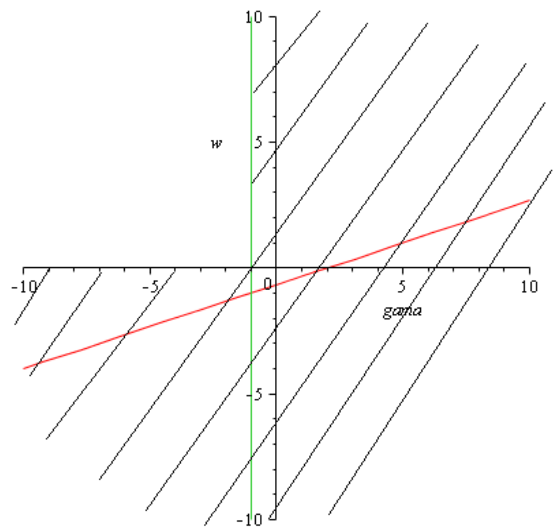


Figure 2-12: The variation of w as a function of γ

If $\gamma > -1 \Rightarrow$ we have an accelerated expansion for 4D and an extra-dimension (when $\gamma > 0$). The Fig (2-13) displays the allowed values of w and γ.

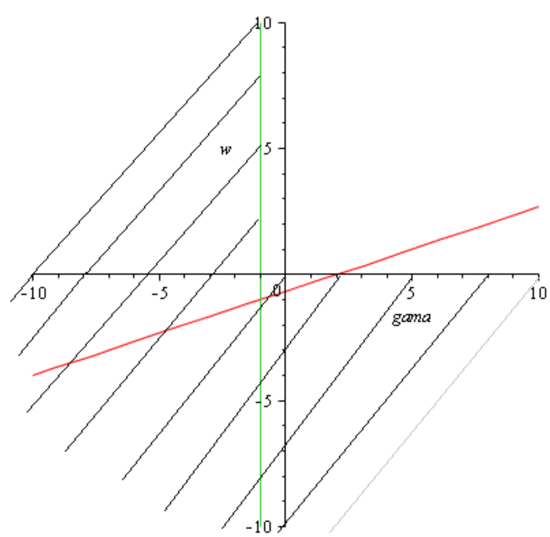


Figure 2-13: The variation of w as a function of γ

Fig (2-14) displays the $\{H_R, H_A\}$ For $\gamma = 1, w = 1, f_0 = 1, n = 3$, we find critical points $\{H_R = 1, H_A = -3\}$,Finally we obtain the saddle point

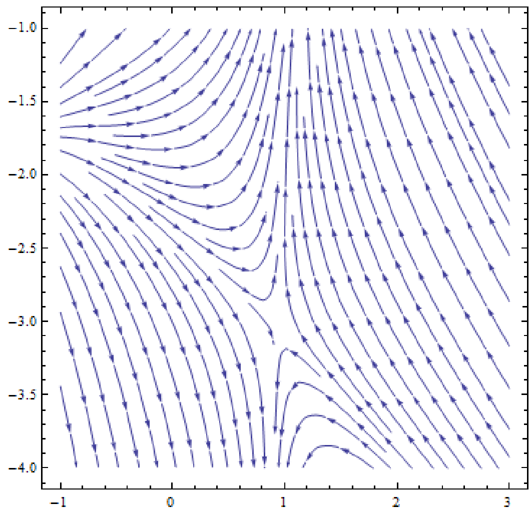


Figure 2-14: Phase portrait

Part II

Gravitational Effect on the Spin Entanglement

Chapter 3

Introduction

In 1935, a famous article was prepared by Einstein, Podolsky, and Rosen (EPR)[56]. It was designed to show that the quantum theory is incomplete. In quantum mechanics, the quantum state of a system should be understood as a catalogue of what an observer has done to the system and what has been observed. When Einstein, Podolsky, and Rosen considered a system of two particles in the state $|\phi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$, if the first subsystem is in a state of $|0\rangle$ or $|1\rangle$ with probability $\frac{1}{2}$, then the measurement made on the first particle has an impact on the second particle. The same results are obtained for the second particle. If these two particles traveled away from each other millions of light years, by measuring the first particle, if it obtained the state of $|0\rangle$ then the second one should take the state of $|1\rangle$. This looks like the state knowledge of the second particle came to the observer of the first particle faster than the speed of light. Einstein rejected this view and proposed a series of arguments to show that the quantum state is simply an incomplete characterization of a quantum system and that the universe is described by a more fundamental theory (Hidden Variables Theory).

For thirty years, the study of entanglement was ignored until the Irish physicist John Stewart Bell came along, the first one who looked at entanglement in simpler systems when he put the correlations between two-valued dynamical quantities, such as polarization or spin, of two separated systems in an entangled state. Bell designed a thought experiment showing that the quantum description of physical reality is not complete [57], Bell introduced a concept that satisfy the assumptions of local realism, and then showed that for certain quantum states they are violated. This inequality is now known as Bell's inequality. In this way, the EPR paradox has become the basis for defining the term entanglement of states, which is a type of correlation between particles that have no classical formalism.

Many physicists have used this quantum property in their work on quantum information [58, 59] Entanglement plays a vital role in quantum information and forms the basis of applications such as quantum cryptography [60, 61], teleportation [62], and quantum computation [63]. The good thing about quantum information is that it's been backed by experiments [64, 70].

Entanglement has also given us a new understanding of many physical phenomena, such as superconductivity [71], quantum phase transitions [72]. Recently, since relativity is an indispensable component of any complete theoretical model, it is convenient to study entanglement from a relativistic point of view, so the concept of quantum entanglement was applied to relativistic effects, which led to the emergence of many works, such as [73, 78],with a particular interest in describing how entanglement is perceived by different observers in relative motion, for both inertial [73, 80] and non-inertial observers [81, 82]. And several groups have focused their investigations on quantum information behavior in the proximity of black hole [83, 86], in addition, the entanglement in an expanding cosmos was also investigated [87, 88].

Aim of the work

The aim of this work is to study the spin entanglement of a system of two particles moving in a curved space-time due to a massive body. We also try to study the entanglement in different black hole metrics.

3.1 Definition of Entanglement

Entanglement is a unique form of correlation that exists only in quantum systems that can be decomposed into two or more subsystems, and an entanglement measure is a functional state quantifying the amount of entanglement across two subsystems. The quantum state of each particle cannot be described independently of the state of the others, even when the particles are separated by a large distance. Instead, a quantum state must be described for the system as a whole.

3.1.1 Pure and Mixed States

Now we can introduce a broader class of states represented by the density operator, the so-called mixed states, and pure states.

A- Pure States:

Let's begin with the pure states. If all the objects are in the same state, the ensemble is represented by a pure state. Consider an ensemble of given objects in the state $|s\rangle$. For an observable F, we have

$$F\,|\varphi_n\rangle = f_n\,|\varphi_n\rangle \tag{3.1}$$

with respect to the eigenstates of the operator F

$$|s\rangle = \sum_n C_n\,|\varphi_n\rangle \tag{3.2}$$

where

$$C_n = \langle\varphi_n\,|s\rangle \tag{3.3}$$

the probability of obtaining f_n as a result of the measure of F on the system in its state $|s\rangle$ is

$$P(f_n) = |C_n|^2 \tag{3.4}$$

by introducing the operator p_n of the nth eigenvector $P_n = |\varphi_n\rangle \langle\varphi_n|$, we can write:

$$P(f_n) = \langle p_n \rangle_s \tag{3.5}$$

the expectation value of F on the pure state $|s\rangle$ is then given:

$$\begin{aligned}
\langle F \rangle_s &= \langle s| F |s\rangle \\
&= \langle s| \sum_n C_n f_n |\varphi_n\rangle \\
&= \sum_n C_n^* \langle \varphi_n| \sum_n C_n f_n |\varphi_n\rangle \\
&= \sum_n |C_n|^2 f_n \\
&= \sum_n \langle s |\varphi_n\rangle \langle \varphi_n |s\rangle f_n \\
&= \sum_n \langle \varphi_n |s\rangle \langle s |\varphi_n\rangle f_n \\
&= \sum_n \langle \varphi_n |s\rangle \langle s| F |\varphi_n\rangle
\end{aligned}$$

thus, defining the operator

$$\rho = |s\rangle \langle s| \tag{3.6}$$

so it can be written

$$\langle F \rangle_s = \sum_n \langle \varphi_n| \rho F |\varphi_n\rangle \tag{3.7}$$

The operator eq. (3.6) represents the density operator of a pure ensemble, which allows the study of quantum systems uniformly. The density operator, then, provides a uniform procedure for calculating expectation values and the probabilities of individual returns for both pure and mixed state systems. The expression (3.7) shows that the expectation value for F can be obtained by computing the trace of ρF matrix

$$\langle F\rangle_s = Tr(\rho F) = Tr(F\rho) \tag{3.8}$$

using the eq. (3.8), we can find $P(f_n)$ in the same way

$$P(f_n) = \langle p_n\rangle_s = Tr(\rho p_n) \tag{3.9}$$

B- Mixed States:

Now let us define a more general type of states, a mixed ensemble is a set of pure ensembles, represented by their pure states $\left|s^{(i)}\right\rangle$, the average value of an observable F over a mixed state is

$$\langle F\rangle = \sum_i p_i \langle F\rangle_i \tag{3.10}$$

where $\langle F\rangle_i = \left\langle s^{(i)}\right| F \left|s^{(i)}\right\rangle$, and p_i are nonzero and satisfy the relations

$$0 \le p_i \le 1 \quad \text{and} \quad \sum_i p_i = 1 \tag{3.11}$$

p_i is probability for states $\left|s^{(i)}\right\rangle$. The probability that a measure of F over a mixed state provides eigenvalue f_n is

$$P(f_n) = \sum_i p_i P^{(i)}(f_n) \tag{3.12}$$

where $P^{(i)}(f_n) = \left|C_n^{(i)}\right|^2 = \left|\langle\varphi_n \left|s^{(i)}\right\rangle\right|^2$, and $\left|s^{(i)}\right\rangle = \sum_n C_n^{(i)} \left|\varphi_n\right\rangle$, We will now define the density operator of a mixed state by making explicit the expectation value of F in the state $\left|s^{(i)}\right\rangle$

$$\langle F\rangle_i = \left\langle s^{(i)}\right| F \left|s^{(i)}\right\rangle = \sum_n f_n \left|\langle\varphi_n \left|s^{(i)}\right\rangle\right|^2 = \sum_n f_n \langle\varphi_n \left|s^{(i)}\right\rangle \left\langle s^{(i)} \right|\varphi_n\rangle \tag{3.13}$$

which substituted in $\langle F\rangle$

$$\begin{aligned}\langle F\rangle &= \sum_i p_i \sum_n f_n \langle\varphi_n \left|s^{(i)}\right\rangle \left\langle s^{(i)} \right|\varphi_n\rangle \\ &= \sum_n \langle\varphi_n| \sum_i p_i \left|s^{(i)}\right\rangle \left\langle s^{(i)}\right| F |\varphi_n\rangle\end{aligned} \tag{3.14}$$

permits us to define the density operator of a mixed state as follows

$$\rho = \sum_i p_i \left|s^{(i)}\right\rangle \left\langle s^{(i)}\right| \tag{3.15}$$

and permits us to achieve the analogous expression of eq. (3.8) in the mixed state case

$$\langle F\rangle = \sum_n \langle\varphi_n| \rho F |\varphi_n\rangle = Tr(\rho F) = Tr(F\rho) \tag{3.16}$$

One can see in expression the (3.16) all information about the mixture is factorized and contained in ρ.We point out that, as for the density operator of a pure state, the density operator of a mixed state is defined in a unique way, since arbitrary phase factors would vanish if factorized out of the projectors $\left|s^{(i)}\right\rangle \left\langle s^{(i)}\right|$.

Properties of the density operator:

Each property is satisfied both by density operators of pure and mixed states.

1- ρ is Hermetain: $\rho^* = \rho$

2- Normalization: $Tr(\rho) = 1$

$$\begin{aligned} Tr(\rho) &= \sum_k \langle\alpha_k| \sum_i p_i \left|s^{(i)}\right\rangle \left\langle s^{(i)}\right| \alpha_k\rangle \\ &= \sum_i p_i \sum_k \left|\left\langle s^{(i)}\right| \alpha_k\rangle\right| \\ &= \sum_i p_i \end{aligned} \tag{3.17}$$

3- A density operator is positive

4- We also note that an operator is positive if and only if all of its eigenvalues are greater than or equal to zero, which implies that the eigenvalues of any density operator must satisfy this property. In addition, because the trace of a density matrix is one and the trace is just the sum of the eigenvalues, we have that if λ_j is an eigenvalue of a density matrix, then $0 \leq \lambda_j \leq 1$.

3.1.2 Density matrices

The density matrix or density operator is an alternate representation of the state of a quantum system. It can encode all the information about a quantum mechanical system, where describing a quantum system with the density matrix is equivalent to using the wave-function. Suppose we have an arbitrary orthonormal basis $\{|\alpha_k\rangle\}$, the generic matrix element is given by

$$\rho_{kl} = \langle\alpha_k| \rho |\alpha_l\rangle = \sum_i p_i \langle\alpha_k| s^{(i)}\rangle \left\langle s^{(i)}\right| \alpha_l\rangle \tag{3.18}$$

where

$$\left|s^{(i)}\right\rangle = \sum_k C_k^{(i)} |\alpha_k\rangle \qquad \text{and } C_k^{(i)} = \langle\alpha_k| s^{(i)}\rangle \tag{3.19}$$

finally, we find

$$\rho_{kl} = \sum_i p_i C_k^{(i)} C_l^{(i)*} \tag{3.20}$$

the diagonal elements are $\rho_{kk} = \sum_i p_i \left| C_k^{(i)} \right|^2$, ρ_{kk} is the global probability of finding generic elements of the mixture is in the state $|\alpha_k\rangle$, notice that $\sum_k \rho_{kk} = 1$ due to property 2 of density operators.

3.1.3 Reduced density matrix

Now, if F_A is an observable on subsystem A, then the operator corresponding to it in the total Hilbert space is $F_A \otimes I_B$, where I_B is the identity on $\mathbf{H}_B$. If $|\psi_{AB}\rangle$ is the state of the entire system, then the expectation value of X_A is given by

$$\begin{aligned} \langle F_A \rangle &= \langle \psi_{AB}| F_A \otimes I_B |\psi_{AB}\rangle \\ &= \sum_i \sum_j \langle \psi_{AB}| F_A \otimes I_B(|i_A\rangle |j_B\rangle)(\langle i_A| \langle j_B|) |\psi_{AB}\rangle \\ &= \sum_i \langle i_A| (\sum_j \langle j_B |\psi_{AB}\rangle \langle \psi_{AB} |j_B\rangle)F_A |i_A\rangle \end{aligned} \tag{3.21}$$

now, we define

$$\begin{aligned} \rho_A &= \sum_j \langle j_B |\psi_{AB}\rangle \langle \psi_{AB} |j_B\rangle \\ &= Tr_B(|\psi_{AB}\rangle \langle \psi_{AB}|) \end{aligned} \tag{3.22}$$

and

$$\langle F_A \rangle = Tr_A(\rho_A F_A) \tag{3.23}$$

The operator ρ_A is known as the reduced density operator for subsystem A, and it can be used to evaluate the expectation value of any observable that pertains only to subsystem A. The reduced density operator works only on the subspace containing subset A and satisfies all the properties already listed for a generic density operator.

3.2 Entanglement Measures

To better understand the structure of entangled states, it is useful to introduce the entanglement measure. Where Bell inequalities [92], and entanglement witnesses [93, 94, 95] are early approaches to realize the entanglement characterization. At present, there is no unique measure of entanglement, but all of them should clearly satisfy the following properties:

- Must be maximum for maximally entangled states (Bell states).
- Must be zero for separable states.
- Must be non-zero for all non-separable states.

In 1996, Bennett et al. proposed an entanglement measurement method named "entanglement of formation" (EOF) [96, 97], for a general quantum system consisting of two parts, here named A and B, an arbitrary pure state can be written as [98]

$$|\psi_{AB}\rangle = \sum_i c_i \left|\psi_i^A\right\rangle \otimes \left|\psi_i^B\right\rangle \tag{3.24}$$

where $\{\left|\psi_i^A\right\rangle, \left|\psi_i^B\right\rangle\}$are sets of orthonormal states for A and B, the values of c_i precisely show the features of the state $|\psi_{AB}\rangle$. The EOF for the state in eq. (3.24) is defined as

$$E(\psi_{AB}) = S(Tr_B |\psi_{AB}\rangle \langle\psi_{AB}|) = S(Tr_A |\psi_{AB}\rangle \langle\psi_{AB}|) = -\sum_i c_i^2 \log_2 c_i^2 \tag{3.25}$$

where S is the von Neumann entropy. On the other hand, the definition of EOF can be extended to a mixed state:

$$E_f(\rho) = \sum_j p_j E(\psi_{AB}) \tag{3.26}$$

it was proven that there is a general formula of $E_f(\rho)$ [97, 99, 100], which is based on an exactly calculable quantity, that is concurrence C. For a pure state $|\psi_{AB}\rangle$ in a two-qubit system, $C(|\psi_{AB}\rangle)$ is defined as

$$C(\rho) = \left|\langle\psi_{AB}\left|\widetilde{\psi}_{AB}\right\rangle\right| \tag{3.27}$$

where $\left|\widetilde{\psi}_{AB}\right\rangle = (\sigma_y \otimes \sigma_y)\left|\psi^*_{AB}\right\rangle$, σ_y is the Pauli operator $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $|\psi^*_{AB}\rangle$ is the complex conjugate of $|\psi_{AB}\rangle$ for a two-qubit system, the relation between the concurrence and EOF can be written as:

$$E(\rho) = \varepsilon(C(\rho)) \tag{3.28}$$

where

$$\varepsilon(C) = -\frac{1+\sqrt{1-C^2}}{2}\log_2\frac{1+\sqrt{1-C^2}}{2} - \frac{1-\sqrt{1-C^2}}{2}\log_2\frac{1-\sqrt{1-C^2}}{2} \tag{3.29}$$

On the other hand, according to [100], there is an explicit formula for concurrence as

$$C(\rho) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}) \tag{3.30}$$

$\lambda_i (i = 0, 1, 2, 3, 4)$ is the non-negative eigenvalue of the Hermitian matrix $R^2 = \sqrt{\rho}\widetilde{\rho}\sqrt{\rho}$,where $\widetilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$, ρ^* is the complex conjugate of ρ

Chapter 4

Entanglement and Inertial Observers

4.1 Massive Particles with Spin

4.1.1 Particle State

The particle states for massive particles are defined by their spin and momentum, which is given by

$$|\overrightarrow{p},\sigma\rangle \tag{4.1}$$

for massless particles, the states are given by $|\overrightarrow{p},\lambda\rangle$ where λ indicates the helicity states of the particle ($\lambda=\pm 1$ for photons, $\lambda=\pm\frac{1}{2}$ for massless fermions). Under a Lorentz transformation (LT) Λ the single particle state for a massive particle transforms under the unitary transformation $U(\Lambda)$ as

$$U(\Lambda)\,|\overrightarrow{p},\sigma\rangle=\sum_{\sigma'}D_{\sigma'\sigma}(W(\Lambda,\overrightarrow{p}))\left|\Lambda\overrightarrow{p},\sigma'\right\rangle \tag{4.2}$$

for more detail see the Annex.

4.1.2 Single-Two Particle States Entanglement

A general one-particle state is written as

$$|\psi\rangle = \sum_{\sigma} \int d\mu(p) f_{\sigma}(p) \, |p, \sigma\rangle \tag{4.3}$$

where $d\mu(p) = \frac{1}{(2\pi)^3} \frac{d^3p}{2E_p}$ is the Lorentz-invariant measure introduced to normalize the basic states, such that

$$\begin{aligned} \sum_{\sigma} \int d\mu(p) \, \langle p, \sigma \, |p, \sigma\rangle &= 1 \\ \sum_{\sigma} \int d\mu(p) \, |f_{\sigma}(p)|^2 &= 1 \end{aligned} \tag{4.4}$$

Peres, Scudo, and Terno [73] did the first study of entanglement in a relativistic frame in the context of single particle states. When they considered a state with a Gaussian distribution $f_{\sigma}(p)$, they discovered that two observers connected by a Lorentz boost will disagree on the entropy of the reduced spin state, resulting in different entanglement between spin and momentum. The two-particle Hilbert space is given by $\mathbf{H}_{12} = \mathbf{H}_1 \otimes \mathbf{H}_2$, where the states

$$|p_1, \sigma_1; p_2, \sigma_2\rangle = |p_1, \sigma_1\rangle \otimes |p_2, \sigma_2\rangle \tag{4.5}$$

normalized as

$$\sum_{\sigma_1 \sigma_2} \int d\mu(p_1, p_2) \, \langle p_1, \sigma_1; p_2, \sigma_2| \, p_1, \sigma_1; , p_2, \sigma_2\rangle = 1 \tag{4.6}$$

a general state is given by

$$|\psi\rangle = \sum_{\sigma_1\sigma_2} \int d\mu(p_1, p_2) f_{\sigma_1\sigma_2}(p_1, p_2) \, |p_1, \sigma_1; p_2, \sigma_2\rangle \tag{4.7}$$

also normalized as

$$\sum_{\sigma_1\sigma_2} \int d\mu(p_1, p_2) \, |f_{\sigma_1\sigma_2}(p_1, p_2)|^2 = 1 \tag{4.8}$$

4.2 Entanglement in Curved Space-Time

4.2.1 Mathematical Formalism

In a curved space-time, the spin of the particle is not well defined. Thus, one has to define it locally. For this, one introduces a local inertial frame at each point by using a Vierbein (or tetrad) e_a^μ defined by

$$g_{\mu\nu} e_a^\mu e_b^\nu = \eta_{ab} \tag{4.9}$$

where $g_{\mu\nu}$ is the metric of the curved η_{ab} is Minkowski space-time. P^μ is the four energy-momentum tensor of the particle, the corresponding spin quantum state is denoted by $|P, \sigma\rangle$ where $\sigma(=\uparrow, \downarrow)$. If this particle moves to another point in space-time, its state in the local frame becomes [103, 104].

$$\sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda, P)) \left|\Lambda P, \sigma'\right\rangle \tag{4.10}$$

where Λ is the Lorentz transformation matrix and $(W(\Lambda, P))$ is the Wigner rotation operator corresponding to Λ, $D_{\sigma'\sigma}$ denotes the two-dimensional representation of the Wigner rotation operator.

Now, let us consider a system consisting of two spin $\frac{1}{2}$ non-interacting particles, where the center (centroid) of this system is described by a wave packet in the local frame where the initial state is written as

$$\left|\Psi^{(i)}\right\rangle = \sum_{\sigma_1\sigma_2} \iint d^3p_1 d^3p_2 \psi_{\sigma_1\sigma_2}(p_1, p_2) \left|P_1, \sigma_1; P_2, \sigma_2\right\rangle \tag{4.11}$$

where P_1 and P_2 are the four-momentum of particles 1 and 2 respectively, $\psi_{\sigma_1\sigma_2}(p_1, p_2)$ are wave functions determining momentum and spin distribution. It can be used to express momentum entanglement, spin entanglement, and even entanglement between spins and momenta. The normalization condition

$$\sum_{\sigma_1\sigma_2} \iint d^3p_1 d^3p_2 \left|\psi_{\sigma_1\sigma_2}(p_1, p_2)\right|^2 = 1 \tag{4.12}$$

On the other hand, the change in the local inertial frame is given by

$$\delta e^a_\mu(x) = u^\nu(x) d\tau \nabla_\nu e^a_\mu(x) = -u^\nu(x) w^a_{vb} e^b_\mu(x) d\tau = \chi^a_b(x) e^b_\mu(x) d\tau \tag{4.13}$$

where

$$w^a_{vb} = -e^\mu_b(x) \nabla_v e^a_\mu(x) = e^a_\mu(x) \nabla_v e^\mu_b(x) \tag{4.14}$$

and

$$\chi^a_b(x) = -u^\nu(x) w^a_{vb} \tag{4.15}$$

where w^a_{vb} is the spin connection elements, $\chi^a_b(x)$ is its change along the direction of $u^\nu(x)$ (the 4-vector velocity of the centroid), $\bigtriangledown_\nu$ is the covariant derivative. The change δq^μ in the momentum has the form

$$\delta q^\mu(x) = u^\nu(x) d\tau \nabla_\nu q^\mu(x) = m a^\mu(x) d\tau \tag{4.16}$$

where m is the rest mass of the particle and

$$a^\mu(x) = u^\nu(x) \bigtriangledown_\nu u^\mu(x) \tag{4.17}$$

$a^\mu(x)$ is the 4-vector acceleration produced by the classical force as measured in the local frame. Thus, we can rewrite the eq. (4.16) as

$$\delta q^\mu(x) = -\frac{1}{mc^2}[a^\mu(x)q_\nu(x) - q^\mu(x)a_\nu(x)]q^\nu(x)d\tau \tag{4.18}$$

where

$$q^\mu(x)q_\mu(x) = -m^2c^2, \quad q^\mu(x)a_\mu(x) = 0 \tag{4.19}$$

we obtain

$$\delta q^a(x) = \lambda_a^b(x)q^b(x)d\tau \tag{4.20}$$

where

$$\lambda_a^b(x) = -\frac{1}{mc^2}[a^a(x)q_b(x) - q^a(x)a_b(x)] + \chi_b^a(x) \tag{4.21}$$

the first term exists even in special relativity, and the second term is due to the space-time curvature, so it exists only in general relativity. When the particle moves, the momentum in the local inertial frame transforms under the local Lorentz transformation. It can be written as

$$\Lambda_b^a(x) = \delta_b^a + \lambda_b^a(x)d\tau \tag{4.22}$$

where τ is proper time. There is a unitary operator denoted by $U(\Lambda_b^a(x))$ (see annex), that corresponds to the local LT $\Lambda_b^a(x)$, we can apply $U(\Lambda(x))$ on the state $|p^a, \sigma\rangle$, we find

$$U(\Lambda(x))\,|p^a, \sigma\rangle = \sum_{\sigma'} D^{(s)}_{\sigma'\sigma}(W(\Lambda(x), p))\left|\Lambda p^a, \sigma'\right\rangle \tag{4.23}$$

The infinitesimal Wigner rotation is represented as

$$W^a_b(\Lambda(x), p) = \delta^a_b + w^a_b d\tau \tag{4.24}$$

where $w^0_0(x) = w^0_i(x) = w^i_0(x) = 0$, and

$$w^i_k(x) = \lambda^i_k(x) + \frac{\lambda^i_0(x)p_k(x) - \lambda_{k0}(x)p^i(x)}{p^0(x) + mc} \tag{4.25}$$

where i and j run over the three spatial inertial frame labels (1, 2, 3). When the centroid moves along a path $x^\mu(\tau)$ from $x^\mu_i(\tau) = x^\mu(\tau_i)$ to $x^\mu_f(\tau) = x^\mu(\tau_f)$, we can obtain a global Lorentz transformation $\Lambda(x_f, x_i)$ [101]

$$\Lambda(x_f, x_i) = T\exp(\int_{x_i}^{x_f} \lambda(x(\tau))d\tau) \tag{4.26}$$

$\lambda(x)$ is a matrix whose elements are given by eq. (4.21) and T is the time-ordering operator, the Wigner rotation operator can be expressed as

$$W(\Lambda(x_f, x_i), P) = T\exp(\int_{x_i}^{x_f} w(x(\tau))d\tau) \tag{4.27}$$

Suppose that in the local inertial frame at the initial point x^μ_i , the wave packet that is given by eq. (4.11), with corresponding reduced density matrix. Then, in the local inertial frame at the final point x^μ_f, the wave packet will be

$$\begin{aligned} \left|\Psi^{(f)}\right\rangle \;=\; & \sum_{\sigma_1\sigma_2\sigma'_1\sigma'_2} \int d^3p_1 d^3p_2 \sqrt{\frac{(\Lambda_1 P_1)^0(\Lambda_2 P_2)^0}{P^0_1 P^0_2}}\psi_{\sigma_1\sigma_2}(p_1, p_2) \\ & D_{\sigma'_1\sigma_1}(W(\Lambda_1, P_1)D_{\sigma'_2\sigma_2}(W(\Lambda_2, P_2)\left|\Lambda_1 P_1, \sigma'_1; \Lambda_2 P_2, \sigma'_2\right\rangle \end{aligned} \tag{4.28}$$

4.2.2 Generalization of the Model

Now, in order to measure entanglement between 2 particles in a gravitational field, let us consider the following space-time where the metric ds^2 has the form

$$ds^2 = F(r)dt^2 + G(r)dr^2 + H(r,\theta)d\theta^2 + I(r,\theta)d\varphi^2 \tag{4.29}$$

$F(r)$, $G(r)$, $H(r,\theta)$, $I(r,\theta)$ are arbitrary functions that have a linear relation with the coordinates $(r) or (r,\theta)$, let us make a diagonal choice of the tetrad

$$e_0^t = \frac{1}{\sqrt{F(r)}}, \quad e_1^r = \frac{1}{\sqrt{G(r)}}, \quad e_2^\theta = \frac{1}{\sqrt{H(r,\theta)}}, \quad e_3^\varphi = \frac{1}{\sqrt{I(r,\theta)}} \tag{4.30}$$

By using the metric (4.29), the non-vanishing Christoffel connection elements are

$$\begin{aligned}
\Gamma^t_{tt} &= \frac{1}{2F}\dot{F} \qquad \Gamma^r_{rr} = \frac{1}{2G}G' \qquad \Gamma^\theta_{\theta\theta} = \frac{1}{2H}\partial_\theta H \\
\Gamma^r_{tt} &= -\frac{1}{2g}F' \qquad \Gamma^\varphi_{\theta\varphi} = \frac{1}{2I}\partial_\theta I \qquad \Gamma^t_{rr} = -\frac{1}{2F}\dot{G} \\
\Gamma^\theta_{\varphi\varphi} &= -\frac{1}{2H}\partial_\theta I \qquad \Gamma^r_{\varphi\varphi} = -\frac{1}{2G}I' \qquad \Gamma^r_{\theta\theta} = -\frac{1}{2G}hH' \\
\Gamma^t_{\varphi\varphi} &= -\frac{1}{2F}\dot{I} \qquad \Gamma^\theta_{r\theta} = \frac{1}{2H}H' \qquad \Gamma^t_{rt} = \frac{1}{2F}F' \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{2I}I'
\end{aligned} \tag{4.31}$$

where $\dot{X} = \frac{\partial X}{\partial t}$ and $X' = \frac{\partial X}{\partial r}$. For a circular motion and constant angular velocity $\frac{d\varphi}{dt}$ on the equatorial plane where $\theta = \frac{\pi}{2}$ are given by

$$\begin{aligned}
w^0_{t1} &= \frac{1}{2}\frac{F'}{\sqrt{GF}} \qquad w^0_{\varphi 3} = \frac{\dot{I}}{2\sqrt{IF}} \qquad w^1_{\theta 2} = -\frac{1}{2\sqrt{GH}}H' \\
w^1_{\varphi 3} &= -\frac{1}{2}\frac{I'}{\sqrt{GI}} \qquad w^2_{\varphi 3} = -\frac{1}{2\sqrt{HI}}\partial_\theta I
\end{aligned} \tag{4.32}$$

Furthermore the non-vanishing components $\chi^a_b(x)$, u^μ and λ^a_b

$$\begin{aligned} \chi_1^0 &= -u^t w_{t1}^0 \qquad \chi_3^0 = -u^\varphi w_{\varphi 1}^0 \qquad \chi_2^1 = -u^\theta w_{\theta 2}^1 \\ \chi_3^1 &= -u^\varphi w_{\varphi 3}^1 \qquad \chi_3^2 = -u^\varphi w_{\varphi 3}^2 \end{aligned} \tag{4.33}$$

$$u^t = \frac{\gamma c}{\sqrt{F}} \qquad u^\varphi = \frac{1}{\sqrt{I}} \gamma r \frac{d\varphi}{dt} \tag{4.34}$$

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ is the Lorentz factor.

$$\begin{aligned} \lambda_1^0 &= \frac{1}{mc^2}[p^0 a_1] + \chi_1^0 \qquad \lambda_3^1 = -\frac{1}{mc^2}[a^1 p_3] + \chi_3^1 \\ \lambda_3^2 &= \chi_3^2 \qquad \lambda_2^0 = \chi_2^0 \end{aligned} \tag{4.35}$$

where $\lambda_b^a = \lambda_a^b$. finally

$$\Theta_b^a : \Theta_0^0 = \Theta_i^0 = \Theta_0^i = 0 \tag{4.36}$$

and

$$\Theta_k^i = \lambda_k^i + \frac{\lambda_0^i p_k - \lambda_{k0} p^i}{p^0 + mc^2} \tag{4.37}$$

$$\Theta_3^1 = \lambda_3^1 + \frac{\lambda_0^1 p_3}{p^0 + mc^2} \qquad \Theta_2^1 = \Theta_3^2 = 0 \qquad \Theta_2^3 = \lambda_2^3 + \frac{\lambda_0^3 p_2 - \lambda_{20} p^3}{p^0 + mc^2} \tag{4.38}$$

where $w_k^i(x) = \Theta_k^i$, and $\Theta = \Theta_3^1 \tau$ so Θ can be rewritten as

$$\Theta = -\frac{\alpha}{2G}\{(AD^2 + B(D^2-1) - I'\sqrt{\frac{G}{HI}}) + \frac{D}{C}p(AD^2 + B(D^2-1) + A\sqrt{G})\} \tag{4.39}$$

where $A = \frac{F'}{F}$, $B = \frac{I'}{H}$, $D = \sqrt{1+q^2}$, $C = 1 + \sqrt{1+p^2}$, $\alpha = \frac{2\tau}{mc}$.

4.2.3 Spin Entanglement in de Sitter–Schwarzschild Space-Time

In this part, we discuss the effect of the gravitational field of a massive body (black hole) on the spin entanglement of a two-particle system in Schwarchild- de Sitter space-time and the role of the cosmological constant and its influence on the entanglement of the singlet and triplet states.

The first solution to Einstein's equations is Karl Schwarzschild's solution, the Schwarzschild solution is the metric that corresponds to the gravitational field created by the distribution of static matter and spherical symmetry, that describes the exterior region of the spherically symmetric distribution of matter or energy such as planet, star, or non-rotating black hole with zero electric charge and angular momentum. The Schwarzschild metric is replaced in the presence of the cosmological constant by the de Sitter–Schwarzschild or Anti-de Sitter–Schwarzschild metric, according to the sign. Let us take the case of a positive cosmological constant and determine the corresponding de Sitter–Schwarzschild line element. Consider a system of two particles (bipartite state) moving in the gravitational field of the Schwarzschild-de Sitter space-time where the metric is given by

$$ds^2 = -c^2(f(r) - \frac{\Lambda r^2}{3})dt^2 + (f(r) - \frac{\Lambda r^2}{3})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{4.40}$$

where $f(r) = 1 - \frac{r_s}{r}$ and $r_s = 2GM$ is the Schwarzschild radius and $r > r_s$. In the presently used relativistic units ($c = G = 1$), in the limiting case $\Lambda = 0$ the metric reduces to the Schwarzschild metric. Let us introduce a local inertial frame by choosing the tetrad

$$e_0^t = \frac{1}{\sqrt{f(r) - \frac{\Lambda r^2}{3}}}, e_1^r = \sqrt{f(r) - \frac{\Lambda r^2}{3}}, e_2^\theta = \frac{1}{r}, e_3^\phi = \frac{1}{r\sin\theta} \tag{4.41}$$

For a circular motion and constant angular velocity $\frac{d\phi}{dt}$ on the equatorial plane where $\theta = \frac{\pi}{2}$, the four-velocity of the centroid is given by

$$u^t(x) = \frac{\cosh\xi}{\sqrt{f(r) - \frac{\Lambda r^2}{3}}} \quad and \quad u^\phi(x) = \frac{\sinh\xi}{r} \tag{4.42}$$

and ξ is the rapidity in the local inertial frame defined by $\tanh\xi = \frac{v}{c}$ and v is the velocity. Notice that the condition

$$(1 - \frac{r_s}{r} - \frac{\Lambda r^2}{3}) > 0 \tag{4.43}$$

we denote by

$$z = \frac{r}{r_s}, w = \Lambda r_s^2, \Sigma = \frac{c\tau}{r_s} \tag{4.44}$$

we find after some calculations

$$\Theta = q\sqrt{q^2+1}\left[\sqrt{q^2+1} - \frac{qp}{\sqrt{p^2+1}+1}\right]\Sigma\frac{z^{\frac{1}{2}}}{(z-1-\frac{z^3 w}{3})^{\frac{1}{2}}}(z-3) \tag{4.45}$$

where $\Theta = w_3^1\tau$

The singlet entangled state

The wave packet in the momentum representation for a system of two spin 1/2 non-interacting particles observed in a local frame (4.11) in the singlet state $\psi_{\sigma_1\sigma_2}(p_1, p_2)$ is given by

$$\psi_{\sigma_1\sigma_2}(p_1,p_2) = \frac{1}{\sqrt{2}}(\delta_{\sigma_1\uparrow}\delta_{\sigma_2\downarrow} - \delta_{\sigma_1\downarrow}\delta_{\sigma_2\uparrow})f(p_1)f(p_2) \tag{4.46}$$

where $f(p)$ is a Gaussian distribution about the momentum q of the centroid

$$|f(p)|^2 = (2\pi)^{-\frac{3}{2}}(\alpha m)^{-3}\exp(-\frac{1}{2}(\frac{p-q}{\alpha m})^2) \tag{4.47}$$

The reduced matrix density for the initial state has the form

$$\varrho^i = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 \\ 0 & -1 & +1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.48}$$

for the final state, the reduced matrix density elements are

$$\begin{aligned} \varrho^f_{\sigma_1'\sigma_2'\sigma_1\sigma_2} = {} & \frac{1}{2}\iint dp_1 dp_2\, |f(p_1)|^2\, |f(p_2)|^2\, [D_{\sigma_1'\uparrow}(\Theta_1)D_{\sigma_2'\downarrow}(\Theta_2)D_{\sigma_1\uparrow}(\Theta_1)D_{\sigma_2\downarrow}(\Theta_2) \\ & -D_{\sigma_1'\downarrow}(\Theta_1)D_{\sigma_2'\uparrow}(\Theta_2)D_{\sigma_1\uparrow}(\Theta_1)D_{\sigma_2\downarrow}(\Theta_2) \\ & -D_{\sigma_1'\uparrow}(\Theta_1)D_{\sigma_2'\downarrow}(\Theta_2)D_{\sigma_1\downarrow}(\Theta_1)D_{\sigma_2\uparrow}(\Theta_2) \\ & +D_{\sigma_1'\downarrow}(\Theta_1)D_{\sigma_2'\uparrow}(\Theta_2)D_{\sigma_1\downarrow}(\Theta_1)D_{\sigma_2\uparrow}(\Theta_2)] \end{aligned} \tag{4.49}$$

Then, straightforward calculations show that the wooters concurrence takes the following form

$$C(\varrho^f) = \max(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}) \tag{4.50}$$

$$= \langle\cos\Theta\rangle^2 + \langle\sin\Theta\rangle^2 \tag{4.51}$$

where $\langle X\rangle = \int dp\, |f(p)|^2\, X$. and

$$\langle \cos \Theta \rangle = \int dp \, |f(p)|^2 \cos \Theta \tag{4.52}$$

Eq. (4.50) has to be solved numerically. Fig (4-1) shows the wootters concurrence $C(\varrho^f)$ as a function of z for fixed values of q, Σ and w

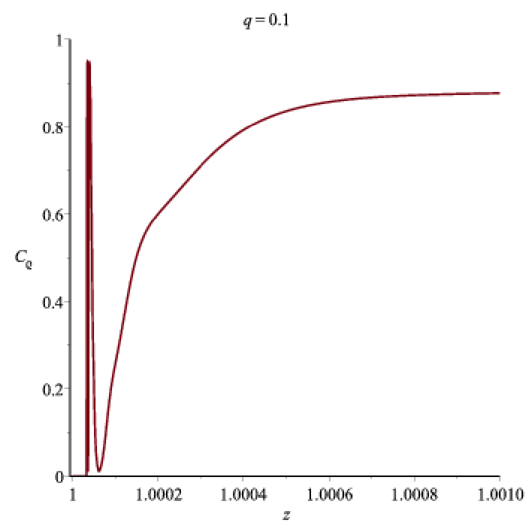


Figure 4-1: $C(\varrho^f)$ as a function of z for fixed q, Σ and w

Notice that as z (equivalently the distance r) increases i.e. going far away from the horizon of the black hole, the entanglement increases and becomes more robust. This is due to the fact that the gravitational field decreases.

Fig (4-2) displays the variation of the concurrence $C(\varrho^f)$ as a function of the centroid momentum q for fixed values of z , Σ and w . Notice the damping periodic oscillatory behavior due to the sine and cosine functions in the Wigner matrix. Furthermore, as q

increases, the maximum of the oscillations decreases due to the fact that as the velocity increases the spin decoherence phenomenon increases.

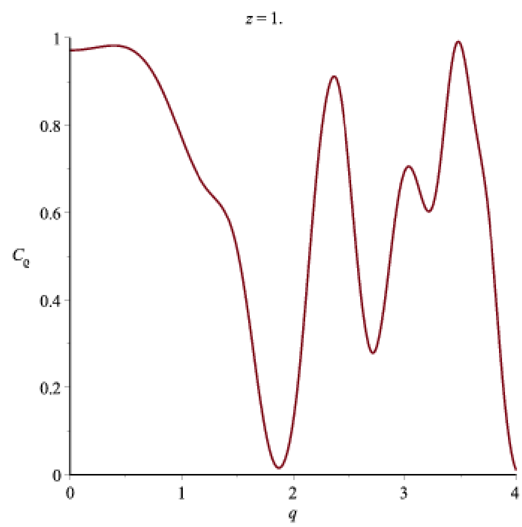


Figure 4-2: Variation of $C(\varrho^f)$ as a function of q for fixed z , Σ and w

Fig (4-3) shows the variation of the concurrence $C(\varrho^f)$ as a function of w for fixed values of z, Σ and q. Notice that if the cosmological constant Λ increases, we will approach more the cosmological horizon where the cosmological curvature becomes stronger and therefore the entanglement decreases. According to the Hawking-Unruh effect, an accelerating particle will radiate and loose information and entanglement decreases.

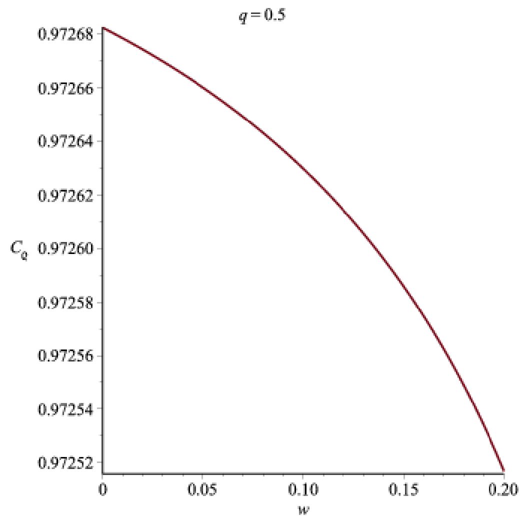


Figure 4-3: $C(\varrho^f)$ as a function of w for fixed values of z, q and Σ

The triplet entangled state

In the triplet state $\psi_{\sigma_1\sigma_2}(p_1, p_2)$ takes the form::

$$\psi_{\sigma_1\sigma_2}(p_1, p_2) = \frac{1}{\sqrt{2}}(\delta_{\sigma_1\uparrow}\delta_{\sigma_2\downarrow} + \delta_{\sigma_1\downarrow}\delta_{\sigma_2\uparrow})f(p_1)f(p_2) \tag{4.53}$$

In this case, one can show that after straightforward but tedious calculations, the Wootters concurrence has the following expression:

$$C(\varrho^f) = \sqrt{\langle\cos 2\Theta\rangle^2 + \langle\sin 2\Theta\rangle^2} \tag{4.54}$$

Figs (4-4), (4-5) and (4-6) are the same as figs (4-2), (4-1) and (4-3) respectively but with a triplet entangled state.

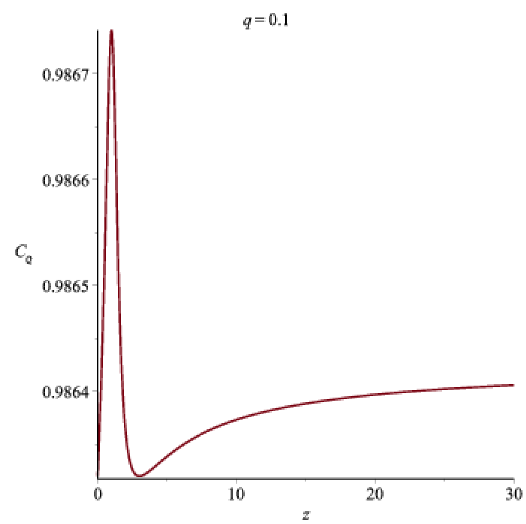


Figure 4-4: The same as fig 4-2 but in triplet entangled state.

Notice the concurrence in the quantum entangled singlet state is more robust than in the triplet case. that the same with [85]. Moreover, the effect of the cosmological constant is very remarkable even in extended regions around the black hole.

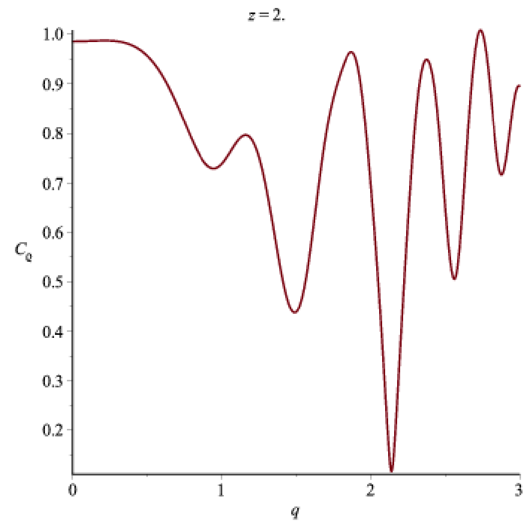


Figure 4-5: Same as fig 4-1 in triplet entangled state

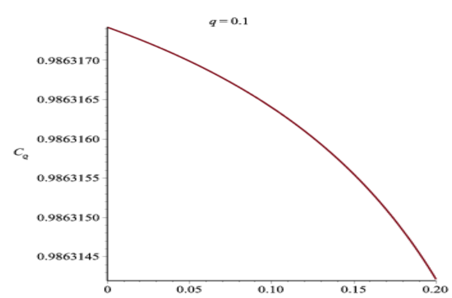


Figure 4-6: Same as fig 4-3 in triplet entangled state

Conclusion

In this part, which falls within the framework of quantum information in a curved static space-time, the Wootters concurrence behavior of two spin$-1/2$ entangled particles moving in curved space-time, is studied in the context of the de Sitter–Schwarzschild metric. We have found that the spin entanglement in the singlet quantum state is more important than in the triplet case. Furthermore, the cosmological constant plays an important role in the region around the black hole.

4.2.4 Spin Entanglement in Kerr Space-Time

We consider the Kerr metric. In Boyer-Lindquist coordinates, it has the following expression [89]

$$ds^2 = \frac{\Delta}{\rho^2} d\widehat{t}^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - \frac{\sin^2\theta}{\rho^2} d\widehat{\varphi}^2 \tag{4.55}$$

where

$$\begin{aligned} \rho &= r^2 + a^2\cos^2\theta \\ a &= \frac{J}{Mc} \\ \Delta &= r^2 - r_s r + a^2 + Q^2 \\ r_s &= \frac{2GM}{c^2} \end{aligned} \tag{4.56}$$

and

$$d\widehat{t}^2 = (dt - a\sin^2\theta d\phi)^2, \qquad d\widehat{\varphi}^2 = ((r^2+a^2)d\phi - adt)^2 \tag{4.57}$$

here M, J and Q are the mass, angular momentum and charge of the black hole. G, c are the Newton gravitational constant (where in nature units $G = c = 1$), velocity of light. In what follows, we deal with a non-charged black hole where $Q = 0$. In this case. Then tetrad are

$$e_t^0 = \sqrt{\frac{\Delta}{\rho^2}},\ e_r^1 = \sqrt{\frac{\rho^2}{\Delta}},\ e_\theta^2 = \sqrt{\rho^2},\ e_\varphi^3 = \sqrt{\frac{1}{\rho^2}} \tag{4.58}$$

Finally, we find

$$\Theta = \tau\sqrt{\frac{r^2}{\Delta}}[\frac{(r_s - 2r)}{2r^2}]q\sqrt{(q^2+1)}[\sqrt{(q^2+1)} - \frac{qp}{\sqrt{(p^2+1)}+1}] \tag{4.59}$$

we put $\frac{r}{r_s} = z; \frac{a}{r_s} = \Sigma; \frac{\tau}{r_s} = \alpha$

$$\Theta = \alpha\sqrt{\frac{z^2}{z^2 - z + \Sigma}}[\frac{(1-2z)}{2z^2}]q\sqrt{(q^2+1)}[\sqrt{(q^2+1)} - \frac{qp}{\sqrt{(p^2+1)}+1}] \tag{4.60}$$

Fig (4-7) displays the variation of the concurrence as a function of the dimensionless parameter $\Sigma \in [0.1]$ and fixed $\alpha \approx 1, z \approx 1.5$ and $q \approx 0.1$. The concurrence is an increasing function, this is due to the fact that the gravitational potential g_{00} decreases as J (or ρ) increases and thus information (or concurrence) increases until a saturated bound of the maximal entanglement $(C(\varrho^f) \sim 1)$.

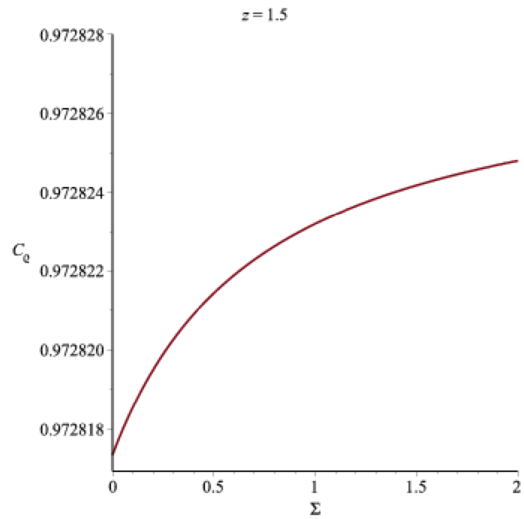


Figure 4-7: Variation of the concurrence as a function of Σ with fixed $\alpha \approx 1, z \approx 1.5$ and $q \approx 0.1$

Fig (4-8) shows the concurrence variation as a function of q for fixed values of $\alpha = 2$, $z = 1.5$ and Σ at smaller value of $q(q \to 0)$, the entangled is max and if $q \nearrow$ the center of the wave packet travels more on the circular trajectory and therefore one has more decoherence (less entanglement) and consequently the concurrence decreases for example if $q = 1$, $C(\varrho^f) \sim 0.7$and if $q = 1.6$, $C(\varrho^f) \sim 0.4$, the oscillator periodic behavior can be explained (as it was pointed out in [90]) by the fact that when q increases, the exponential in the integral that present in the expression of the concurrence approaches unity, so the cosine and sine terms behavior dominates. It is worth to mention that this behavior (minima and maxima) changes if the other parameters such as Σ and z change.

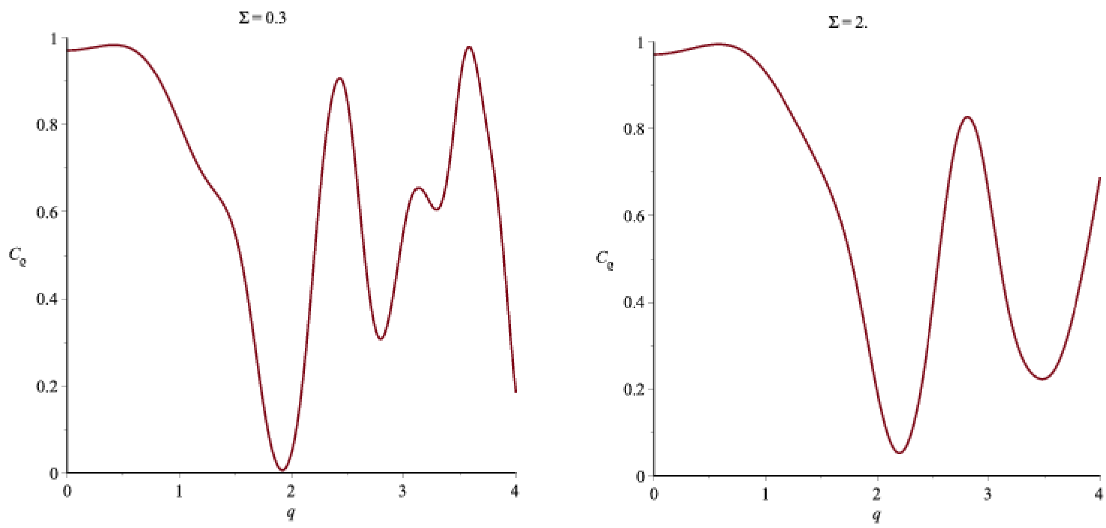


Figure 4-8: The concurrence as a function of q for fixed $\alpha = 2$, $z = 1.5$.

Fig (4-8) shows that if Σ decreases to 0.3 the number and shape of picks change and they become more pronounced, similar behavior is shown in Fig (4-9) if z changes.

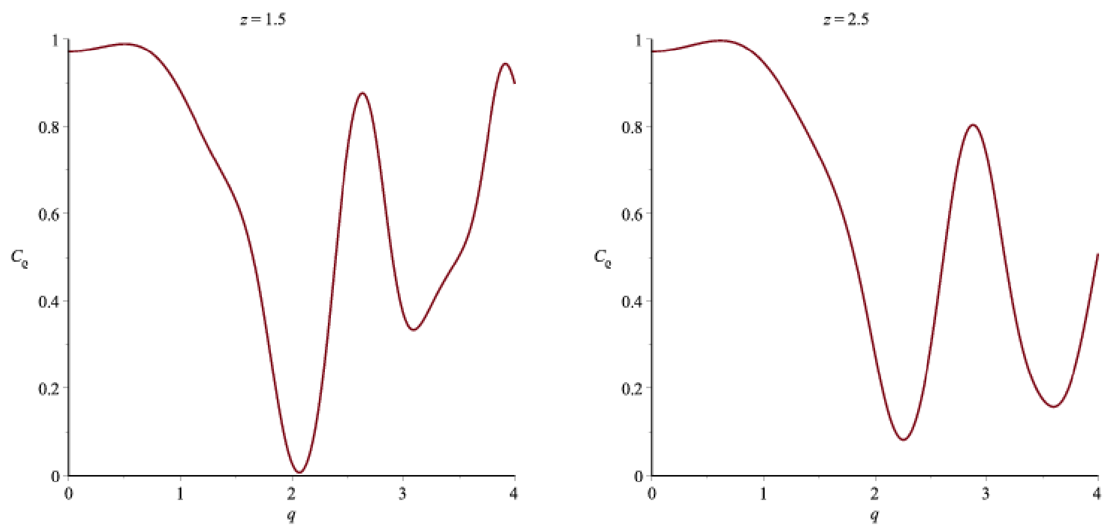


Figure 4-9: The concurrence as a function of qfor fixed $\alpha = 1, \Sigma = 1$.

Fig (4-10) and Fig (4-11) display the variation of the concurrence as a function of z (or circular motion radius) for fixed values of q, $\Sigma, \alpha = 1$, notice that for smaller values of $r(r \rightarrow 0$ near black hole singularity) where the gravitational field is infinite, the entanglement is minimal ($C(\varrho^f) \sim 0$). If we go far from the singularity (z increases) the gravitational field decreases and therefore the information increases and thus the concurrence ($C(\varrho^f) \sim 1$). The shape and number of picks and minima depend strongly on the values of the parameters q and Σ. Fig (4-10), Fig (4-11) and Fig (4-12) show the behavior of the concurrence with variation of z.

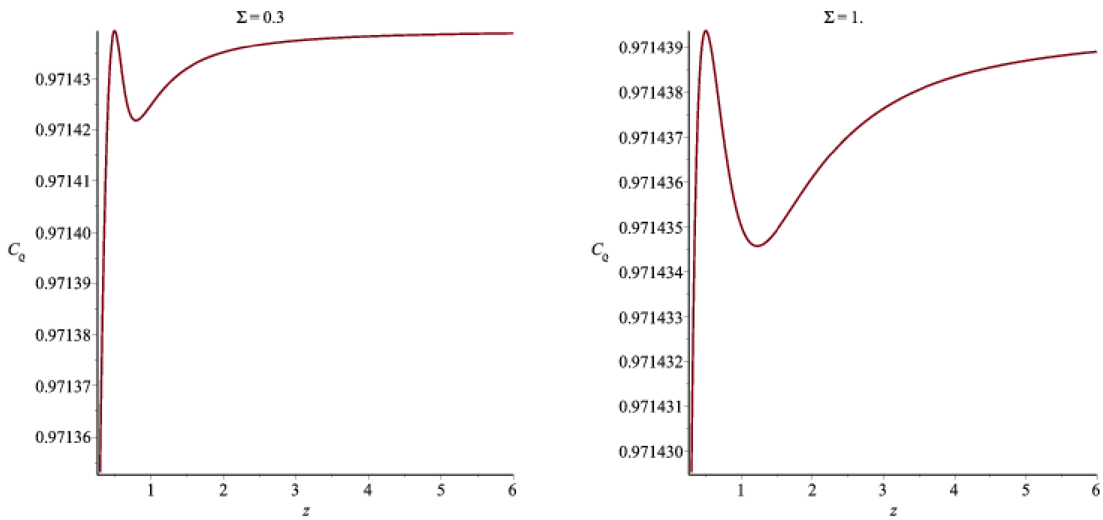


Figure 4-10: The concurrence as a function of z for fixed $\alpha = 1, q = 0.1$.

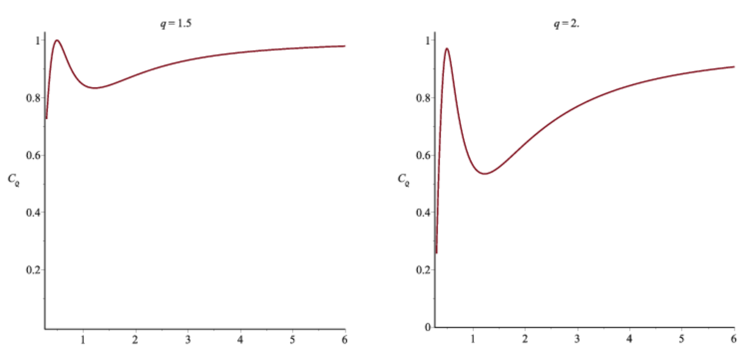


Figure 4-11: The concurrence as a function of z for fixed $\alpha = 1, \Sigma = 1$.

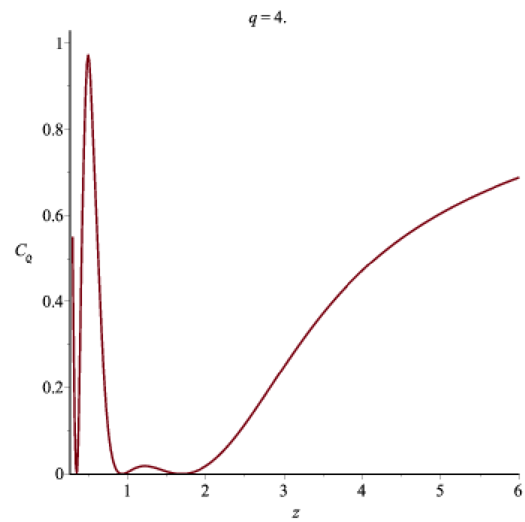


Figure 4-12: The concurrence as a function of z for fixed $\alpha = 1, \Sigma = 0.1$.

Conclusion

Throughout this study, we have studied the singlet state of spin entanglement of two particle systems quantified by Wootters concurrence. We have considered the Kerr space-time. In fact, we have studied the variation of the Wootters concurrence (WC) as a function of the various parameters such as q center of mass momentum of the wave packet and Σ (black hole rotation parameter), z (distance from the black hole). It turns out that the behavior of the WC depends strongly on those parameters (see Figures (4-7), (4-8), (4-9), (4-10), (4-11) and (4-12)), the effect of black hole rotation on quantum entanglement is studied, It can be seen that as the amount of Σ increases, so does the $C(\varrho^f)$. If we have enough information, we may also be able to determine the effect of disk-accretion on quantum entanglement because the area of the event horizon changes as the black hole's angular momentum increases. On the other hand, when we studied WC as a function of q, we found periodic behavior. This can be explained by the fact that the centroid of the wave packet travels more along its circular trajectory in the gravitational field and this leads to more decoherence. The origin of this periodic behavior is the sine and cosine functions in the Wigner rotation, and this confirms previous results about the direct effect of Wigner rotation on quantum entanglement.

4.2.5 Spin Entanglement in Reissner-Nordström Non-Commutative Space-Time

we consider the Reissner-Nordström metric for a charged non-rotating black hole in a commutative space-time. It is given by [108]

$$ds^2 = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2})dt^2 + (1 - \frac{2M}{r} + \frac{Q^2}{r^2})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{4.61}$$

with M and Q are mass and charge respectively, Following ref [109], the Seiberg Witten vierbein $\widehat{e}^a_\mu$ in a non-commutative gauge gravity is given by

$$\widehat{e}^a_\mu = e^a_\mu(x) - i\widetilde{\eta}^{v\rho}e^a_{\mu v\rho}(x) + \widetilde{\eta}^{v\rho}\widetilde{\eta}^{\lambda\tau}e^a_{\mu v\rho\lambda\tau}(x) + O(\widetilde{\eta}^3) \tag{4.62}$$

where

$$e^a_{\mu v\rho} = \frac{1}{4}[w^{ac}_v\partial_\rho e^d_\mu + (\partial_\rho w^{ac}_\mu + R^{ac}_{\rho\mu})e^d_v]\eta_{cd} \tag{4.63}$$

$$
\begin{aligned}
e^{a}_{\mu v\rho\lambda\tau} \;=\;& \frac{1}{32}[2\{R_{\tau v}, R_{\mu\rho}\}^{ab}e^{c}_{\lambda} - w^{ab}_{\lambda}(D_{\rho}R^{cd}_{\tau\mu} + \partial_{\rho}R^{cd}_{\tau\mu})e^{m}_{v}\eta_{dm} \\
& -\{w_{\nu}, (D_{\rho}R_{\tau\mu} + \partial_{\rho}R_{\tau\mu})\}^{cd}e^{c}_{\lambda} - \partial_{\tau}\{w_{\nu}, (\partial_{\rho}w_{\mu} + R_{\rho\mu})\}^{ab}e^{c}_{\lambda} \\
& -w^{ab}_{\lambda}\partial_{\tau}(w^{cd}_{\nu}\partial_{\rho}e^{m}_{\mu} + (\partial_{\rho}w^{cd}_{\mu} + R^{cd}_{\rho\mu})e^{m}_{\mu})\eta_{dm} + 2\partial_{v}w^{ab}_{\lambda}\partial_{\rho}\partial_{\tau}e^{c}_{\mu} \\
& -2\partial_{\rho}(\partial_{\tau}w^{ab}_{\mu} + R^{cd}_{\tau\mu})\partial_{v}e^{c}_{\lambda} - \{w_{\nu}, (\partial_{\rho}w_{\lambda} + R_{\rho\lambda})\}^{ab}\partial_{\tau}e^{c}_{\mu} \\
& -(\partial_{\tau}w^{ab}_{\mu} + R^{ad}_{\tau\mu})(w^{cd}_{\nu}\partial_{\rho}e^{m}_{\lambda} + (\partial_{\rho}w^{cd}_{\lambda} + R^{cd}_{\rho\lambda})e^{m}_{v}\eta_{dm})]\eta_{bc}
\end{aligned}
\tag{4.64}
$$

where $\widetilde{\eta}^{v\rho}$ is non-commutativity anti-symmetric matrix elements defined as

$$
[\widetilde{x}^{\mu}, \widetilde{x}^{v}] = i\widetilde{\eta}^{\mu\nu} \tag{4.65}
$$

and $\widetilde{x}^{\mu}$ are the non-commutative space-time coordinates operators. Here w^{ab}_{λ} (resp.D_{ρ}) is the commutative spin connection (resp. covariante derivative) and $R^{ad}_{\mu\nu} = e^{a}_{\alpha}e^{b}_{\beta}R^{\alpha\beta}_{\mu\nu}$, where $R^{\alpha\beta}_{\mu\nu}$ is the Riemann tensor. The commutative space-time vierbein and Minkowski metric are denoted by e^{a}_{μ} and $\eta_{\mu b}$ respectively. The non-commutative metric is given by

$$
\widehat{g}_{\mu\nu} = \frac{1}{2}(\widehat{e}^{a}_{\mu} * \widehat{e}_{av} + \widehat{e}^{a}_{v} * \widehat{e}_{a\mu}) \tag{4.66}
$$

where "$*$" is the Moyal star product [110], Straightforward calculations using the Maple 13 and setting $z = \frac{r}{r_s}$, $y = \frac{Q^2}{r_s^2}$ and $\lambda = \frac{\widetilde{\eta}^2}{r_s^2}$, with choosing the only non-vanishing components of the NC parameter $\widetilde{\eta}^{01} = \widetilde{\eta}^{01} = \widetilde{\eta}$, one has

$$
\begin{aligned}
F &= \widehat{g}_{00} = -(1-z+y) - (2z - 9y - \frac{11}{4}z^2 + 15zy - 14y^2)z^2\widehat{\theta}^2 \\
G &= \widehat{g}_{11} = \frac{1}{(1-z+y)} + \frac{(-z + \frac{3}{4}z^2 + 3y - 3yz + 2y^2)z^2\widehat{\theta}^2}{(1-z+y)^2} \\
H &= \widehat{g}_{22} = (1 + \frac{(1 - \frac{17}{2}z + \frac{17}{2}z^2 + 27y - \frac{75}{2}zy + 30y^2)z^2\widehat{\theta}^2}{4(1-z+y)})\frac{r_s^2}{z^2} \\
I &= \widehat{g}_{33} = (1 + \frac{(-2z + 8y + z^2 - 8zy + 8y)z^2\widehat{\theta}^2}{4(1-z+y)})\frac{r_s^2}{z^2}
\end{aligned} \tag{4.67}
$$

By using the eq. (4.39), (3.29) and (4.67), we can study the behavior of quantum entanglement.

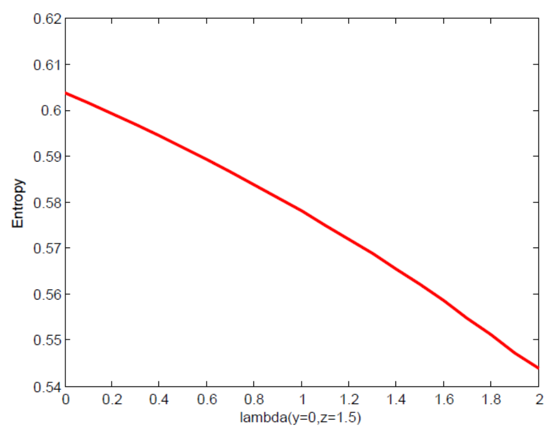


Figure 4-13: $E(\varrho)$ as a function of λ for fixed $z = 1.5$, $y = 0$, $\alpha = 1$, $q = 0.01$.

FIG (4-13) displays the variation of the entanglement $E(\varrho)$ as a function of the NC parameter $\widetilde{\eta}^2$, for a non-charged ($Q = 0$) black hole and fixed $z = 1.5$, $y = 0$, $\alpha = 1$, $q = 0.01$. Notice that if $\widetilde{\eta}^2$ increases, $E(\varrho)$ decreases. Thus, $\widetilde{\eta}$ plays an important role in the value changing of entanglement. In fact, as it was pointed out in ref [109], the NC parameter $\widetilde{\eta}$ can be considered as a magnetic field contributing to the matter density ρ and therefore affecting the curvature of the space-time through its contribution to GF.

Consequently if $\widetilde{\eta}^2$ increases, the GF increases and the information decreases. Including the contribution of NC of space-time, it generates additional terms proportional to $\widetilde{\eta}^2$. In fact the gravitational potential $\widehat{g}_{00}$ will be of the form

$$\widehat{g}_{00} = \widehat{A} + \widehat{B}Q^2 + \widetilde{\eta}^2(\widehat{D}Q^4 + \widehat{C}Q^2 + \widehat{F}) \tag{4.68}$$

where

$$\begin{aligned} \widehat{A} &= -1 + \frac{1}{z}, \qquad \widehat{B} = -\frac{1}{zr_s^2}, \qquad \widehat{D} = \frac{7}{z^6 r_s^6} \\ \widehat{C} &= (9z - 15)\frac{1}{4z^5 r_s^4}, \qquad \widehat{F} = (-2z + \frac{11}{4})\frac{1}{4z^4 r_s^4} \end{aligned} \tag{4.69}$$

The behavior of the entanglement $E(\varrho)$ depends strongly on the sign of $(\widehat{D}Q^4 + \widehat{C}Q^2 + \widehat{F})$. Considering $\widehat{A}$ and $\widehat{B}$ negative:

1) If $Q >> 1$, the term $\widehat{D}Q^4$ dominates. Since $\widehat{D} > 0$, and if $\widetilde{\eta}^2$ increases the GF decreases leading to an increase in $E(\varrho)$ (as is the case in FIG (4-14).

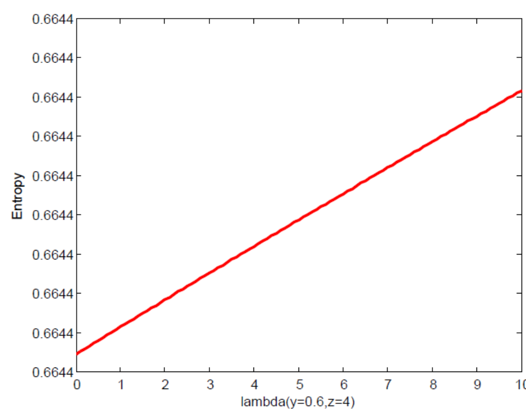


Figure 4-14: $E(\varrho)$ as a function of λ for fixed $z = 4$, $y = 0.6$, $\alpha = 1$, $q = 0.01$.

2) If $Q << 1$, then $\widehat{F}$ dominates and its sign will determine the behavior of $E(\varrho)$

as a function of $\widetilde{\eta}^2$. If $\widehat{F} > 0$ GF increases and $E(\varrho)$ decreases then we return to the case in FIG (4-13).

FIG (4-15) represents the variation of $E(\varrho)$ as a function of z for fixed $y = 0$, $\lambda = 0$, $\alpha = 1$, $q = 0.01$, (the case of commutative Schwarzschild space-time). Notice that we will reproduce the same behavior as in ref [85].

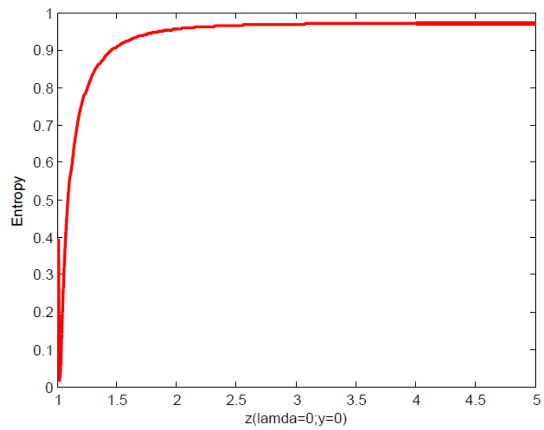


Figure 4-15: The variation of $.E(\varrho)$ as a function of z for fixed $\lambda = 0$, $\alpha = 1$, $y = 0$, $q = 0.01$

FIG (4-16) shows the variation of $E(\varrho)$ as a function of z for fixed $Q \neq 0$, $\widetilde{\eta} = 0$, (the case of commutative Reissner-Nordstrom space-time). Notice that the same behavior as in ref [111] is obtained.

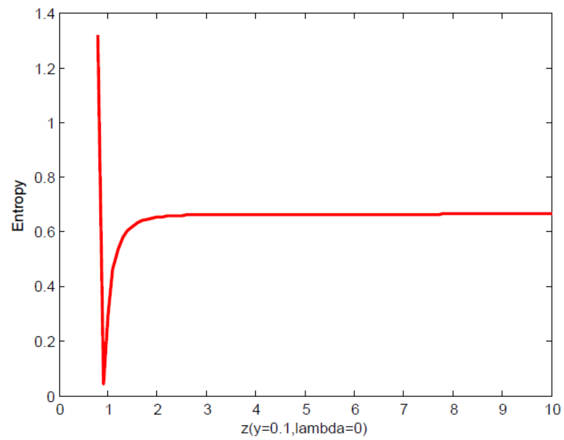


Figure 4-16: The variation of $E(\varrho)$ as a function of z for fixed $y = 0.1, \lambda = 0$, $\alpha = 1$, $q = 0.01$.

FIG (4-17) shows the variation of $E(\varrho)$ as a function of z and fixed $\lambda = 0.01$, $y = 0$, $\alpha = 1$, $q = 0.01$, this case is the Schwarchild black hole in non-commutative space-time.

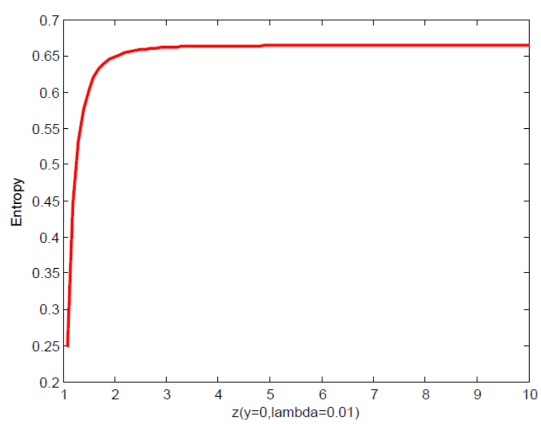


Figure 4-17: $E(\varrho)$ as a function of z and fixed 0.01, $y = 0$, $\alpha = 1$, $q = 0.01$.

FIG (4-18)

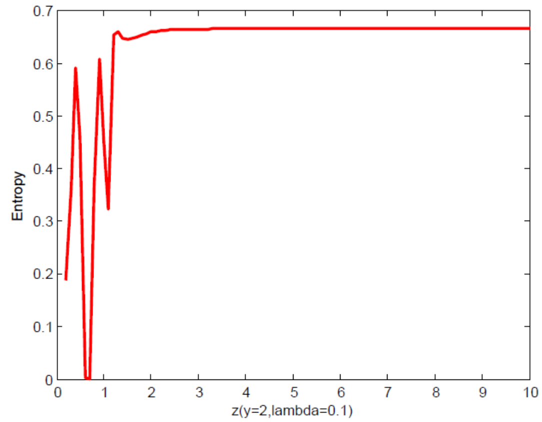


Figure 4-18: The variation of $E(\varrho)$ as a function of z for fixed $\lambda = 0.1$, $y = 2$, $\alpha = 1$, $q = 0.01$.

FIG (4-18) represents the variation of $E(\varrho)$ as a function of z for fixed $\lambda = 0.1$, $y = 2$, $\alpha = 1$, $q = 0.01$, it is the case of Reissner Nordstrom Black Hole in non-commutative space-time. Notice that far from the oscillatory behavior region, when z (or r) increases, the GF $\widehat{g}_{00}$ decreases until reaching a saturation value (~ 1) where $E(\varrho)$ is maximal, notice that for smaller values of r ($\rightarrow 0$ near black hole singularity) where the gravitational field is infinite, the entanglement is minimal, if we go far from the singularity (z increases) the gravitational field decreases and therefore the information increases and thus the entanglement. The oscillatory behavior disappears when we enter the stability region where $E(\varrho) \sim 0.67$. The number of picks and minima depends strongly on the values of the various parameters λ, y, α and q. Concerning the non-commutavity effect on the $E(\varrho)$, it is clear from eq. (4.68) that for smaller values of z, as $\widetilde{\eta}$ increases the gravitational field $\widehat{g}_{00}$ becomes more important (increases) and therefore $E(\varrho)$ decreases. For larger values of z, the effect is almost negligible since the terms in order of $\frac{1}{z^4}$, $\frac{1}{z^5}$, $\frac{1}{z^6}$ decrease faster than the commutative terms in order of $\frac{1}{z}$. Notice also that y increases, the GF increases (the term $\widetilde{\eta}^2$ $(\widehat{D}Q^4)$ dominates at larger value of Q). Thus, the NC effect on the $E(\varrho)$ becomes more important for charged black hole than the neutral ones (if the charge Q increases $E(\varrho)$ decreases).

Table (4.1) summarizes the effect of the black hole charge on the $E(\varrho)$. It is worth mentioning that in order to keep the perturbative expansion with respect to $\widetilde{\eta}^2$ reliable, one must have

$$\left|\widetilde{\eta}^2 A_1\right| < |A_0| \tag{4.70}$$

where $A_0 = \widehat{A} + \widehat{B}Q^2$ and $A_1 = (\widehat{D}Q^4 + \widehat{C}Q^2 + \widehat{F})$, this implies new constraints on the space parameters z, y.

z	$E(\varrho)(y=0)$	$E(\varrho)(y=10)$
2	0.64694	0.6072
4	0.664	0.6567
5	0.6644	0.6626
6	0.6646	0.6641

Table 4.1: Summarizes the effect of the black hole charge on the $E(\varrho)$

4.2.6 Comparison between singlet and triplet state of entanglement

To gain a thorough understanding, we compare the entanglement behavior in the triplet and singlet states, by using concurrence

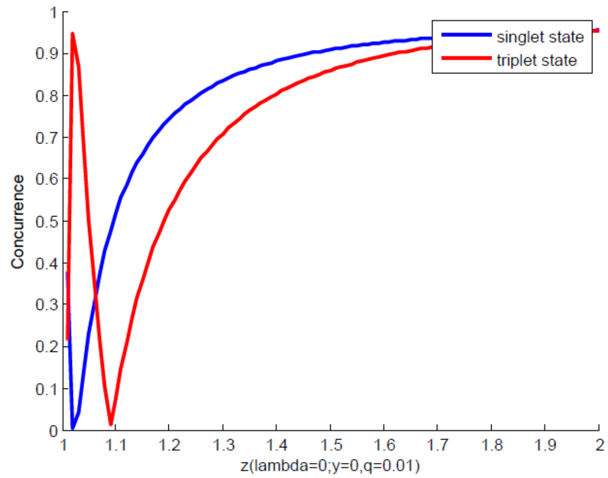


Figure 4-19: The variation of $C(\varrho^f)$ as a function of w for fixed $\lambda = 0$, $y = 0$, $\alpha = 1$, $q = 0.01$.

FIG (4-19) show how the concurrence varies as a function of z for fixed values of q, y, α and λ in singlet and triplet state, respectively. We found the same behavior with FIG (4-18), where for smaller values of $r(r \rightarrow 0$ near the black hole horizon), the gravitational field is infinite, the entanglement is minimal ($C(\varrho^f) \sim 0$). If we go far away from the singularity, (z increases) the gravitational field decreases, so the information increases until a saturated bound of the maximal entanglement ($C(\varrho^f) \sim 1$). By monitoring both of the curves, we notice that in singlet state when z is at the value of 1.23, $C(\varrho^f)$ takes the value of 0.7789. While in triplet state it gives $z = 1.23$, $C(\varrho^f) = 0.5962$.

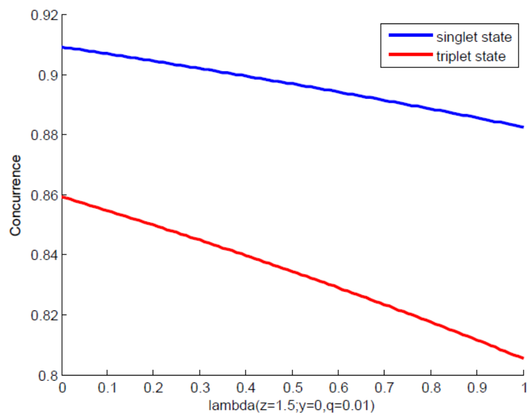


Figure 4-20: The variation of $C(\varrho^f)$ as a function of λ for fixed $z = 1.5$, $y = 0$, $\alpha = 1$, $q = 0.01$.

FIG (4-20) displays the variation of the concurrence as a function of λ by fixing $\alpha = 1$, $z = 1.5$ and $q = 0.01$, for both state singlet and triplet, the concurrence is a decreasing function. This is due to the fact that the gravitational potential g_{00} increases, as we mentioned in FIG (4-14). Take note of this for singlet state when $z = 0.1$, $C(\varrho^f) = 0.9069$, and triplet state when $z = 0.1$, $C(\varrho^f) = 0.8547$.

As it is displayed in FIGs (4-19) and (4-20), we can say that the information (entanglement) for the first is greater or equal to the second, and that the singlet state is more resistant to changes induced by motion than the triplet state, this is due to the fact that for the single state, there is a minimum number of parameters and as mentioned before, gravity decreases the information (the effect of gravity on the single state is less than that of the triplet state).

Chapter 5

General Conclusion

We have seen in this thesis some cosmological models. The first chapter provided a brief overview of General Relativity in space-time. In the second chapter, we were interested in explaining the phase of accelerated expansion in the late universe without the dark energy concept, by introducing the exra-dimension into some cosmological models. At first, we started with the Friedmann-Robertson-Walker Model in five dimensions. We adopted two equations of state, $p = w\rho$ and $p_5 = \gamma\rho$, for perfect fluid in both 4D and 5D. We noticed that we can have an accelerated expansion in four dimensions by considering the constant speed in extra-dimension for the case of $w = 0, \frac{1}{3}$. We extended the study by using Dynamical Study with this model. Unfortunately, the results obtained from this model were not consistent with the fact that the universe is expanding at an accelerating rate. For that, we considered this model with non-perfect fluid, because incorporating viscosity concepts into the cosmic fluid from a hydrodynamic standpoint, it would appear the most natural. Where we took $\overline{p}$ the pressure in 4D, where $\overline{p} = p + h(t)H_R$, to find the Hubble parameter expression in 4D by considering the extra-dimension has constant speed. We observed a similar behavior regarding the perfect fluid case, with a slight change in the rate. The difference is more obvious in a universe dominated by radiation than in one dominated by matter. That can explain why the inflation model was discussed with shear and bulk viscosity, the fact that shear viscosity may affect the expansion history of the universe [118], [111]. The Dynamical Study was taken into account in this case. We found that $w > 0$, but $\gamma < 0$, so we can have an accelerated expansion in 4D with positive pressure, but the 5D shows a negative pressure, which is explained by dark energy. Another model was under investigation. Interesting results for Kantowski–Sachs space-time with the cosmological constant have been found. To obtain an accelerated expansion in both 4D and 5D with positive pressure (w, $\gamma > 0$), the cosmological constant should be positive, under the condition $w < \frac{1}{3}$. Finally, there is one more model with extra dimensions, F(R) gravity, where $f(R) = f_0 R^n$, we can have an accelerated expansion in both 4D and 5D, with positive pressure, as demonstrated in this model. This last one gave interesting results compared with the other models that have been discussed in part one of this thesis.

In second part, we were interested in understanding the effect of the gravitational field on the quantum spin entanglement. By considering a system composed of two particles moving in this field, the system is described by packets of centroid waves, and when applying the idea of local inertial frames, both the increasing speed of the centroid and the shape of the gravitational field cause a Wigner rotation that influences the wave packet. In the fourth chapter, we attempted to discuss the entanglement of this wave packet by determining its expression, by finding of a general formalism capable of describing the entanglement in a general metric. The beginning was with de Sitter–Schwarzschild metric in order to see the cosmological constant effect on the quantum entanglement. As we saw, it has a small effect, but when the cosmological constant is increasing, it makes the entanglement decreases. And the effect is more pronounced with a small positive value of Λ. As well as considering Kerr space-time, we have studied the singlet state of spin entanglement by using Wootters' concurrence. We looked at how the Wootters' concurrence varied. (WC) as a function of several characteristics, such as the wave packet's q center of mass momentum, black hole rotation parameter , and distance from the black hole. It turns out that the WC's behavior is heavily influenced by those parameters. The effect of black hole rotation on the quantum entanglement is being investigated. It can be seen that the concurrence increases as the amount of rotation increases. For the last model, the non-commutative Reissner-Nordström model, we studied spin entanglement. Regarding the non-commutative case, the variation of the quantum entanglement as a function of the NC parameter and the black hole charge y is discussed. We have noticed that the NC effect becomes more important in a charged black hole, so the behavior depends on the black hole's characteristics and not only on the kind of particles (bosons or fermions) [119]. We found that as NC parameter increases, it decreases, as if NC parameter is playing the role of gravity. On the other hand, the NC parameter was considered as having antigravity properties (quintessence, dark energy, etc.), so the NC parameter can induce two terms with opposing signs and that was confirmed in [119], [120].

Appendix A

Calculation

Application about vector calculation: In curvilinear coordinates:

1- Divergence (covariante): we have

$$D_\nu A^\mu = \partial_\nu A^\mu + \Gamma^\mu_{\lambda\nu} A^\lambda \tag{A.1}$$

$$\Rightarrow D_\mu A^\mu = \partial_\mu A^\mu + \Gamma^\mu_{\lambda\mu} A^\lambda$$

$$\begin{aligned} \Gamma^\mu_{\lambda\mu} &= g^{\mu\nu}\Gamma_{\lambda\mu,\nu} = \frac{1}{2}g^{\mu\nu}\left[\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\nu\lambda} - \partial_\nu g_{\mu\lambda}\right] \\ &= \frac{1}{2}g^{\mu\nu}\partial_\lambda g_{\mu\nu} = \frac{1}{2}G^{-1}\partial_\lambda G \\ &= \frac{1}{2}g^{-1}\partial_\lambda g \\ &= \frac{1}{2}g^{-1}dg \end{aligned} \tag{A.2}$$

and

$$\begin{aligned} D_\mu A^\mu &= \partial_\mu A^\mu + \Gamma^\mu_{\lambda\nu} A^\lambda = \partial_\mu A^\mu + \frac{1}{2} g^{-1} \partial_\lambda g A^\lambda \\ &= \partial_\mu A^\mu + \frac{1}{2} g^{-1} \partial_\lambda g A^\lambda \\ &= \partial_\mu A^\mu + |g|^{-\frac{1}{2}} \partial_\mu |g|^{\frac{1}{2}} A^\mu \\ &= |g|^{-\frac{1}{2}} \partial_\mu \left[|g|^{\frac{1}{2}} A^\mu \right] \end{aligned} \tag{A.3}$$

So we can write

$$D_\mu A^\mu = \frac{1}{\sqrt{|g|}} \partial_\mu \left[\sqrt{|g|} A^\mu \right] \tag{A.4}$$

Laplacian or d'Alembert operator

$$\Box \Phi(x) = D_\mu D^\mu \Phi(x) \tag{A.5}$$

$$\Box \Phi(x) = \frac{1}{\sqrt{|g|}} \partial_\mu \left[\sqrt{|g|} g^{\mu\nu} \partial_\nu A^\mu \right] \tag{A.6}$$

curl operator

$$D_\mu A_\nu - D_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{A.7}$$

Demonstration

$$\begin{aligned} D_\mu A_\nu - D_\nu A_\mu &= \left(\partial_\mu A_\nu - \Gamma^\lambda_{\nu\mu} A_\lambda \right) - \left(\partial_\nu A_\mu - \Gamma^\lambda_{\mu\nu} A_\lambda \right) \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu \end{aligned} \tag{A.8}$$

Red shift

An atom on the sun emits radiation of frequency $\nu_0 = \dfrac{1}{\tau_0}$ we have

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} = g_{00}dt^2 \Rightarrow \tau_0 = \sqrt{g_{00}}\tau \tag{A.9}$$

And we have

$$\nu_0 = \frac{1}{\tau_0} = \frac{1}{\sqrt{g_{00}}\tau} = \frac{\nu}{\sqrt{g_{00}}} \tag{A.10}$$

$$\begin{aligned} \Rightarrow \quad & \frac{\Delta\nu}{\nu} = \frac{\nu - \nu_0}{\nu_0} = \frac{\nu_0\sqrt{g_{00}} - \nu_0}{\nu_0} \\ = \quad & = \sqrt{g_{00}} - 1 = 1 - 2\frac{GM}{r} - 1 = -2\frac{GM}{r} = 2 \cdot 10^{-6} \end{aligned} \tag{A.11}$$

Riemann tensor $R^{\nu}_{\rho\mu\lambda}$

We know that

$$D_{\mu}U^{\nu} = \partial_{\mu}U^{\nu} + \Gamma^{\nu}_{\rho\mu}U^{\rho} \tag{A.12}$$

So:

$$\begin{aligned} D_{\lambda}\left(D_{\mu}U^{\nu}\right) &= \partial_{\lambda}\left(D_{\mu}U^{\nu}\right) - \Gamma^{\sigma}_{\mu\lambda}D_{\sigma}U^{\nu} + \Gamma^{\nu}_{\sigma\lambda}D_{\mu}U^{\sigma} \\ &= \partial_{\lambda}\left(\partial_{\mu}U^{\nu} + \Gamma^{\nu}_{\rho\mu}U^{\rho}\right) - \Gamma^{\sigma}_{\mu\lambda}\left[\partial_{\sigma}U^{\nu} + \Gamma^{\nu}_{\rho\sigma}U^{\rho}\right] + \Gamma^{\nu}_{\sigma\lambda}\left[\partial_{\mu}U^{\sigma} + \Gamma^{\sigma}_{\rho\mu}U^{\rho}\right] \\ &= \partial_{\lambda}\partial_{\mu}U^{\nu} + \left(\partial_{\lambda}\Gamma^{\nu}_{\rho\mu}\right)U^{\rho} + \Gamma^{\nu}_{\rho\mu}\partial_{\lambda}U^{\rho} - \Gamma^{\sigma}_{\mu\lambda}\partial_{\sigma}U^{\nu} - \Gamma^{\sigma}_{\mu\lambda}\Gamma^{\nu}_{\rho\sigma}U^{\rho} + \Gamma^{\nu}_{\sigma\lambda}\partial_{\mu}U^{\sigma} + \Gamma^{\nu}_{\sigma\lambda}\Gamma^{\sigma}_{\rho\mu}U^{\rho} \end{aligned} \tag{A.13}$$

$$D_{\mu}\left(D_{\lambda}U^{\nu}\right) = \partial_{\mu}\left(D_{\lambda}U^{\nu}\right) - \Gamma^{\sigma}_{\lambda\mu}D_{\sigma}U^{\nu} + \Gamma^{\nu}_{\sigma\mu}D_{\lambda}U^{\sigma} \tag{A.14}$$

We search for symmetric terms in μ and ν we find the last result:

$$[D_\lambda, D_\mu]\, U^\nu = \left[\partial_\lambda \Gamma^\nu_{\rho\mu} - \partial_\mu \Gamma^\nu_{\rho\lambda} + \Gamma^\nu_{\sigma\lambda}\Gamma^\sigma_{\rho\mu} - \Gamma^\nu_{\sigma\mu}\Gamma^\sigma_{\rho\lambda}\right] \tag{A.15}$$

So we can write

$$[D_\lambda, D_\mu]\, U^\nu = R^\nu_{\rho\mu\lambda} U^\rho \tag{A.16}$$

in the last we find

$$R^\nu_{\rho\mu\lambda} = \left[\partial_\lambda \Gamma^\nu_{\rho\mu} - \partial_\mu \Gamma^\nu_{\rho\lambda} + \Gamma^\nu_{\sigma\lambda}\Gamma^\sigma_{\rho\mu} - \Gamma^\nu_{\sigma\mu}\Gamma^\sigma_{\rho\lambda}\right] \tag{A.17}$$

Bianchi identity:

$$D_\mu R^\rho_{\sigma\nu\lambda} + D_\nu R^\rho_{\sigma\lambda\mu} + D_\lambda R^\rho_{\sigma\nu\mu} = 0 \tag{A.18}$$

Demonstration:

1-

$$\begin{aligned}
[D_\lambda, D_\mu]\, U_\sigma &= [D_\lambda, D_\mu]\, g_{\sigma\nu} U^\nu \qquad \text{(A.19)}\\
&= g_{\sigma\nu}\,[D_\lambda, D_\mu]\, U^\nu = g_{\sigma\nu} R^\nu_{\rho\mu\lambda} U^\rho\\
&= R_{\sigma\rho\mu\lambda} U^\rho = -R_{\rho\sigma\mu\lambda} U^\rho\\
&= -R^\rho_{\sigma\mu\lambda} U_\rho
\end{aligned}$$

So

$$[D_\lambda, D_\mu]\,(A_\rho B^\sigma) = R^\sigma_{\mu\lambda\tau} A_\rho B^\tau - R^\tau_{\rho\mu\lambda} A_\tau B^\sigma \tag{A.20}$$

and

$$[D_\lambda, D_\nu]\,(D_\mu U^\sigma) = -R^\sigma_{\mu\nu\lambda} D_\sigma U^\rho + R^\rho_{\sigma\nu\lambda} D_\mu U^\sigma \tag{A.21}$$

2- $D_\mu\left([D_\lambda, D_\nu]\,U^\rho\right) = D_\mu\left(R^\rho_{\sigma\nu\lambda}U^\sigma\right)$

$$= \left(D_\mu R^\rho_{\sigma\nu\lambda}\right)U^\sigma + R^\rho_{\sigma\nu\lambda}D_\mu U^\sigma$$

in the end we get:

$$[D_\mu,[D_\lambda,D_\nu]]\,U^\rho = D_\mu R^\rho_{\sigma\nu\lambda}U^\sigma + R^\sigma_{\mu\nu\lambda}D_\sigma U^\rho$$

$$+\,[D_\lambda,[D_\nu,D_\mu]]\,U^\rho = D_\lambda R^\rho_{\sigma\mu\nu}U^\sigma + R^\sigma_{\lambda\mu\nu}D_\sigma U^\rho$$

$$+\,[D_\nu,[D_\mu,D_\lambda]]\,U^\rho = D_\nu R^\rho_{\sigma\lambda\mu}U^\sigma + R^\sigma_{\nu\lambda\mu}D_\sigma U^\rho$$

$$0 \qquad = [\text{Bianchi}] + (\qquad = 0)\,D_\sigma U^\rho$$

Appendix B

phase portrait

Phase portrait is a geometric representation of the trajectories of a dynamical system in phase space.

B.1 phase portrait of a linear 2-dimensionnal systems

We assume the following system

$$\begin{cases} \dot{x}_1 = ax_1 + bx_2 \\ \dot{x}_2 = cx_1 + dx_2 \end{cases} \tag{B.1}$$

and we try to find its phase portrait, we put

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I_2) = \lambda^2 - (a+b)\lambda + (ad - bc) \\ &= \lambda^2 - \lambda Tr(A) + \det(A) \\ &= \lambda^2 - \lambda T + D \end{aligned} \tag{B.2}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{B.3}$$

Case 1 $\Delta = T^2 - 4D > 0$, there is tow solutions

$$\lambda_{1,2} = \frac{T \pm \sqrt{T^2 - 4D}}{2} \tag{B.4}$$

the general solution is

$$x(t) = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2 \tag{B.5}$$

· if $D < 0$, $(\lambda_1 > 0 > \lambda_2)$: the trajectories are "parabola" which approach to the fixed point called "saddle point" , (see Fig (B-1).

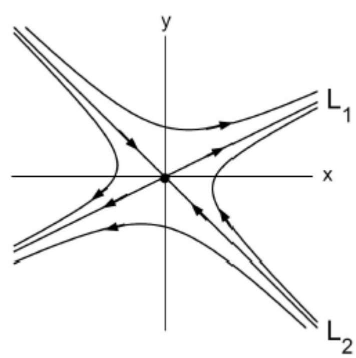


Figure B-1: Generic trajectories for Saddle point

· if $D > 0$ and $T > 0$, eigenvalues are positive, the trajectories are " parabola " which approach to the fixed point called "nodal source", this point is unstable (see fig.B-2)).

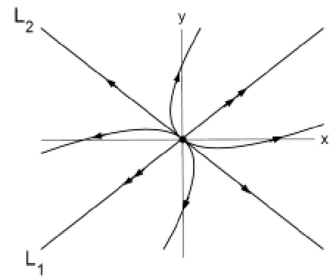


Figure B-2: Generic trajectories for Nodal Source ($\twoheadrightarrow$ fast escape to ∞)

· if $D > 0$ and $T < 0$: eigenvalues are negative the trajectories are " parabola " which approach to the fixed point called "Nodal Sink" this point is stable (see fig.B-3).

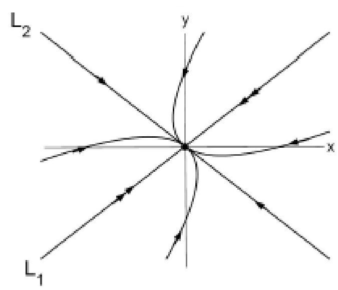
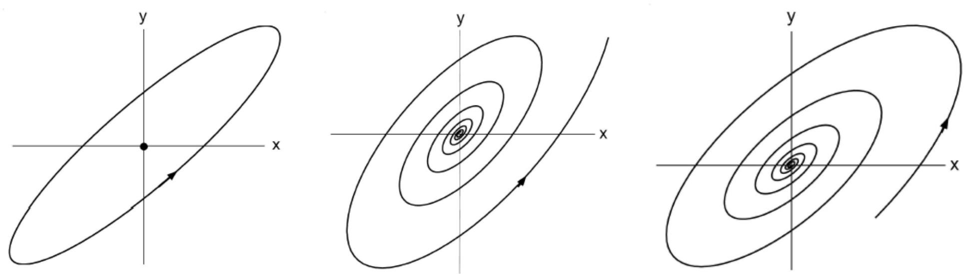


Figure B-3: Generic trajectories for Nodal Sink ($\twoheadleftarrow$ fast aproach to 0)

Case 2: $\Delta = T^2 - 4D < 0$, then, $\lambda = \alpha + i\beta$, where $\alpha = \frac{T}{2}, \beta = \sqrt{4D - \frac{T^2}{2}}$. We have λ is complex then the eigenvector ($V = u + iw$) is complex too, the general solution has the form

$$x(t) = e^{\alpha t}[c_1(u \cos \beta t - w \sin \beta t) + c_2(u \sin \beta t + w \cos \beta t)] \tag{B.6}$$



$\alpha = 0$, Center $\alpha > 0$, Spiral Source $\alpha < 0$, Spiral Sink

Figure B-4: Subcases according to α

Note: the pictures are from Math 3331 Differential Equations 9.3 Phase Plane Portraits, by Jiwen He, Department of Mathematics, University of Houston.

Appendix C

Relativistic Quantum Mechanics

C.1 Poincaré Group and its representations

We have two equivalent definitions , the mathematical definition of Poincaré , group P, that Groups isometries of Minkowski space-time (an isometry is a transformation that keeps the lengths) , while the physical definition is to Group conversions between inertial frames of reference in Minkowski space-time. The translations group and the Lorentz group produce the Poincaré group. The full Lorentz group $SO(3,1)$ (the "1" indicates the additional, time-like dimension) is defined as

$$\pounds := \left\{ \Lambda \in GL(4,\mathbb{R}); \Lambda^T \eta \Lambda = \eta \right\} \tag{C.1}$$

where η is the Minkowski metric $diag(-+++)$. Where Λ are the entries of a 4-matrix representing a so-called Lorentz transformation. The Lorentz group leaves the norm x^2 of a vector invariant is not enough because on physical grounds we need the line element $\eta_{\mu\nu} dx^\mu dx^v = -c^2 dt^2 + dx^2$ to be invariant. This guarantees that the speed of light is the same in every inertial frame, and it allows us to add constant translations to the Lorentz transformation

$$x^{'} = T(\Lambda, a)x = \Lambda x + a \tag{C.2}$$

where a are representing a translation in space-time. If we have two consecutive Poincaré transformations

$$T(\Lambda^{'}, a^{'})T(\Lambda, a) = T(\Lambda^{'}\Lambda, a^{'} + \Lambda^{'}a) \tag{C.3}$$

the inverse element

$$T^{-1}(\Lambda, a) = T(\Lambda^{-1}, -\Lambda^{-1}a) \tag{C.4}$$

Consider now the representations $U(\Lambda, a)$ of the Poincaré group on some vector space, and its elements can be written as

$$U(\Lambda, a) = \exp(\frac{i}{2}\varepsilon_{\mu v}M_{\mu v})\exp(ia_{\mu}P^{\mu}) = 1 + \frac{i}{2}\varepsilon_{\mu v}M_{\mu v} + ia_{\mu}P^{\mu} + ... \tag{C.5}$$

where the explicit forms of $U(\Lambda, a)$ and the generators $M_{\mu v}$ and P^{μ} depend on the representation. Since $\varepsilon_{\mu v}$ is totally antisymmetric, $M_{\mu v}$ can also be chosen antisymmetric. It contains the six generators of the Lorentz group, whereas the momentum operator P^{μ} is the generator of space-time translations. $M_{\mu v}$ and P^{μ} form a Lie algebra whose commutator relations can be derived from

$$U(\Lambda, a)U(\Lambda^{'}, a^{'})U^{-1}(\Lambda, a) = U(\Lambda\Lambda^{'}\Lambda^{-1}, a + \Lambda a^{'} - \Lambda\Lambda^{'}\Lambda^{-1}a) \tag{C.6}$$

which follows from the composition rules C.3 and C.4. Inserting infinitesimal transformations C.5 for each $U(\Lambda = 1 + \varepsilon, a)$, with $U^{-1}(\Lambda, a) = U(\Lambda = 1 - \varepsilon, -a)$, keeping only linear terms in all group parameters $\varepsilon, \varepsilon^{'}, a$ and $a^{'}$, and comparing coefficients of the terms $\sim \varepsilon\varepsilon^{'}, a\varepsilon^{'}, \varepsilon a^{'}$ and $aa^{'}$ leads to the identities

$$\begin{aligned} i\left[M^{\mu v}, M^{\rho v}\right] &= \eta^{v\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{v\sigma} - \eta^{\sigma\mu}M^{\rho v} + \eta^{\sigma v}M^{\rho\mu} \\ i\left[P^{\mu}, M^{\rho\sigma}\right] &= \eta^{\mu\rho}P^{\sigma} - \eta^{\mu\sigma}P^{\rho} \\ \left[P^{\mu}, P^{v}\right] &= 0 \end{aligned} \tag{C.7}$$

The representation of the Poincaré group on the state vectors of infinite (because it's non-compact) dimensional Hilbert space is unitary (connected Lie group) and can be written as

$$\left|\psi^{'}\right\rangle = U(\Lambda, a)\left|\psi\right\rangle \tag{C.8}$$

The Poincaré algebra P has two Casimir element, vectors commuting with the generators of the algebra.

Particle states

The one-particle state should be defined as $\{|m, s, p^0, \mathbf{p},\sigma\rangle\}$, we can simplify notation due to the invariance of the Casimir operators and set

$$\left|m, s, p^0, \mathbf{p},\sigma\right\rangle = |p, \sigma\rangle \tag{C.9}$$

such that the basis states are labelled by their four-momentum and the Z-component of the spin (later simply called spin). This correspond to a basis of plane waves and, thus, transform under translations as

$$U(I, a)\,|p, \sigma\rangle = \exp(-ipa)\,|p, \sigma\rangle \tag{C.10}$$

Using the transformation property of the four-momentum operator $U^{-1}(\Lambda)P^{\mu}U(\Lambda) = \Lambda^{\mu}_{\lambda}P^{\lambda}$ we can write

$$\begin{aligned} P^{\mu}U(\Lambda)\left|p,\sigma\right\rangle &= U(\Lambda)U^{-1}(\Lambda)P^{\mu}U(\Lambda)\left|p,\sigma\right\rangle \\ &= U(\Lambda)\Lambda^{\mu}_{\lambda}P^{\lambda}\left|p,\sigma\right\rangle \\ &= \Lambda^{\mu}_{\lambda}p^{\lambda}U(\Lambda)\left|p,\sigma\right\rangle \end{aligned} \tag{C.11}$$

Notice that general Lorentz transformation takes the momentum $p^{\mu} \rightarrow \Lambda^{\mu}_{\lambda}p^{\lambda}$ and $U(\Lambda)\left|p,\sigma\right\rangle$ must be a linear combination of all the states with momentum Λp

$$U(\Lambda)\left|p,\sigma\right\rangle = \sum_{\sigma'} D_{\sigma'\sigma}(\Lambda,p)\left|\Lambda p,\sigma'\right\rangle \tag{C.12}$$

Since $U(\Lambda)$ is a representation it respect the group multiplication imposing conditions on the values of $D_{\sigma'\sigma}$.Those conditions are satisfied when we restrict $D_{\sigma'\sigma}(\Lambda,p)$ to $D_{\sigma'\sigma}(W,p)$ where W are Lorentz transformations that leave invariant a chosen standard momentum k^{μ}. As a consequence of the above equation, under a Lorentz transformation $U(\Lambda)$, the momentum label p goes to Λp, and the spin transform under the representation $D_{\sigma'\sigma}$ of the little group W (In general if you have a group G which acts on a space X, and an element x in X, the little group of x is the subgroup of G that leaves x invariant) . We Consider the Lorentz transformation that takes k^{μ} to $p^{\mu} = L^{\mu}_{v}(p)k^{\nu}$ such that

$$\left|p,\sigma\right\rangle = U(L(p))\left|k,\sigma\right\rangle \tag{C.13}$$

where $L(p)$ is a standard boost taking the standard rest frame 4-momentum $k \equiv (m,0,0,0)$ to an arbitrary 4-momentum p, is an arbitrary LT taking $p \rightarrow \Lambda.p \equiv \Lambda p$. We apply a Lorentz transformation Λ on state $\left|p,\sigma\right\rangle$

$$\begin{aligned} U(\Lambda)\left|p,\sigma\right\rangle &= U(\Lambda)U(L(p))\left|k,\sigma\right\rangle \\ &= U(I)U(\Lambda)U(L(p))\left|k,\sigma\right\rangle \\ &= U(L(\Lambda p))U(L^{-1}(\Lambda p))U(\Lambda)U(L(p))\left|k,\sigma\right\rangle \\ &= U(L(\Lambda p))U(L^{-1}(\Lambda p)\Lambda L(p))\left|k,\sigma\right\rangle \\ &= U(L(\Lambda p))U(W(\Lambda,p))\left|k,\sigma\right\rangle \end{aligned} \tag{C.14}$$

where $U(W(\Lambda,p)) = U(L^{-1}(\Lambda p)\Lambda L(p))$ leaves the standard momentum k invariant, and $L^{-1}(\Lambda p)$ is an inverse standard boost taking the final 4-momentum p back to the particle's rest frame. $W(\Lambda,p)$ is rotation corresponding the Lorentz transformation Λ and momentum p, and it is an element of the little group (called also the stability group) of the standard four-momentum k^{μ}, this rotation do not change the standard four-momentum: $W_{v}^{\mu}k^{v} = k_{\mu}$. On the other hand $U(L(\Lambda p))$ take k to p without touching the spin, then

$$\begin{aligned} U(\Lambda)\left|p,\sigma\right\rangle &= U(W(\Lambda,p))\left|\Lambda p,\sigma\right\rangle \\ &= \sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda,p))\left|\Lambda p,\sigma'\right\rangle \end{aligned} \tag{C.15}$$

Where $D_{\sigma'\sigma}(W(\Lambda,p))$ is a spin-s representation of the rotation $W(\Lambda,p)$. For massive particles where the standard momentum is $k^{\mu}(m,0,0,0)$ in its rest frame, and

$$D^{s}_{\sigma'\sigma}(W) = \left\langle s,\sigma'\right| \exp(i\delta\mathbf{s}.\mathbf{n})\left|s,\sigma\right\rangle \tag{C.16}$$

So, if we are dealing with massive spin $-\frac{1}{2}$ particles

$$D_{\sigma'\sigma}(W) = I\cos\frac{\delta}{2} + i(\widehat{\sigma}.\widehat{n})\sin\frac{\delta}{2} \tag{C.17}$$

by using the relation

$$(\sigma.a)(\sigma.b) = (a.b) + i\sigma(a.b) \tag{C.18}$$

where δ is the Wigner angle and σ are the Pauli matrices. This rotation is a consequence of the non-closeness of the boost generator algebra, two boost is equivalent to a boost and a rotation. That rotation is closely related to Thomas precession and is called the Wigner rotation, for more detail see [112, 113, 114, 115, 116].

Wigner rotation

Wigner is a transformation which leaves k^μ invariant, and $D(W(\Lambda, p))$ represents its action on the state. The explicit form of $L(p)$ is dependent on the class of the four-momenta. For massive particles, $p^\mu p_\mu = -m^2$, and a convenient choice for the standard vector is $k^\mu(m, 0, 0, 0)$. It is then obvious that the set of Wigner transformations leaving k^μ unchanged is just the rotation group $SO(3)$. Furthermore, $L(p)$ can then be taken as the pure Lorentz boost [103]

$$\begin{aligned} L^0_0(p) &= \cosh\chi \\ L^0_i(p) &= L^i_0(p) = \widehat{p}_i \sinh\chi \\ L^j_i(p) &= \delta^j_i + (\cosh\chi - 1)\widehat{p}^j\widehat{p}_i \end{aligned} \tag{C.19}$$

with $\tanh\chi = \frac{|p|}{\sqrt{|p|^2+m^2}}$. In this parametrization, $L(p) = \exp(-i\chi\widehat{\mathbf{p}}.\mathbf{k})$ where $k_i = M^{0i}$ is the boost generator [103].

Appendix D

Articles

PAPER • OPEN ACCESS

The cosmological constant effect on the quantum entanglement

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The cosmological constant effect on the quantum entanglement

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Abstract. In Schwarzschild-de Sitter space-time, the effect of a gravitational field near a massive body black hole on the spin entanglement has been studied for two-qubits in the case of triplet and singlet states of a two particles system in a circular geodesic motion. It is found that the effect of the cosmological constant on the robustness of the Wooters concurrence is more important in the triplet state than in the singlet one even in the extended region around the black hole

1. Introduction

Quantum entanglement is a physical phenomenon observed in quantum mechanics and it plays an important role in the field of quantum information processing such as quantum teleportation [1] and cryptography [2] ; meanwhile, it is at the heart of philosophical discussions on the interpretation of quantum mechanics. The first idea appeared with a famous article prepared by Einstein, Podolsky and Rosen (EPR) [3] generally referred to EPR paradox. This phenomenon occurs when pairs or a set of particles interact in a such way that the quantum state of each particle cannot be described independently of the others even when the particles are separated by a very large distance. Measurements of some physical properties such as position, momentum, spin etc.., performed on entangled particles are found to be correlated.

Recently, there are a number of articles that treat the effect of gravitational field on the quantum information by introducing the idea of local inertial frames namely the work of Terashima and Ueda who studied the spin rotation caused by the space time curvature for spin 1/2 particles moving in a gravitational field [4] and the gravitational spin entropy production for particles with arbitrary spin [5]. Many other issues on the problem were widely discussed in the literature [5,6,7]. On the other hand, cosmology is currently confronted with two unknown components: dark matter and energy. Dark matter is introduced to obtain the gravitational field needed to describe observations like the galactic rotation curves, gravitational lensing or the structure of the cosmic microwave background while dark energy is needed to explain the observed accelerated expansion of the universe. The repulsive force necessary to obtain an accelerated expansion of the universe can be provided by the inclusion of a vacuum energy. This corresponds to the well-known modification of the Einstein equations consisting of the addition of a cosmological term $\Lambda\, g_{\mu\nu}$. The Schwarzschild-de Sitter space-time describes the static gravitational field of a spherically symmetric mass in a universe with a cosmological constant Λ.

In fact, de Sitter space with a positive cosmological constant is spherically symmetric and has a cosmological horizon surrounding any observer. The Schwarzschild solution is a spherically symmetric solution of the Einstein equations with zero cosmological constant and it describes a black hole event horizon. Then, Schwarzschild-de Sitter space-time is a combination of the two [8].

In this paper we discuss the effect of the gravitational field of a massive body (black hole) on the spin entanglement of a two particles system in a circular geodesic motion in a Schwarchild-de Sitter space-time and the role of the cosmological constant and its influence on the entanglement of the singlet and triplet states.In section2, we present the mathematical formalism. In section3, we deduce the form of the corresponding Wigner rotational matrix. In section4, we calculate and discuss the Wooters concurrence to quantify the spin entanglement of a mixed state bipartite singlet and triplet states and finally in section5, we draw our conclusions.

2. Mathematical formulation

On a curved space-time, the spin of the particle is not well defined. Thus, one has to define it locally. For this, one introduces a local inertial frame at each point by using a Vierbein (or tetrad) $e_a{}^{\mu}$ defined by:

$$g_{\mu v} e_a{}^{\mu} e_b{}^{v} = \eta_{ab} \tag{1}$$

where $g_{\mu v}$ (resp. η_{ab}) is the metric of the curved (resp. Minkowski) space-time. If p^{μ} is the four energy-momentum tensor of the particle, the corresponding spin quantum state is denoted by $|p, \sigma\rangle$ where $\sigma(=\uparrow, \downarrow)$. If this particle moves to another point of the space-time, its state in the local frame becomes [9,10]

$$\sum_{\sigma'} D_{\sigma'\sigma}(\tilde{\Lambda}, p)\left|\tilde{\Lambda} p, \sigma'\right\rangle \tag{2}$$

where $\tilde{\Lambda}$ is the Lorentz transformation matrix and $D_{\sigma'\sigma}$ a 2x2 Wigner rotation matrix elements corresponding to a momentum dependent change of the spin state of a relativistic particle with a change of the referential frame[11,12,13].

Now, let us consider a system consisting of two spin 1/2 non interacting particles (separable state) where the center (centroid) of this system is described by a wave packet in the local frame where the initial state is written as:

$$\left|\psi^i\right\rangle = \sum_{\sigma_1\sigma_2} \int dp_1 dp_2\, g_{\sigma_1\sigma_2}(p_1, p_2) |p_1, \sigma_1; p_2, \sigma_2\rangle \tag{3}$$

where:

$$\sum_{\sigma_1\sigma_2} \int dp_1 dp_2 \left|g_{\sigma_1\sigma_2}(p_1, p_2)\right|^2 = 1 \tag{4}$$

with p_1 and p_2 are the four-momenta of the two particles. When the system reaches a final point, the wave packet $\left|\psi^f\right\rangle$ in the local inertial frame becomes:

$$\left|\psi^f\right\rangle = \sum_{\sigma_1\sigma_2\sigma_1'\sigma_2'} \int dp_1 dp_2 \sqrt{\frac{(\Lambda p_1)^0 (\Lambda p_2)^0}{p_1{}^0 p_2{}^0}}\, g_{\sigma_1\sigma_2}(p_1, p_2) \times D_{\sigma_1'\sigma_1}(\Lambda, p_1) D_{\sigma_2'\sigma_2}(\Lambda, p_2) |\Lambda p_1, \sigma_1'; \Lambda p_2, \sigma_2'\rangle \tag{5}$$

On the other hand. the change in the local inertial frame is given by:

$$\delta e_{\mu}{}^{a}(x) = u^{v}(x) d\tau \nabla_{v} e_{\mu}{}^{a}(x) = -u^{v}(x) w_{vb}{}^{a} e_{\mu}{}^{b}(x) d\tau = \chi_b{}^{a}(x) e_{\mu}{}^{b}(x) d\tau \tag{6}$$

where

$$w_{\nu b}{}^{a} = -e_{b}{}^{\mu}(x)\nabla_{\nu}e_{\mu}{}^{a}(x) = e_{\mu}{}^{a}(x)\nabla_{\nu}e_{b}{}^{\mu}(x) \tag{7}$$

and

$$\chi_{b}{}^{a}(x) = -u^{\nu}(x)w_{\nu b}{}^{a} \tag{8}$$

where $w_{\nu b}{}^{a}$ is the spin connection elements and $\chi_{b}{}^{a}(x)$ denotes the change in the local inertial frame along $u^{\nu}(x)$. The change δp^{μ} in the momentum has the form:

$$\delta p^{\mu}(x) = u^{\nu}(x)d\tau\nabla_{\nu}p^{\mu}(x) = ma^{\mu}(x)d\tau \tag{9}$$

where m is the rest mass of the particle and

$$a^{\mu}(x) = u^{\nu}(x)\nabla_{\nu}u^{\mu}(x) \tag{10}$$

is the acceleration due to an external force and ∇_{ν} is the covariant derivative. Thus, we can rewrite eq.(9) as:

$$\delta p^{\mu}(x) = -\frac{1}{mc^{2}}[a^{\mu}(x)p_{\nu}(x) - p^{\mu}(x)a_{\nu}(x)]p^{\nu}(x)d\tau \tag{11}$$

with c the velocity of light and

$$p^{\mu}(x)p_{\mu}(x) = -m^{2}c^{2}; \quad p^{\mu}(x)a_{\mu}(x) = 0 \tag{12}$$

We deduce that in the local frame:

$$\delta p^{a}(x) = \lambda_{a}{}^{b}(x)p^{b}(x)d\tau \tag{13}$$

where

$$\lambda_{a}{}^{b}(x) = -\frac{1}{mc^{2}}[a^{a}(x)p_{b}(x) - p^{a}(x)a_{b}(x)] + \chi_{b}{}^{a}(x) \tag{14}$$

3. Wigner rotational matrix in the Schwarzschild-de Sitter space-time

Consider a system of two particles (bipartite state) moving in the gravitational field of the Schwarzschild-de Sitter space-time where the metric is given by:

$$ds^{2} = -c^{2}\left(f(r) - \frac{\Lambda r^{2}}{3}\right)dt^{2} + \left(f(r) - \frac{\Lambda r^{2}}{3}\right)^{-1}dr^{2} + r^{2}(d\theta^{2} + sin^{2}\theta\, d\varphi^{2}) \tag{15}$$

Here $f(r) = 1 - (\frac{r_s}{r})$ and $r_s = 2GM$ is the Schwarzschild radius and $r > r_s$. Let us introduce a local inertial frame by choosing the tetrad:

$$e_{0}{}^{t} = \frac{1}{c\sqrt{f(r) - \frac{\Lambda r^{2}}{3}}}, \quad e_{1}{}^{r} = \sqrt{f(r) - \frac{\Lambda r^{2}}{3}}, e_{2}{}^{\theta} = \frac{1}{r}, e_{3}{}^{\varphi} = \frac{1}{rsin\theta} \tag{16}$$

Now, for a particle in a circular motion on the equatorial plane where $\theta = \frac{\pi}{2}$, the four-velocity of the centroid is given by:

$$u^{t}(x) = \frac{cosh\xi}{\sqrt{f(r) - \frac{\Lambda r^{2}}{3}}} \quad and \quad u^{\varphi}(x) = \frac{csinh\xi}{r} \tag{17}$$

Where ξ is the rapidity in the local inertial frame defined by $tanh\xi = \frac{V}{c}$ and V denotes the velocity. Notice that the condition

$$\left(1 - \frac{r_s}{r} - \frac{\Lambda r^2}{3}\right) > 0 \tag{18}$$

Implies $\Lambda {r_s}^2 < \frac{4}{9}$ .In what follows, we denote by:

$$z = \frac{r}{r_s}\ ; \quad w = \Lambda {r_s}^2\ ; \quad \Sigma = \frac{c\tau}{r_s} \tag{19}$$

In this case, Θ takes the form:

$$\Theta = q\sqrt{q^2+1}\left[\sqrt{q^2+1} - \frac{qp}{\sqrt{p^2+1}+1}\right]\Sigma \frac{z^{1/2}}{\left(z-1-\frac{z^3 w}{3}\right)^{1/2}}(z-3) \tag{20}$$

and the two dimensional representation of the Wigner rotation matrix $D(\Theta)$ reads:

$$D(\Theta) = e^{-iJ_2\Theta} = \begin{pmatrix} cos\frac{\Theta}{2} & -sin\frac{\Theta}{2} \\ sin\frac{\Theta}{2} & cos\frac{\Theta}{2} \end{pmatrix} \tag{21}$$

4. Measure of entanglement

The most widely used measures to quantify the entanglement for a mixed state is the so called Wooters concurrence defined as[14,15,16,17]:

$$C(\rho) = \max\,(0, \sqrt{\lambda_1}, -\sqrt{\lambda_2}, -\sqrt{\lambda_3}, -\sqrt{\lambda_4}) \tag{22}$$

Where ρ is the density matrix of the state and $\sqrt{\lambda_i}$'s are the eigenvalues of $\rho\tilde{\rho}$ with

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y) \tag{23}$$

and σ_y is the Pauli matrix

4.1. The singlet entangled state

the wave packet in the momentum representation for a system of two spin 1/2 non-interacting particles observed in a local frame and located at an initial point can be written as:

$$|\psi^i\rangle = \sum_{\sigma_1\sigma_2} \int\int dp_1 dp_2\, g_{\sigma_1\sigma_2}(p_1, p_2)|p_1, \sigma_1; p_2, \sigma_2\rangle \tag{24}$$

where

$$\sum_{\sigma_1\sigma_2} \int\int dp_1 dp_2 \left|g_{\sigma_1\sigma_2}(p_1, p_2)\right|^2 = 1 \tag{25}$$

In the singlet state, $g_{\sigma_1\sigma_2}(p_1, p_2)$ is given by

$$g_{\sigma_1\sigma_2}(p_1, p_2) = \frac{1}{\sqrt{2}}\left(\delta_{\sigma_1\uparrow}\delta_{\sigma_2\downarrow} - \delta_{\sigma_1\downarrow}\delta_{\sigma_2\uparrow}\right)f(p_1)f(p_2) \tag{26}$$

and the reduced matrix density for the initial state has the form[11,12]:

$$\rho^i = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 \\ 0 & -1 & +1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (27)$$

For the final state, the reduced matrix density elements are:

$$\rho^f_{\sigma_1'\sigma_2'\sigma_1\sigma_2} = \frac{1}{2}\iint dp_1\, dp_2 |f(p_1)|^2 |f(p_2)|^2 [D_{\sigma_1'\uparrow}(\Theta_1) D_{\sigma_2'\downarrow}(\Theta_2) D_{\sigma_1\uparrow}(\Theta_1) D_{\sigma_2\downarrow}(\Theta_2) - D_{\sigma_1'\downarrow}(\Theta_1) D_{\sigma_2'\uparrow}(\Theta_2) D_{\sigma_1\uparrow}(\Theta_1) D_{\sigma_2\downarrow}(\Theta_2) - D_{\sigma_1'\uparrow}(\Theta_1) D_{\sigma_2'\downarrow}(\Theta_2) D_{\sigma_1\downarrow}(\Theta_1) D_{\sigma_2\uparrow}(\Theta_2) + D_{\sigma_1'\downarrow}(\Theta_1) D_{\sigma_2'\uparrow}(\Theta_2) D_{\sigma_1\downarrow}(\Theta_1) D_{\sigma_2\uparrow}(\Theta_2)] \quad (28)$$

Then, straightforward calculations show that the wooters concurrence takes the following form:

$$C(\rho^f) = \langle cos\Theta \rangle^2 + \langle sin\Theta \rangle^2 \quad (29)$$

where $\langle X \rangle = \int dp |f(p)|^2 X$. We note that the expectation values values in eq.(29) have to be solved numerically using a Gaussian distribution. Fig.1 shows the wooters concurrence $C(\rho^f)$ as a function of z for fixed values of q, Σ and w

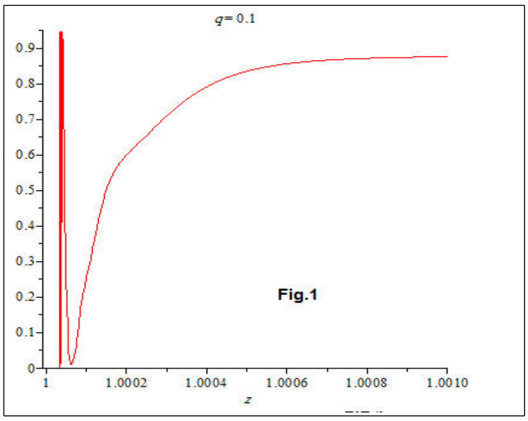

Figure 1. $C(\rho^f)$ as a function of z for fixed q, Σ and w

Notice that as z (equivalently the distance r) increases i.e.going far away from the horizon of the black hole, the entanglement increases and becomes more robust. This is due to the fact that the gravitational field decreases. Fig.2, displays the variation of the concurrence $C(\rho^f)$ as a function of the centroid momentum q for fixed values of z , Σ and w . Notice the damping periodic oscillatory behavior due to the sine and cosine functions in the Wigner matrix. Furthermore, as q increases, the maximum of the oscillations decreases due to the fact that as the velocity increases the spin

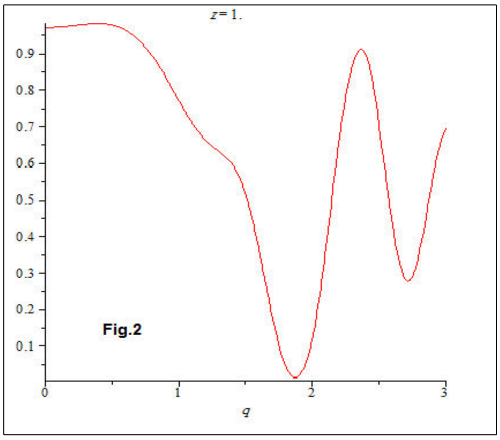

Figure 2. variation of $C(\rho^f)$ as a function of q for fixed z , Σ and w

decoherence phenomenon increases. Fig.3, shows the variation of the concurrence $C(\rho^f)$ as a function of w for fixed values of z, Σ and q. Notice that if the cosmological constant Λ increases, we will approach more the cosmological horizon where the cosmological curvature becomes stronger and

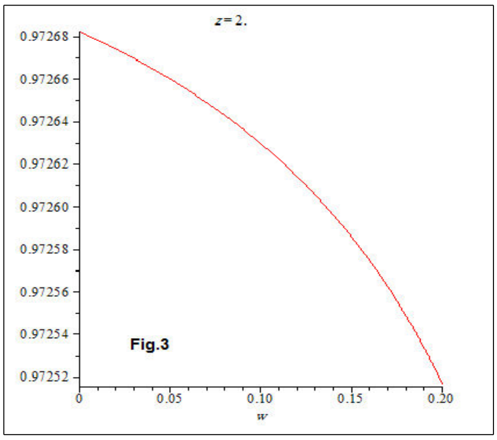

Figure 3. $C(\rho^f)$ as a function of w for fixed values of z, Σ, q.

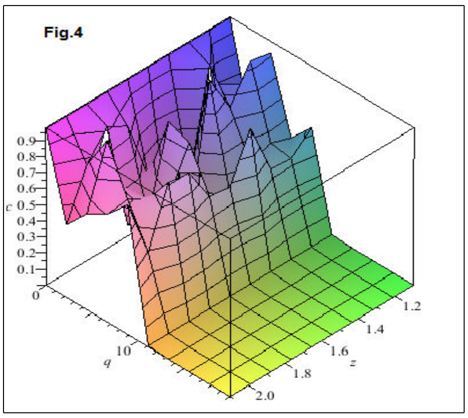

Figure 4. 3D plot of Wooters concurrence as a function of z and q for fixed Σ and w

therefore the entanglement decreases. According to the Hawking-Unruh effect, an accelerating particle will radiate and loose information and entanglement decreases. Fig.4 displays a 3D plot representing the Wooters concurrence as a function of z and q for fixed Σ and w. Fig.5 displays a 3D plot representing the Wooters concurrence as a function of z and w for fixed values of Σ and q

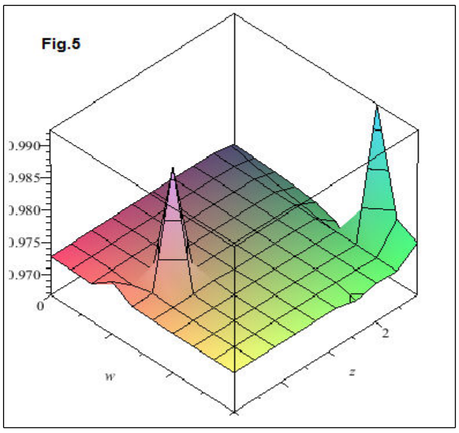

Figure 5. 3D plot of Wooters concurrence as a function of z and w for fixed Σ and q

4.2. The triplet entangled state

In the triplet state, $g_{\sigma_1\sigma_2}(p_1,p_2)$ takes the form:

$$g_{\sigma_1\sigma_2}(p_1,p_2) = \frac{1}{\sqrt{2}}\left(\delta_{\sigma_{1\uparrow}}\delta_{\sigma_{2\downarrow}} + \delta_{\sigma_{1\downarrow}}\delta_{\sigma_{2\uparrow}}\right)f(p_1)f(p_2) \tag{30}$$

In this case, one can show that after straightforward but tedious calculations, the Wooters concurrence has the following expression:

$$C(\rho^f) = \sqrt{\langle cos2\Theta\rangle^2 + \langle sin2\Theta\rangle^2} \tag{31}$$

Fig.6, fig.7 and fig.8 are the same as fig.2, fig.1 and fig.3 respectively but with a triplet entangled state.

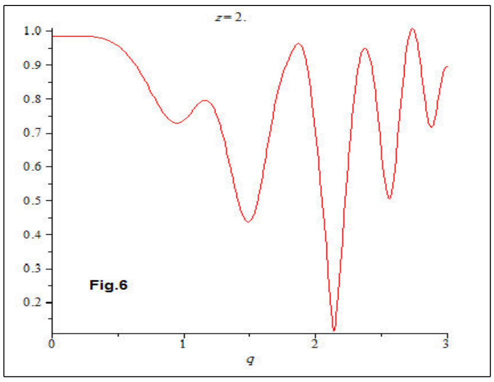

Figure 6. same as fig.2 but with a triplet entangled state.

Notice that contrary to ref. [8], the concurrence in the quantum entangled triplet state is more robust than in the singlet case. Moreover, the effect of the cosmological constant is very remarkable even in extended regions around the black hole

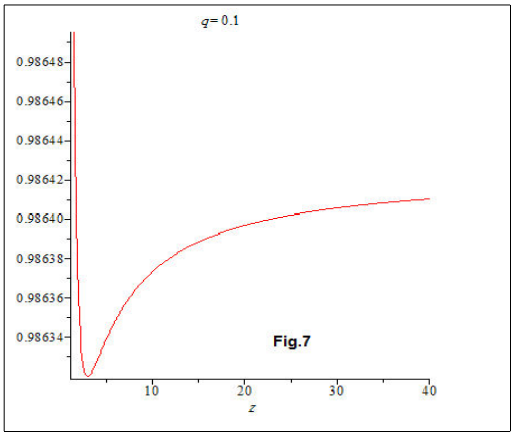

Figure 7. same as fig.1 but with a triplet entangled state

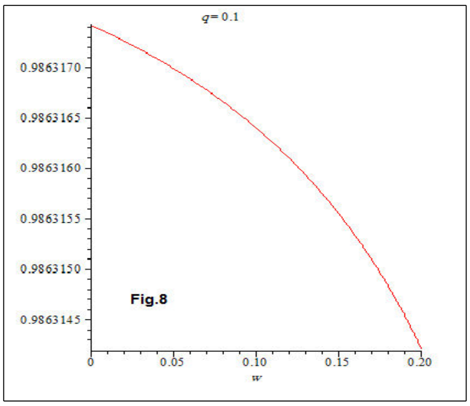

Figure 8. same as fig.3 but with a triplet entangled state

5. Conclusion

In this work, which falls within the framework of quantum information in a curved static space-time, Wooters concurrence behavior of two spin 1/2 entangled particles moving in a circular motion is studied in the congtext of the Schwarzschild-de Sitter metric. We have found that the spin entanglement in the triplet quantum state is more important than in the singlet case. Furthermore, the cosmological constant plays an important role in the neighborhood region around the black hole (more study is under investigation).

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References

[1] Nielsen, M.A. and Chuang, I.L. 2000 *Quantum Computation and Quantum Information*. (Cambridge University Press, Cambridge)
[2] Bennett, C.H. and Wiesner,S.J. 1992 *Phys. Rev. Lett.* **69** 2881
[3] Einstein,A., Podolsky, B. and Rosen,N. 1935 *Phys. Rev.* **47** 777 , Bohm, D. *Quantum Theory* 1989 (Dover, New York,) 611
[4] Terashima, H. And Ueda, M. *2005 J. Phys.* **A38** 2029
[5] H. Terashima and M. Ueda. 2003 *Quantum Inf. Comput* **3** 224.

IOP Conf. Series: Journal of Physics: Conf. Series **1269** (2019) 012014 doi:10.1088/1742-6596/1269/1/012014

[6] Nasr Esfahani, B 2010 *J. Phys.* **A43** 455305
[7] H H. Terashima and M. Ueda. 2004 *Phys. Rev.* **A69** 032113.
[8] Ahmadi, F and Mehrafarin, M. 2014, *Quantum Information Processing* **13**, 639
[9] Weinberg, S. 1995 *The Quantum Theory of Fields*. (Cambridge University Press)
[10] Huang, K., *Quantum Field Theory: From Operators to Path Integrals* 1998 (Wiley-VCH, Portland, ME, U.S.A)
[11] Mebarki,N. ,Mohadi,A. and Boussahel, M. 2016 *Entanglement of spin 1/2 particles with PT symmetric Hamiltonian in a Schwarchild-De Sitter space-time* , Progress in Quantum Physics with Non-Hermitian Operators PHHQP16 YITP Workshop, Yukawa Institute for Theoretical Physics, Kyoto University, Japan August 8–12, 2016, Mebarki, N. , Morchedi,O. And Aissaoui,H., 2015 *International Journal of Theoretical Physics* **54**(11):1
[12] Schlosshauer; M. 2007 *Decoherence and The Quantum to Classical Translation* (Springer-Verlag, Berlin Heidelberg)
[13] Saldanha, P.L. and Vedral, V. Wigner 2013 *Phys. Rev.* **A87** 042102
[14] Rindler, W and Ishak, M 2007 *Phys.Rev* **D76** 043006.
[15] Wootters,W.K. 1998 *Phys.Rev.Lett.***80**, 2245.
[16] Hill,S. and Wootters,W.K. 1997 *Phys.Rev.Lett.***78** 5022.
[17] Beringer, J. et al 2012 *Phys. Rev*. **D86** 010001.

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ORIGINAL PAPER

Spin quantum entanglement in non-commutative curved space–time

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Abstract: The elaboration of a general formalism on quantum spin entanglement in curved space–time is presented by a system of two particles described by wave packets moving in a gravitational field (GF). This formulation allows us to study different models in curved space–time. In this work, the non-commutative Reissner–Nordström model is considered. The spin entanglement of a system of two spin 1/2 particles is discussed. With particularity that contains multiple and various physical parameters, allowing for a detailed study of this purely quantum phenomenon in different frames of space and geometry or both at the same time.

Keywords: Non-commutative space; Quantum information; Curved space; Spin Entanglement

1. Introduction

During the last decade, great interest has been devoted to quantum entanglement and information theory [1–5]. The spin quantum entanglement of a bipartite system plays an important role in most physical systems, such as condensed matter. Recently, the effect of relativistic motion on the entanglement correlation of quantum spin states has been the focus of many physicists, where the spin entanglement of massive particles can change under Lorentz transformations. The entangled momentum of rotation in a flat space–time is discussed by Peres, Scudo and Terno [6], in the same year Gingrich and Adami [7] showed that the entanglement between the spins is affected by the Wigner rotation. This latter in special relativity is known as the product of two Lorentz boost in different directions. Furthermore, this study is extended to a curved space–time [8–13], where Terashima and Ueda [8, 9] studied the EPR (Einstein–Podolsky–Rosen) correlation and Bell's inequality in the Schwarzschild space–time. By considering accelerated particles in the gravitational field (GF), they showed that the acceleration and the gravity deteriorate the perfect anti-correlation of a pair of EPR spins in the same direction. On the other hand, in [9] they showed when the spin entropy of a spin-1/2 particle moving in the gravitational field can be generated. Considering that if the spin state of the particle is pure at one point in space–time, it becomes mixed at another point. Because the local inertial frames of reference at different points are different in general. Moreover, they showed that the spin entropy of particles in a circular motion is quickly incremented close to the event horizon of the Schwarzschild black hole. Also, the spin entanglement can be more powerful against changes brought about by motion in the singlet state than in the triplet state [10].

The very early quantum space–time model based on non-commutative (NC) algebra was suggested by Snyder in 1949 [14] to ameliorate short-distance singularities in quantum field theory. This idea was the motivation behind studying non-commutative space with cosmological models [15–17], where NC Seiberg Witten space–time has played an important role in studying many phenomena in particle physics and cosmology [18–24], where some authors [25, 26] have suggested some non-commutative models in classical cosmology to explain the accelerated expansion of our universe, and NC opened the door for a new explanation of dark matter and dark energy as well as the cosmic microwave background (C.M.B) and its anisotropies [27–32].

Emerging of the entanglement entropy concept and its application to black hole entropy issues [33, 34], another exciting area has attracted many physicists: the relationship between the structure of space–time and entanglement. Where it was considered, the non-commutativity can

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induce entanglement [35–37]. Abhishek Muhuri and others [38] showed that even in non-commutative space, the entanglement is generated only if the harmonic oscillator is anisotropic.

The model we present here is one that tries to understand quantum entanglement behavior, which can be a better alternative to experiment or to verify the effects of the NC space on quantum entanglement, as was done in studies [39, 40] the effects of the passing gravitational wave on the quantum states of a system of N spin-1/2 particles have been investigated by Ye Yeo et al.

Based on previous work, this article discusses the effect of the gravitational field (near or far from the black hole) on the quantum spin entanglement (QSE) of a bipartite system. The system is described by packets of centroid waves as a momentum representation [11]. Using the idea of local inertial frames, both the increasing speed of the centroid and the shape of the gravitational field cause a Wigner rotation that influences the wave packet. As a result of this fact, we try to extend our study to a metric or to different metrics in general. In order to be able to study the effects of both the GF shape and various parameters of the black hole, either in a commutative framework of geometry or even non-commutative. In Sect. 2, we present a general mathematical formalism. In Sect. 3, the non-commutative Reissner–Nordström space–time is considered. In Sect. 4, we compare the behavior of entanglement in singlet and triplet state, and in Sect. 5, we draw our conclusion by focusing on the SE of the centroid packet and how it is affected by various parameters like the acceleration of the centroid, the distance from a massive body, and the NC of space.

2. Mathematical formalism

In order to study the spin of a particle in curved space–time, one has to use an inertial local frame at each point. This can be done at the tangent at a point of curved space–time using the vierbein (or tetrad) e_a^μ (μ (resp.) is a curved (resp. flat) index) defined by:

$$g_{\mu\nu}e_{\mathrm{a}}^{\mu}e_{\mathrm{b}}^{\nu}=\eta_{\mathrm{ab}} \tag{1}$$

Where $g_{\mu\nu}$ and $\eta_{\mathrm{ab}}(\eta_{\mathrm{ab}}=\mathrm{diag}(-1,1,1,1))$ are the curved and Minkowski space–time metric, respectively. Let us introduce one fermionic particle $|P,\sigma\rangle$ with a 4-momentum P^μ and spin $\sigma(=\uparrow,\downarrow)$ at some point of the space–time. If we move from one point to another, this state becomes (in a local frame) [10, 11]:

$$\sum_{\sigma'}D_{\sigma'\sigma}(W(\Lambda,P))|\Lambda P,\sigma'\rangle \tag{2}$$

Where Λ is the Lorentz transformation matrix and $W(\Lambda,P)$ is the Wigner rotation operator corresponding to Λ $D_{\sigma'\sigma}$ denotes the two-dimensional representation of the Wigner rotation operator [41].

Let us consider a system of two non-interacting spin 1/2 particles, where its center of mass system can be described by an initial wave packet $|\psi^i\rangle$ given in a local frame by [11] [9]:

$$|\psi^i\rangle=\sum_{\sigma_1\sigma_2}\int d^3p_1d^3p_2\psi_{\sigma_1\sigma_2}(p_1,p_2)|P_1,\sigma_1;P_2,\sigma_2\rangle \tag{3}$$

With the normalization condition:

$$\sum\nolimits_{\sigma_1\sigma_2}\int d^3p_1d^3p_2\left|\psi_{\sigma_1\sigma_2}(p_1,p_2)\right|^2=1 \tag{4}$$

Here, P_1 and P_2 are 4-momentum of the particles 1 and 2, respectively. $\psi_{\sigma_1\sigma_2}(p_1,p_2)$ are wave functions determining momentum and spin distribution, It can be used to express momentum entanglement, spin entanglement, and even entanglement between spins and momenta. Now, it is easy to show that when the system reaches another point of the inertial local frame, the wave packet becomes $|\psi^f\rangle$ like this:

$$\begin{aligned}|\psi^f\rangle&=U(\Lambda_1(x_f,x_i))\otimes U(\Lambda_2(x_f,x_i))|\psi^i\rangle\\&=\sum_{\sigma_1\sigma_2\sigma_1'\sigma_2'}\int d^3p_1d^3p_2\sqrt{\frac{(\Lambda_1P_1)^0(\Lambda_2P_2)^0}{P_1^0P_2^0}}\psi_{\sigma_1\sigma_2}(p_1,p_2)\\&\times D_{\sigma_1'\sigma_1}(W(\Lambda_1,P_1))D_{\sigma_2'\sigma_2}(W(\Lambda_2,P_2))|\Lambda_1P_1,\sigma_1';|\Lambda_2P_2\sigma_2'\rangle\end{aligned} \tag{5}$$

Where $U(\Lambda_1(x_f,x_i))$ is a unitary operator, x_f,x_i are the centroid location at a final and initial point, respectively. The Wigner rotation operator can have the following formula [9]

$$W(\Lambda_1,P_1)=Texp\left[\int_{x_i}^{x_f}w(x(\tau))d\tau\right] \tag{6}$$

T here is the time-ordering operator,τ proper time and w is a matrix whose elements are given by

$$w_{\mathrm{k}}^i=\lambda_{\mathrm{k}}^i+\frac{\lambda_0^ip_{\mathrm{k}}-\lambda_{\mathrm{k}0}p^i}{p^0+\mathrm{mc}^2} \tag{7}$$

Where $i,k=(1,2,3)$, with m being the mass of the particle. Where the infinitesimal Lorentz transformation matrix elements $\lambda_b^a(x)$ have the form:

$$\lambda_{\mathrm{b}}^a(x)=-\frac{1}{\mathrm{mc}^2}[a^a(x)q_{\mathrm{b}}(x)-q^a(x)a_{\mathrm{b}}(x)]+\chi_{\mathrm{b}}^a(x) \tag{8}$$

With:

$$\chi_{\mathrm{b}}^{a}(x) = -u^{\mu}(x)\omega_{\mu\mathrm{b}}^{a} \tag{9}$$

And:

$$\omega_{\mu\mathrm{b}}^{a} = -e_{\mathrm{b}}^{v}(x)\nabla_{\mu}e_{\mathrm{v}}^{a}(x) \tag{10}$$

Here, $\omega_{\mu b}^{a}$ is a spin connection, where $\chi_{\mathrm{b}}^{a}(x)$ represents its change along the direction of the 4-vector velocity of the centroid, $u^{\mu}(x)$ is the four-velocity of the centroid, ∇_{μ} stands for the covariant derivative and $a^{a}(x)$ the 4-vector acceleration produced by a classical force as measured in the local frame which is given by

$$a^{a}(x) = e_{\mu}^{a}(x)(u^{v}(x)\nabla_{\mathrm{v}}u^{\mu}(x)) \tag{11}$$

To mention where q comes from, let us consider a system of two non-interacting spin 1/2 particles (wave packet) whose center of mass is described by an equatorial plane with $\theta = \pi/2$. The motion has a radius with constant speed v. After obtaining a central force motion, the components of the centroid 4-momentum in the local inertial frame are given by [12]

$$q^{0} = \gamma\mathrm{mc} \qquad q^{1} = q^{2} = 0 \qquad q^{3} = \gamma\mathrm{mv} \tag{12}$$

Where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ is the Lorentz factor.

Now, in order to measure entanglement between 2 particles in a gravitational field, let us consider the following space–time where the metric ds^2 has the form

$$ds^2 = F(r)dt^2 + G(r)dr^2 + H(r,\theta)d\theta^2 + I(r,\theta)d\varphi^2 \tag{13}$$

$F(r), G(r), H(r,\theta), I(r,\theta)$ are arbitrary functions that have a linear relation with the coordinates (r) or (r,θ), let us make a diagonal choice of the tetrad

$$e_{0}^{t} = \frac{1}{\sqrt{F(r)}}, e_{1}^{r} = \frac{1}{\sqrt{G(r)}}, e_{2}^{\theta} = \frac{1}{\sqrt{H(r,\theta)}}, e_{3}^{\varphi} = \frac{1}{\sqrt{I(r,\theta)}} \tag{14}$$

Thus, the non-vanishing spin connection elements are

$$\omega_{t1}^{0} = \frac{1}{2}\frac{F}{\sqrt{\mathrm{GF}}}, \omega_{\varphi 3}^{0} = \frac{\dot{I}}{2\sqrt{\mathrm{IF}}}, \omega_{\theta 2}^{1} = -\frac{1}{2\sqrt{\mathrm{GH}}}H', \omega_{\varphi 3}^{1} = -\frac{1}{2}\frac{I'}{\sqrt{\mathrm{GI}}}, \omega_{\varphi 3}^{2} = -\frac{1}{2}\frac{1}{\sqrt{\mathrm{HI}}}\partial_{\theta}I \tag{15}$$

Where $\dot{I} = \frac{\partial I}{\partial t}$ and $I' = \frac{\partial I}{\partial r}, H' = \frac{\partial H}{\partial r}\partial_{\theta} = \frac{\partial}{\partial \theta}$. Furthermore, the non-vanishing components $u^{v}, \chi_{b}^{a}(x)$ and λ_{a}^{b}, for a circular motion and constant angular velocity $\frac{d\varphi}{dt}$ on the equatorial plane where $\theta = \pi/2$ are given by

$$u^{t}(x) = \frac{\gamma c}{\sqrt{F}} u^{\varphi}(x) = \frac{1}{\sqrt{I}}\gamma r\frac{d\varphi}{dt} \tag{16}$$

$$\chi_{1}^{0}(x) = -u^{t}(x)\omega_{t1}^{0}, \chi_{3}^{0} = -u^{\varphi}\omega_{\varphi 1}^{0}, \chi_{2}^{1}(x) = -u^{0}\omega_{\theta 2}^{1}, \chi_{3}^{1}(x) = -u^{\varphi}(x)\omega_{\varphi 3}^{1}, \chi_{3}^{2}(x) = -u^{\varphi}\omega_{\varphi 3}^{2} \tag{17}$$

And

$$\lambda_{1}^{0} = \frac{1}{mc^2}\left(p^{0}a_{1}\right) + \chi_{1}^{0}, \lambda_{3}^{1} = -\frac{1}{mc^2}\left(a^{1}p_{3}\right) + \chi_{3}^{1}, \lambda_{3}^{2} = \chi_{3}^{2}, \lambda_{2}^{0} = \chi_{2}^{0} \tag{18}$$

It is important to mention that the two non-vanishing components of the 4-vector velocity u^{t} and u^{φ} can be rewritten as

$$u^{t}(x) = \frac{\cosh\xi}{\sqrt{F}} \quad \text{and} \quad u^{\varphi}(x) = \frac{c\sinh\xi}{\sqrt{I(r,\theta)}} \tag{19}$$

Where ξ is the rapidity in the local inertial frame such that $\frac{v}{c} = tanh\xi$.

To quantify the spin entanglement of the two particles system, we use the Wootters concurrence [42–44] for the mixed state $|P_1,\uparrow,P_2,\downarrow$ defined by

$$C(\rho) = \max\left(0, \sqrt{\lambda_1}, -\sqrt{\lambda_2}, -\sqrt{\lambda_3}, -\sqrt{\lambda_4}\right) \tag{20}$$

Where $\sqrt{\lambda_{\mathrm{i}}}$ are the square roots of the eigenvalues of the matrix $\rho\tilde{\rho}$ with: $\tilde{\rho} = \left(\sigma_y \otimes \sigma_y\right)\rho^{*}\left(\sigma_y \otimes \sigma_y\right)$, σ_y here is the Pauli matrix, and ρ is the state density matrix: $\rho = |\psi\rangle\langle\psi|$, where ψ take this following expression [11]

$$\psi_{\sigma_1\sigma_2}(p_1,p_2) = \varepsilon_{\mathrm{i}}f(p_1)f(p_2) \tag{21}$$

Where ε_{i} is one of the Bell states; this choice allows us to assume a maximum spin entanglement, $f(p)$ is a normalized function which is defined by

$$f(p) = \frac{\sqrt{\delta(p^1)}\sqrt{\delta(p^2)}}{\sqrt{\pi^{\frac{1}{2}}\mathrm{bmc}}}\exp\left(-\frac{(p^3-q^3)^2}{2b^2m^2c^2}\right) \tag{22}$$

Where b is width. To get more simplification of calculations, let $p^1 = 0$, $p^2 = 0$, $b = 1$.

If λ_{i} are positive real numbers, the entanglement can be quantified by the spin entanglement $E(\rho)$ defined as [11]

$$E(\rho) = h\left(\frac{1+\sqrt{1-C^2(\rho^f)}}{2}\right) \tag{23}$$

Where:

$$h(x) = -xlog_2x - (1-x)log_2(1-x) \tag{24}$$

Equation (20) can be shown to have the following expression [10], in the case of spin singlet state in curved space–time

$$C(\rho^f) = \langle\cos\Theta^2\rangle + \langle\sin\Theta^2\rangle \tag{25}$$

In the case of the spin triplet state in curved space–time

$$C(\rho^f) = \sqrt{\langle\cos 2\Theta\rangle^2 + \langle\sin 2\Theta\rangle}^2 \tag{26}$$

With

$$\langle\cos x\rangle = \int dp|f(p)|^2\cos x \tag{27}$$

Θ here is a shorthand notation for $\tau\Theta_3^1$ (Θ_3^1 is the only non-vanishing component of Θ_k^i where $w_k^i(x) = \Theta_k^i$, τ is propre time). The two-dimensional representation of the Wigner rotation matrix $D(\Theta)$ is

$$D(\Theta) = e^{-iJ_2\Theta} = \begin{pmatrix} \cos\frac{\Theta}{2} & -\sin\frac{\Theta}{2} \\ \sin\frac{\Theta}{2} & \cos\frac{\Theta}{2} \end{pmatrix} \tag{28}$$

Where J_2 is the 2-component of the angular momentum operator. By using (7), (8) and (25), Θ can be rewritten as

$$\Theta = \Theta_3^1\tau = -\frac{\alpha}{2G}\left\{\left(AD^2 + B(D^2-1) - I'\sqrt{\frac{G}{HI}}\right) + \frac{D}{C}p\left(AD^2 + B(D^2-1) + A\sqrt{G}\right)\right\} \tag{29}$$

Where:$A = \frac{F'}{F}$, $B = \frac{I'}{H}$, $D = \sqrt{1+q^2}$, $C = 1 + \sqrt{1+p^2}$, $\alpha = \frac{\tau}{r_s}$..

3. Spin entanglement in Reissner–Nordström non-commutative space–time

We consider the Reissner–Nordström metric for a charged non-rotating black hole in commutative space–time. It is given by [45]

$$ds^2 = -c^2\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{30}$$

M and Q are mass and charge, respectively, and t is the time coordinate, r is the radial coordinate, (θ, φ) are the spherical angles, the Schwarzschild radius of the body given by $r_s = 2M$, where r_s does not represent a singularity, but in this case it is only a parameter. And the new singularities are

$$r_\pm = M \pm \sqrt{M^2 - Q^2}$$

There is an external event horizon at r_+. The internal Cauchy horizon is the other horizon r_-.

The extremal case is defined as the limiting case where $Q = M$ and $r_+ = r_-$.

Following ref [46], the Seiberg Witten vierbein $\hat{e}_\mu^a$ in a non-commutative gauge gravity is given by

$$\hat{e}_\mu^a = e_\mu^a(x) - i\tilde{\eta}^{\nu\rho}e_{\mu\nu\rho}^a(x) + \tilde{\eta}^{\nu\rho}\tilde{\eta}^{\lambda\tau}e_{\mu\nu\rho\lambda\tau}^a(x) + O(\tilde{\eta}^3) \tag{31}$$

Where

$$e_{\mu\nu\rho}^a = \frac{1}{4}\left(\omega_\nu^{ac}\partial_\rho e_\mu^d + \left(\partial_\rho^a\omega_\mu^{ac} + R_{\rho\mu}^{ac}\right)e_\nu^d\right)\eta_{\rm cd} \tag{32}$$

And

$$\begin{aligned} e_{\mu\nu\rho\lambda\tau}^a = \Big[&\frac{1}{32}2\{R_{\tau\nu}, R_{\mu\rho}\}^{ab}e_\lambda^c - \omega_\lambda^{ab}\left(D_\rho R_{\tau\mu}^{cd} + \partial_\rho R_{\tau\mu}^{cd}\right)e_\nu^m\eta_{\rm dm} \\ &- \{\omega_\nu, (D_\rho R_{\tau\mu} + \partial_\rho R_{\tau\mu})\}^{cd}e_\lambda^c - \partial_\tau\{\omega_\nu, (\partial_\rho\omega_\mu + R_{\rho\mu})\}^{ab}e_\lambda^c \\ &- \omega_\lambda^{ab}\partial_\tau\left(\omega_\nu^{cd}\partial_\rho e_\mu^m + \left(\partial_\rho\omega_\mu^{cd} + R_{\rho\mu}^{cd}\right)e_\mu^m\right)\eta_{\rm dm} + 2\partial_\nu\omega_\lambda^{ab}\partial_\rho\partial_\tau e_\mu^c \\ &- 2\partial_\rho\left(\partial_\tau\omega_\mu^{ab} + R_{\tau\mu}^{cd}\right)\partial_\nu e_\lambda^c - \{\omega_\nu, (\partial_\rho\omega_\lambda + R_{\rho\lambda})\}^{ab}\partial_\tau e_\mu^c \\ &- \left(\partial_\tau\omega_\mu^{ab} + R_{\tau\mu}^{ab}\right)\left(\omega_\nu^{cd}\partial_\rho e_\lambda^m + \left(\partial_\rho\omega_\lambda^{cd} + R_{\rho\lambda}^{cd}\right)e_\nu^m\eta_{\rm dm}\right]\eta_{\rm bc} \end{aligned} \tag{33}$$

Where $\tilde{\eta}^{\nu\rho}$ is non-commutativity anti-symmetric matrix elements defined as

$$[\tilde{x}^\mu, \tilde{x}^\nu] = i\tilde{\eta}^{\mu\nu} \tag{34}$$

And $\tilde{x}^\mu$ are the non-commutative space–time coordinates operators. Here, ω_λ^{ab} (resp.D_ρ) is the commutative spin connection (resp. covariant derivative) and $R_{\mu\nu}^{ad} = e_\alpha^a e_\beta^b R_{\mu\nu}^{\alpha\beta}$, where $R_{\mu\nu}^{\alpha\beta}$ is the Riemann tensor. The commutative space–time vierbein and the Minkowski metric are denoted by e_μ^a and $\eta_{\mu b}$, respectively. The non-commutative metric is defined by

$$\hat{g}_{\mu\nu} = \frac{1}{2}\left(\hat{e}_\mu^a * \hat{e}_{\rm a\nu} + \hat{e}_\nu^a * \hat{e}_{a\mu}\right) \tag{35}$$

Where " $*$ " is the Moyal star product [47], straightforward calculations using Maple 13 and setting $z = \frac{r}{r_s}$, $y = \frac{Q^2}{r_s^2}$, $\lambda = \frac{\tilde{\eta}^2}{r_s^2}$ (in the case $\theta = \pi/2$, with choosing the only non-vanishing components of the NC parameter $\tilde{\eta}^{01} = -\tilde{\eta}^{10} = \tilde{\eta}$) one has

$$\begin{aligned} F &= -\left(1 - \frac{1}{z} + \frac{y}{z^2}\right) - \frac{\left(2z^3 - 9yz^2 - \frac{11}{4}z^2 + 15zy - 14y^2\right)\lambda}{4z^2} \\ G &= \frac{1}{\left(1 - \frac{1}{z} + \frac{y}{z^2}\right)} + \frac{\left(-z^3 + \frac{3}{4}z^2 + 3yz^2 - 3yz + 2y^2\right)\lambda}{4z^2(z^2 - z + y)^2} \\ H &= (z^2 + \frac{\left(z^4 - \frac{17}{2}z^3 + \frac{17}{2}z^2 + 27yz^2 - \frac{75}{2}zy + 30y^2\right)\lambda}{16z^2(z^2 - z + y)} \\ I &= (z^2 + \frac{(-2z^3 + 8yz^2 + z^2 - 8zy + 8y^2)\lambda}{16(z^2 - z + y)} \end{aligned} \tag{36}$$

We have two singularities $z_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} + y}$.

Figure 1 displays the variation of the entanglement $E(\varrho)$ as a function of the NC parameter $\tilde{\eta}^2$, for a non-charged $(Q=0)$ black hole and fixed $z=1.5, y=0, \alpha=1$, $q=0.01$. Notice that if $\tilde{\eta}^2$ increases, $E(\varrho)$ decreases. Thus, $\tilde{\eta}^2$ plays an important role in the value changing of entanglement. In fact, as it was pointed out in ref [46], the NC parameter $\tilde{\eta}$ can be considered as a magnetic field contributing to the matter density ρ and therefore affecting the curvature of the space–time through its contribution to GF. Consequently if $\tilde{\eta}^2$ increases, the GF increases and the information decreases. Including the contribution of NC of space–time, it generates additional terms proportional to $\tilde{\eta}^2$. In fact, the gravitational potential $\hat{g}_{00}$ will be of the form

$$\hat{g}_{00} = \hat{A} + \hat{B}Q^2 + \tilde{\eta}^2\left(\hat{D}Q^4 + \hat{C}Q^2 + \hat{F}\right) \tag{37}$$

Where

$$\hat{A} = -1 + \frac{1}{z}, \hat{B} = -\frac{1}{zr_s^2}, \hat{D} = \frac{7}{z^6 r_s^6}, \hat{C} = (9z-15)\frac{1}{4z^5 r_s^4}, \hat{F} = \left(-2z + \frac{11}{4}\right)\frac{1}{4z^4 r_s^4} \tag{38}$$

The behavior of the entanglement $E(\varrho)$ depends strongly on the sign of $\left(\hat{D}Q^4 + \hat{C}Q^2 + \hat{F}\right)$. Considering $\hat{A}$ and $\hat{B}$ negative:

(1) If $Q \gg 1$, the term $\hat{D}Q^4$ dominates. Since $\hat{D} > 0$, and if $\tilde{\eta}^2$ increases the GF decreases leading to an increase in $E(\varrho)$(as is the case in Fig. 2).

(2) If $Q \ll 1$, then $\hat{F}$ dominates and its sign will determine the behavior of $E(\varrho)$ as a function of $\tilde{\eta}^2$. If $\hat{F} > 0$ GF increases and $E(\varrho)$ decreases then we return to the case in Fig. 1.

Figure 3 represents the variation of $E(\varrho)$ as a function of z for fixed $Q=0, \tilde{\eta}=0, \alpha=1, \mathrm{q}=0.01$, (the case of commutative Schwarzschild space–time). Notice that we will reproduce the same behavior as in ref [10].

Figure 4 shows the variation of $E(\varrho)$ as a function of z for fixed $Q \neq 0, \tilde{\eta}=0$ (the case of commutative Reissner–Nordstrom space–time). Notice that the same behavior as in ref [11] is obtained.

Figure 5 shows the variation of $E(\varrho)$ as a function of z and fixed $\lambda=0.01, y=0, \alpha=1, q=0.01$; this case is the Schwarzschild black hole in non-commutative space–time.

Figure 6 represents the variation of $E(\varrho)$ as a function of z for fixed $\lambda=0.1, y=2, \alpha=1, q=0.01$ it is the case of Reissner–Nordstrom Black Hole in non-commutative space–time. Notice that far from the oscillatory behavior region, when z (or r) increases, the GF $\hat{g}_{00}$ decreases until reaching a saturation value (~ 1) where $E(\varrho)$ is maximal. Notice that for smaller values of $r(\rightarrow 0$ near black hole singularity) where the gravitational field is infinite, the entanglement is minimal. If we go far from the singularity (z increases), the gravitational field decreases and therefore the information increases and thus the entanglement. The oscillatory behavior disappears when we enter the stability region where $E(\varrho) \sim 0.67$. The number of picks and minima depends strongly on the values of the various parameters λ, y, α and q. Concerning the non-commutativity effect on the $E(\varrho)$, it is clear from Eq. (37) that for smaller values of z, as $\tilde{\eta}$ increases the gravitational field $\hat{g}_{00}$ becomes more important (increases) and therefore $E(\varrho)$ decreases. For larger values of z, the effect is almost negligible since the terms in order of $\frac{1}{z^4}, \frac{1}{z^5}, \frac{1}{z^6}$ decrease faster than the commutative terms in order of $\frac{1}{z}$. Notice also that y increases, the GF increases (the term $\tilde{\eta}^2\left(\hat{D}Q^4\right)$ dominates at larger value of Q). Thus, the NC effect on the $E(\varrho)$ becomes more important for charged black hole than neutral ones (if the charge Q increases, $E(\varrho)$ decreases).

Table 1 summarizes the effect of the black hole charge on the $E(\varrho)$. It is worth mentioning that in order to keep the perturbative expansion with respect to $\tilde{\eta}^2$ reliable, one must have

$$\left|\tilde{\eta}^2 A_1\right| < |A_0| \tag{39}$$

Where $A_0 = \hat{A} + \hat{B}Q^2$ and $A_1 = \left(\hat{D}Q^4 + \hat{C}Q^2 + \hat{F}\right)$, this implies new constraints on the space parameters z, y.

4. Comparison between singlet and triplet state of entanglement

To gain a thorough understanding, we compare the entanglement behavior in the triplet and singlet states, by using concurrence.

Figure 7 shows how the concurrence varies as a function of z for fixed values of q, y, α *and* λ in singlet and triplet state, respectively. We found the same behavior with Fig. 6, where for smaller values of $r(r \rightarrow 0$ near the black hole horizon), the gravitational field is infinite, the entanglement is minimal ($C\left(\rho^f\right) \sim 0$). If we go far away from the singularity, (z increases) the gravitational field decreases, so the information increases until a saturated bound of the maximal entanglement ($C\left(\rho^f\right) \sim 1$). By monitoring both of the curves, we notice that in singlet state when z is at the value of 1.23, $C\left(\rho^f\right)$ takes the value of 0.7789. While in triplet state it gives $z=1.23, C\left(\rho^f\right)=0.5962$.

Figure 8 displays the variation of the concurrence as a function of $\boldsymbol{\lambda}$ by fixing $\alpha=1, z=1.5$ and $q=0.01$ for both state singlet and triplet, the concurrence is a decreasing function. This is due to the fact that the gravitational potential g_{00} increases, as we mentioned in Fig. 2. Take

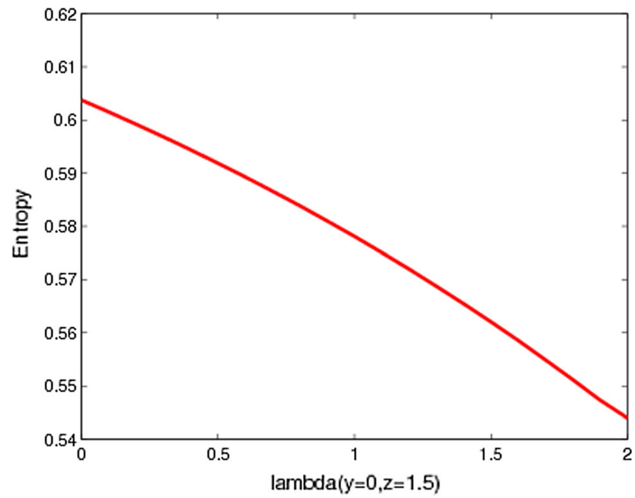

Fig. 1 $\boldsymbol{E}(\varrho)$ as a function of λ for fixed $z = 1.5, y = 0, \alpha = 1, q = 0.01$

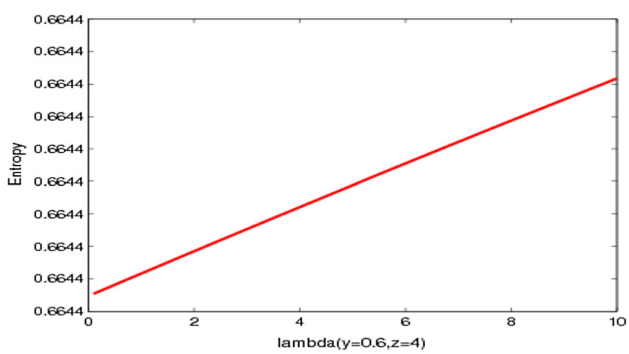

Fig. 2 $\boldsymbol{E}(\varrho)$ as a function of λ for fixed $z = 4, y = 0.6, \alpha = 1, q = 0.01$

note of this for singlet state when $z = 0.1, C\left(\rho^f\right) = 0.9069$, and triplet state when $z = 0.1, C\left(\rho^f\right) = 0.8547$.

As it is displayed in Figs. 7 and 8, we can say that the information (entanglement) for the first is greater or equal to the second, and that the singlet state is more resistant to changes induced by motion than the triplet state, this is due to the fact that for the single state, there is a minimum number of parameters and as mentioned before, gravity decreases the information (the effect of gravity on the single state is less than that of the triplet state).

5. Conclusions

Throughout this paper, we have studied the spin entanglement of two particles system quantified by entanglement

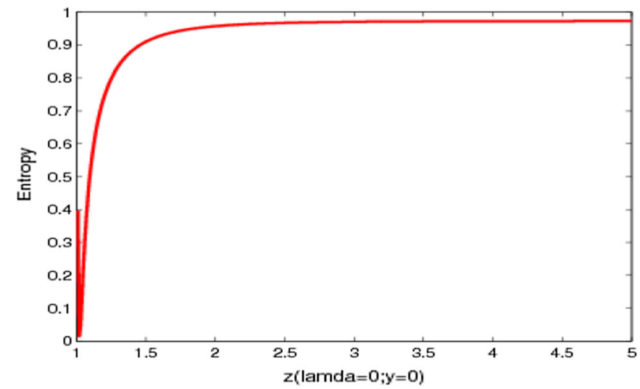

Fig. 3 The variation of $\boldsymbol{E}(\varrho)$ as a function of z for fixed $\lambda = 0, \alpha = 1, y = 0, q = 0.01$

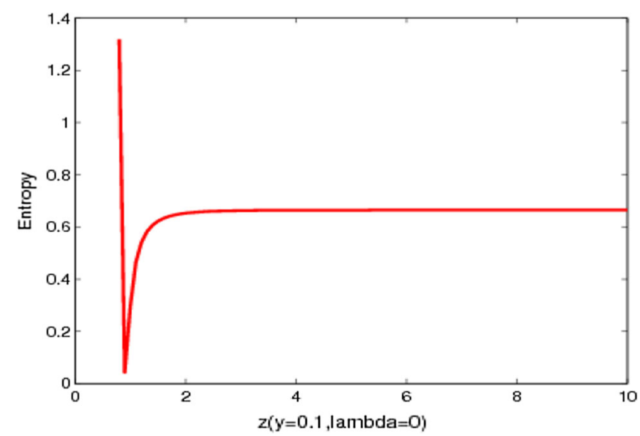

Fig. 4 The variation of $\boldsymbol{E}(\varrho)$ as a function of z for fixed $y = 0.1, \lambda = 0, y = 0, \alpha = 1, q = 0.01$

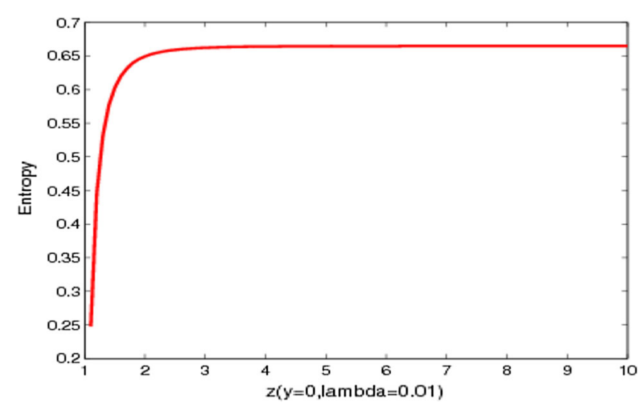

Fig. 5 $\boldsymbol{E}(\varrho)$ as a function of z and fixed $\lambda = 0.01, y = 0, \alpha = 1, q = 0.01$

$E(\varrho)$. Regarding the non-commutative case (as shown in Figs. 1, 2, 3, 4, 5 and 6), the variation of the quantum entanglement as a function of z, the NC parameter λ and

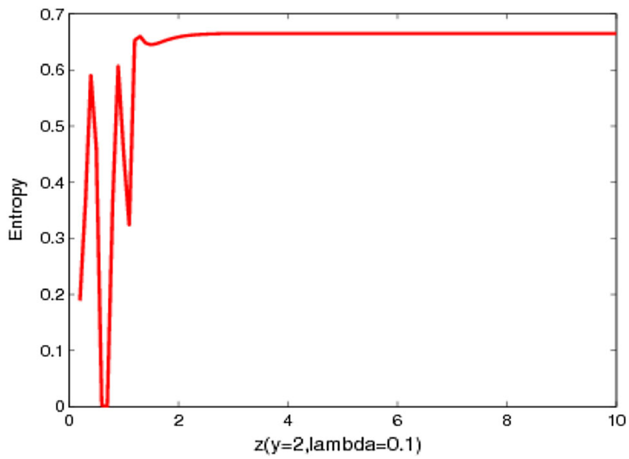

Fig. 6 The variation of $E(\varrho)$ as a function of z for fixed $\lambda = 0.1, y = 2, \alpha = 1, q = 0.01$

Table 1 Illustrative values of $E(\varrho)$ as a function of z for $y = 0$ *and* $y = 10$

z	$E(\varrho)(y = 0)$	$E(\varrho)(y = 10)$
2	0.64694	0.6072
4	0.664	0.6567
5	0.6644	0.6626
6	0.6646	0.6641

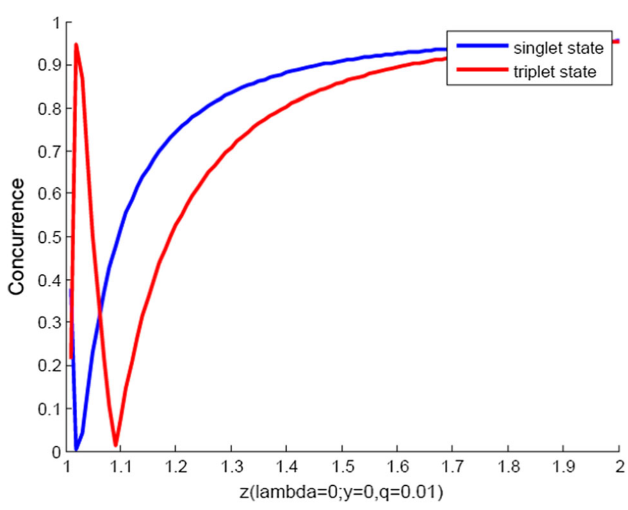

Fig. 7 The variation of $C(\rho^f)$ as a function of z for fixed $\lambda = 0, y = 0, \alpha = 1, q = 0.01$

the black hole charge y is discussed. We have noticed that the NC effect on $E(\varrho)$ becomes more important in a charged black hole, so the behavior of $E(\varrho)$ depends on the black hole's characteristics and not only on the kind of particles (bosons or fermions) [48]. We found as NC

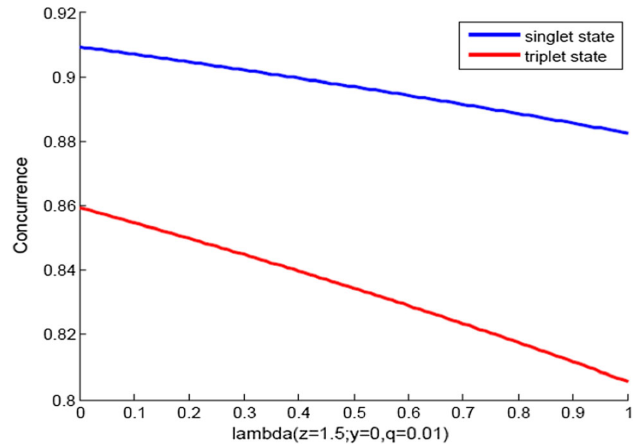

Fig. 8 The variation of $C(\rho^f)$ as a function of λ for fixed $z = 1.5, y = 0, \alpha = 1, q = 0.01$

parameter increases, $E(\varrho)$ decreases, as if NC parameter is playing the role of gravity. On the other hand, as we mentioned in the introduction, NC parameter was considered as having antigravity properties (quintessence, dark energy, etc.), so NC parameter can induce two terms with opposing signs and that was confirmed in [48, 49]

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References

[1] A Einstein *Rev.* **47** 777 (1935)
[2] M A Nielsen and I L Chuang (Cambridge University Press) (2000)
[3] C H Bennett and S J Wiesner *Phys. Rev. Lett.* **69** 2881 (1992)
[4] M Czachor and M Wilczewski *Phys. Rev. A* **68** 010302 (2003)
[5] H Li and J Du *Phys. Rev. A* **68** 022108 (2003)
[6] A Peres, P F Scudo and D R Terno *Phys. Rev. Lett.* **88** 230402 (2002)
[7] R M Gingrich and C Adami *Phys. Rev. Lett.* **89** 270402 (2002)
[8] H Terashima and M Ueda *Phys. Rev. A* **69** 032113 (2004)
[9] H Terashima and M Ueda *J. Phys. A* **38** 2029 (2005)
[10] F Ahmadi and M Mehrafarin *Quan. Infor. Proc.* **13** 639 (2014)
[11] B Nasr Esfahani *J. Phys. A* **43** 455305 (2010)
[12] A Mohadi, N Mebarki and M Boussahel *J. Phys. Conf. Ser.* **1269** 012014 (2019)
[13] N Mebarki, A Mohadi and M Boussahel Yukawa Institute for Theoretical Physics (Kyoto University, Japan) August 8–12 (2016)
[14] H S Snyder *Phys. Rev.* **72** 68 (1947)
[15] P Nicolini *Lett. B* **632** 547 (2006)
[16] A Connes (Academic Pres., New York) (1994)
[17] B Juro, L Muller, S Schraml and P Schupp *Phys. J. C.* **21** 383 (2001)
[18] N Mebarki and F Khelili *J. Theor. Phys.* **6** 20 (2009)
[19] M Chaichian *Lett. B* **660** 573 (2008)
[20] H Bouhallouf *J. Theor. Phys.* **56** 3588 (2017)

[21] N Mebarki *J. Theor. Phys.* **7** 181 (2010)
[22] N Mebarki, S Zaim, L Khodja and H Aissaoui *Phys. Scripta.* **78** 045101 (2008)
[23] S Zaim and A Boudine *J. Phys.* **53** 445 (2008)
[24] N Mebarki and L Khodja *Phys. Scr.* **81** 055103 (2010)
[25] B Vakili *Lett. B* **687** 119 (2010)
[26] G Oliveira-Neto and A R Vaz *Eur. Phys. J. Plus.* **132** 1 (2017)
[27] P K F Kuhfittig *Mod. Phys.* **8** 323 (2017)
[28] S Kawamoto, C Y Wei and W Y Wen *Class. Quant. Grav* **34** 177001 (2017)
[29] M M Ettefaghi *Phys. Rev. D* **79** 065022 (2009)
[30] N Mebarki Dark Energy, Induced Cosmological Constant and Matter – Antimatter Asymmetry From Non Commutative Geometry, in Proc. 3rd Int Meeting. On the Frontiers of Physics (eds.) S P Chia, K Ratnavelu and M R Muhamad, (Melville, New York), p 38 (2009)
[31] A Espinoza-Garcia, J Socorro 10th Workshop of the gravitation and Mathematical physics Division of the Mexican Physical Society (eds.) R B B_arcenas et al 545 (2014)
[32] Z Rezaei, S Peyman Zakeri (e-Print 2007.01501[hep-ph]) (2020)
[33] S Ryu and T Takayanagi *Phys. Rev. Lett.* **96** 181602 (2006)
[34] S N Solodukhin *Liv. Rev. Relat.* **14** 8 (2011)
[35] S C Jing *J. Phys. A* **38** 8409 (2005)
[36] S Adhikari, B Chakraborty, A S Majumdar and S Vaidya *Phys. Rev. A* **79** 042109 (2009)
[37] C Bastos, A Bernardini, O Bertolami, N C Dias and J N Prata *Phys. Rev. D* **88** 085013 (2013)
[38] A Muhuri *Phys. J. Plus.* **136** 1 (2021)
[39] Y Yeol et al *Phys. Lett.* **78** 20006 (2007)
[40] H Yao and X-L Qi *Phys. Rev. Lett.* **105** 080501 (2010)
[41] S Weinberg (Cambridge University Press) (1995)
[42] P L Saldanha and V W Vedral *Phys. Rev. A* **87** 042102 (2013)
[43] W K Wootters *Phys. Rev. Lett.* **80** 2245 (1998)
[44] S Hill and W K Wootters *Phys. Rev. Lett.* **78** 5022 (1997)
[45] P Das *Phys. J. C* **77** 1 (2017)
[46] H A Chamseddine *Phy. Lett. B* **504** 33 (2001)
[47] N Mebarki *J. Theo. Phys.* **3** 55 (2006)
[48] M F Ghiti, N Mebarki and H Aissaoui *Inter. J. Mod Phys. A* **30** 1550141 (2015)
[49] M F Ghiti, N Mebarki and H Aissaoui *J. Phys. Conf. Ser.* **1766** 012023 (2021)

Bibliography

[1] P Coles. Einstein, Eddington and the 1919 Elipse. *arxiv:astro-ph/0102462v1*, 2001.

[2] P A M Dirac. Directions in Physics. *Wiley-Interscience*, 72-86, 1978.

[3] A D Sakharov et al. Vacuum quantum fluctuations in curved space and the theory of gravitation. *Soviet Phys Uspekhi,* **34**(5): 394, 1991.

[4] M Visser Sakharov's induced gravity: a modern perspective. *Mod Phys Lett A,* **17**(15n17): 977-991, 2002.

[5] V A Sanghai and T Clifton. Parameterized post-Newtonian cosmology. *Classical and Quantum Gravity,* **34**(6): 065003, 2017.

[6] V A Sanghai and T Clifton. Cosmological backreaction in the presence of radiation and a cosmological constant. *Physical Review D,* **94**(2): 023505, 2016.

[7] J Foster and J D Nightingale. A Short Course in Generel Relativity. *Springer Science,* 2010.

[8] S M Caroll. Lecture Notes On General Reletivity. *arXiv preprint gr-qc*/9712019, 1997.

[9] Th Kalusa. On the Unification Problem in Physics. *arXiv preprint arXiv*:1803.08616, 2018.

[10] N Arkani-Hamed et al. A small cosmological constant from a large extra dimension. *Phys Lett B,* **480**(1-2):193-199, 2000.

[11] A G Cohen and D B Kaplan. Solving the hierarchy problem with noncompact extra dimensions. *Phys Lett B,* **470**(1-4): 52-58, 1999.

[12] C L Bennett et al. The Microwave Anisotropy Probe* Mission. *Astrophys J*, **583**(1): 1, 2003.

[13] C Eckart. The thermodynamics of irreversible processes. III. Relativistic theory of the simple fluid. *Phys Rev*, **58**(10):919, 1940.

[14] W Israel and J M Stewart. Thermodynamics of nonstationary and transient effects in a relativistic gas. *Phys Lett A*, **58(**4):213-215, 1976.

[15] S Weinberg. Entropy generation and the survival of protogalaxies in an expanding universe. *Astrophys J*, **168**, 175, 1971.

[16] S Weinberg. Gravitation and Cosmology. *Wiley New York*, 1972.

[17] S Floerchinger, N Tetradis and U A Wiedemann. Accelerating Cosmological Expansion from Shear and Bulk Viscosity. *Phys Rev Lett*, **114(9):** 091301, 2015.

[18] A G Riess et al. Observational evidence from supernovae for an accelerating universe and a cosmological constant. *Astron J*, **116**(3):1009, 1998.

[19] J L Tonry et al. Cosmological results from high-z supernovae. *Astrophys J*, **594**(1): 1,2003. R A Knop. Suparnova Cosmology Project Collaboration. *Astrophys J*, **598**:102, 2003.

[20] P Astier et al. The Supernova Legacy Survey: Measurement of Omega_M, Omega_Lambda and w from the First Year Data Set. *IAU Joint Discussion*, **26**:20, 2006.

[21] D N Spergel et al. Three-year Wilkinson Microwave Anisotropy Probe (WMAP) observations: implications for cosmology. *Astrophys J Supplement Series,* **170**(2):377, 2007.

[22] T M Davis et al. Scrutinizing exotic cosmological models using ESSENCE supernova data combined with other cosmological probes, *Astrophys J,* **666**(2):716, 2007.

[23] R R Caldwell and P J Steinhardt. Quintessence. *Phys world,* **13**(11): 31, 2000.

[24] A Pradhan, R Zia and R P Singh. Viscous fluid cosmology with time dependent q and $\Lambda-$term in Bianchi type-I space-time and late time acceleration. *Ind J of Phys,* **87**(11): 1157–1167, (2013).

[25] S P Kandalkar, P P Khade and S P Gawande. Kantowski-Sachs viscous fluid cosmological model with a varying Λ. *Turk J Phys* 33 (2009) ,155 –165.

[26] S Capozziello, S Carloni and A Troisi. Curvature quintessence matched with observational data. *Inter J of Mod Phys D,* 12(10): 1969–1982, 2003.

[27] V Faraoni. f (R) gravity: successes and challenges. *arXiv preprint arXiv*:0810.2602, 2008.

[28] B C Paul. Viscous cosmologies with extra dimensions. *Phys Rev D,* **64**(2): 027302, 2001.

[29] J Miritzis. The recollapse problem of closed Friedmann–Robertson–Walker models in higher-order gravity theories. *J of math phys,* **46**(8): 082502, 2005.

[30] I Y Aref'eva, L V Joukovskaya, and S Y Vernov. Dynamics in nonlocal linear models in the Friedmann-Robertson-Walker metric. *J of Phys A: Math and Theor,* **41**(30): 304003 (2008).

[31] A Malki, N Mebarki and M Y Boudjada. FRW cosmology in Finsler geometry with extra dimensions. *AIP Conf Proc 8*, **1444**(1): 438-442, 2012.

[32] R F Sawyer. Bulk viscosity of a gas of neutrinos and coupled scalar particles, in the era of recombination. *Phys Rev D,* **74**(4): 043527, 2006.

[33] V B Johri and R Sudharsan. BD-FRW cosmology with bulk viscosity. *Austr J of Phys,* **42**(2): 215-222, 1989.

[34] R. Kantowski and R. K. Sachs. Some spatially homogeneous anisotropic relativistic cosmological models. *J of Math Phys,* **7**(3): 443-446, 1966.

[35] C B Collins. Global structure of the"Kantowski–Sachs"cosmological models. *J of Math Phys*, **18**(11): 2116-2124, 1977.

[36] L E. Mendez and A B Henriquez. Inflation in a simple Kantowski-Sachs model. *Phys Lett B,* **254:** 44-48, 1991.

[37] M De Cesare, S S Seahra and E Wilson-Ewing. The singularity in mimetic Kantowski-Sachs cosmology. *J of Cosm and Astropar Phys*, **2020**(07): 018, 2020.

[38] K S Adhav et al. Kantowski-Sachs cosmological models with anisotropic dark energy. *Open Physics*, **9**(4): 919-925, 2011.

[39] V B Raut, K S Adhav and D K Joshi. Anisotropic Kantowski-Sachs Cosmological Model in)(Rf Theory of Gravity with Volumetric Exponential Expansion Law. 2(1). ISSN : 2321-497X • pp. 21-28, 2014

[40] B N Breizman, V Ts Gurovich and V P Sokolov. The Possibility Of Setting Up Regular Cosmological Solutions. *Sov. Phys. - JETP,* **32**: 155, 1971.

[41] M Amarzguioui et al. Cosmological constraints on f (R) gravity theories within the Palatini approach. *Astrono & Astrophys*, **454**(3): 707-714, 2006.

[42] A A Starobinsky. A new type of isotropic cosmological models without singularity. *Phys Lett B*, **91**(1): 99-201, 1980.

[43] S Nojiri and S D Odintsov. Modified gravity with ln R terms and cosmic acceleration. *Gen Rel and Grav*, **36**(8)**:**1765-1780, 2004.

[44] S. M. Carroll et al. Cosmology of generalized modified gravity models. *Phy Rev D,* **71**(6): 063513, 2005.

[45] S Capozziello et al. Cosmological viability of F(R)-gravity as an ideal fluid and its compatibility with a matter dominated phase. *Phys Lett B,* **639**(3-4):135-143, 2006.

[46] S A Appleby and R A Battye. Do consistent F(R) models mimic general relativity plus Λ?. *Phy Lett B,* **654**(1-2): 7-12, 2007.

[47] J D Evans, L M H Hall and P Caillo. Standard cosmological evolution in a wide range of F(R) models. *Phys Rev D,* **77**(8): 083514, 2008.

[48] A Dev et al. Delicate f (R) gravity models with a disappearing cosmological constant and observational constraints on the model parameters. *Phys Rev D,* **78**(8): 083515, 2008.

[49] K Freese and M Lewis. Cardassian expansion: a model in which the universe is flat, matter dominated, and accelerating. *Phys Lett B,* **540**(1-2):(1-8), 2002.

[50] S. M. Capozzielo, M De Laurentis and O Luongo. Connecting early and late universe by F(R) gravity. *Int J of Mod Phys D,* **24**(04): 1541002, 2015.

[51] S Nojiri and S D Odintsov. Newton law corrections and instabilities in F(R) gravity with the effective cosmological constant epoch. *Phys Lett B,* **652**(5-6): 343-348, 2007.

[52] S Nojiri and S D Odintsov. Unifying inflation with LambdaCDM epoch in modified F(R) gravity consistent with Solar System tests. *Phys Lett B.* **657**(4-5): 238-245, 2007.

[53] G Cognola et al. A Class of viable modified F(R) gravities describing inflation and the onset of accelerated expansion. *Phys Rev D,* **77**(4): 046009, 2008.

[54] A A Starobinsky. Disappearing cosmological constant in F(R) gravity. *JETP Lett*, **86**(3): 157-163, 2007.

[55] M C B Abdalla, S Nojiri and S D Odintsov. Consistent modifed gravity: Dark energy, acceleration and the absence of cosmic doomsday. *Class Quantum Grav*, **22**(5): L35, 2005.

[56] A Einstein, B Podolsky and N Rosen. Can quantum-mechanical description of physical reality be considered complete?. *Phys Rev*, 47(10): 777, 1935.

[57] R Horodecki et al. Quantum entanglement. *Rev Mod of Phys*, **81**(2): 865-942, 2009.

[58] M A Nielsen and I L Chuang. Quantum Computation and Quantum Information. *Cambridge University Press, Cambridge, England,* 2000.

[59] M Le Bellac. A short introduction to quantum information and quantum computation. *Cambridge University Press*, 2006.

[60] C Bennett and G Brassard. Quantum cryptography: Public key distribution and coin tossing. *Proceedings of the IEEE international Conference on Computers, systems and signal Proceeding, Bangalore*, 175-179, 1984.

arXiv preprint arXiv:2003.06557, 2020.

[61] A K Ekert. Quantum Cryptography and Bell's Theorem. Quantum Measurements in Optics, :413–418, 1992.

[62] C H Bennett et al. Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels. *Phys Rev Lett*, **70**(13): 1895, 1993.

[63] A Ekert and R Jozsa. Quantum computation and Shor's factoring algorithm. Rev of Mod Phys, **68**(3): 733, 1996.

[64] D Boschi. Experimental realization of teleporting an unknown pure quantum state via dual classical and Einstein-Podolski-Rosen channels. *Phys Rev Lett*, **80**(6):1121-1125, 1998.

[65] D Bouwmeester et al. Experimental quantum teleportation. *Phil Trans R Soc Lond A*, **356**(1733):1743-1737, 1998.

[66] A Furusawa et al. Unconditional quantum teleportation. *Science,* **282**(5389): 706–709, 1998.

[67] K Mattle et al. Dense Coding in Experimental Quantum Communication. *Phys Rev Lett*, **76**(25): 4656, 1996

[68] T Jennewein et al. Quantum Cryptography with Entangled Photons. *Phys Rev Lett,* **84**(20): 4729, 2000.

[69] W Tittel. Quantum Cryptography Using Entangled Photons in Energy-Time Bell States. *Phys Rev Lett*, **84**(20): 4737, 2000.

[70] P W Shor. Scheme for reducing decoherence in quantum computer memory. *Phys Rev A,* **52**(4): R2493, 1995.

[71] Y Shi. Quantum Entanglement in Second-quantized Condensed Matter Systems. *J Phys A,* **37**(26): 6807, 2004.

[72] A Osterloh et al. Scaling of Entanglement close to a Quantum Phase Transitions. *Nat*, **416**(6881): 608-610, 2002.

[73] A Peres, P F Scudo, and D R Terno. Quantum Entropy and Special Relativity. *Phys Rev Lett*, **88**(23):230402, 2002.

[74] H Terashima and M Ueda. Einstein-Podolsky-Rosen correlation seen from moving observers. arXiv preprint quant-ph/0204138, 2003.

[75] P M Alsing and G J Milburn. On Entanglement and Lorentz Transformations. *Quant Inf and Compu*, **2**(6): 487-512, 2002.

[76] H Li and J Du. Relativistic-invariant quantum entanglement between the spins of moving bodies. *Phys Rev A*, **68**(2): 022108, 2003.

[77] W T Kim and E J Son. Lorentz-invariant Bell's inequality. *Phys Rev A,* **71**(1): 014102, 2005.

[78] A S V Bittencourt, A E Bernardini and M Blasone. Global Dirac bispinor entanglement under Lorentz boosts. *Phys Rev A*, **97**(3): 032106, 2018.

[79] D Ahn, H J Lee, Y H Moon and S W Hwang. Relativistic entanglement and Bell's inequality. *Phys Rev A,* 67(1): 012103, 2003.

[80] R M Gingrich and C Adami. Quantum entanglement of moving bodies. *Phys Rev Lett.* **89**(27): 270402, 2002.

[81] J Leon and E Martin-Martinez. Spin and occupation number entanglement of Dirac fields for noninertial observers. *Phys Rev A* **80**(1): 012314, 2009.

[82] P M Alsing and I Fuentes. Observer-dependent entanglement. *Class Quant Grav*, **29**(22):224001, 2012.

[83] B Nasr Esfahani, M Ahmadi and F Ahmadi. Fidelity of two-particle wave packets moving around the Schwarzschild spacetime. *Int J Theor Phys*, 47(12): 3130–3138, 2008.

[84] F Ahmadi and M Mehrafarin. Entanglement of three-qubit spin states moving in the gravitational field of a massive body. *J Phys A,* **45**(48): 485302, 2012.

[85] F Ahmadi and M Mehrafarin. Entangled spin states in geodesic motion around massive body. *Quan infor proc*, **13**(3):639–647, 2014

[86] D Ahn et al. The black hole final state for the Dirac fields in Schwarzschild spacetime. *J High Energy Phys*, **2008**(6):062, 2008.

[87] J L Ball, I Fuentes-Schuller and F P Schuller. Entanglement in an expanding spacetime. *Phys Lett A,* **359**(6): 550-554, 2006.

[88] G Ver Steeg and N C Menicucci. Entangling power of an expanding universe. *Phys Rev D,* **79**(4): 044027, 2009.

[89] C H Lee and Y J Han. Innermost stable circular orbit of Kerr-MOG black hole. *Eur Phys J C,* **77**(10): 1-9, 2017.

[90] A Mohadi, N Mebarki and M Boussahel. The cosmological constant effect on the quantum entanglement. *J Phys: Conf. Ser*, **1269**(1): 012014, 2019.

[91] J S Farnes. A unifying theory of dark energy and dark matter: Negative masses and matter creation within a modified ΛCDM framework. *Astrono & Astrophys,* **620**:A29, 2018.

[92] J S Bell. On the Einstein–Podolsky–Rosen paradox. On the einstein podolsky rosen paradox. *Physics Physique Fizika,* **1**(3): 195, 1964.

[93] M Horodecki, P Horodecki and R Horodecki. Separability of n-particle mixed states: necessary and sufficient conditions in terms of linear maps. *Phys Lett A,* **283**(1-2): 1-7, 2001.

[94] K Bartkiewicz et al. Quantifying entanglement of a two-qubit system via measurable and invariant moments of its partially transposed density matrix. *Phys Rev A* , **91**(2): 022323, 2015

[95] K Bartkiewicz et al. Method for universal detection of two-photon polarization entanglement. *Phys Rev A,* **91**(3): 032315, 2015.

[96] C H Bennett et al. Concentrating partial entanglement by local operations. *Phys Rev A*, **53**(4): 2046, 1996.

[97] C H Bennett et al. Mixed-state entanglement and quantum error correction. *Phys Rev A*, **54**(5): 3824, 1996.

[98] W K Wootters. Entanglement of formation and concurrence. *Quantum Inf Comput*, **1**(1): 27–44, 2001.

[99] S Hill and W K Wootters. Entanglement of a pair of quantum bits. *Phys Rev Lett*, **78**(26): 5022, 1997.

[100] W K Wootters. Entanglement of formation of an arbitrary state of two qubits. *Phys Rev Lett*, **80**(10): 2245, 1998.

[101] H Terashima and M Ueda. Einstein-Podolsky-Rosen correlation in gravitational field. *Phys Rev A*, **69**(3): 032113, 2004.

[102] S Hawking and G Ellis. The Large Scale Structure of Space-Time, Cambridge University Press, 1995.

[103] S. Weinberg. The Quantum Theory of Fields. Cambridge University Press, **1**, 1995.

[104] K Huang. From Operators to Path Integrals. Quantum Field Theory (John Wiley and Sons, Inc., New York, 1998), 1998.

[105] N Mebarki, A Mohadi and M Boussahel. Entanglement of spin 1/2 particles with PT symmetric Hamiltonian in a Schwarchild-De Sitter space-time , Progress in Quantum Physics with Non-Hermitian Operators PHHQP16 YITP Workshop, Yukawa Institute for Theoretical Physics, Kyoto University, Japan August 8–12, 2016.N Mebarki, O Morchedi And H Aissaoui. 2015 International Journal of Theoretical Physics 54(11):1

[106] M Schlosshauer. Decoherence and The Quantum to Classical Translation. Springer Science & Business Media, 2007.

[107] P L Saldanha and V Vedral. Wigner rotations and an apparent paradox in relativistic quantum information. *Phys Rev A,* **87**(4): 042102, 2013.

[108] P Das, R S Ripon and S Ghosh. Motion of charged particle in Reissner–Nordstrom space-time: a Jacobi-metric approach. *Eurp Phys J C*, **77**(11): 1-4, 2017.

[109] A H Chamseddine. Deforming Einstein's gravity. *Phys Lett B,* **504**(1-4): 33-37, 2001.

[110] N Mebarki, F Khelili and J Mimouni. Extended Non Symmetric Gravitation Theory with a Scalar Field in Non Commutative Geometry. *Electr J of Theor Phys,* **3**(13): 55-69, 2006.

[111] B N Esfahani. Spin entanglement of two spin-1/2 particles in a classical gravitational field. *J Phys A,* **43**(45): 455305, 2010.

[112] J D Jackson. Classical Electrodynamics. American Association of Physics Teachers, 1999.

[113] E P Wigner. *Ann Math*, **40**: (149-204), 1939.

[114] H Goldstein. Classical Mechanics. Addison-Wesley Pub. Co., 1980.

[115] R M Eisberg. Fundamentals of Modern Physics. *Amer J of Phys,* **30**(12): 937–938, 1962.

[116] E F Taylor and J A Wheeler. Space-time Physics. San Francisco: WH Freeman, 1966.

[117] M Bastero-Gil, B Arjun and O R Rudnei. Shear viscous effects on the primordial power spectrum from warm inflation. *J of Cosmo and Astropart Phys,* **2011**(07): 030 (2011).

[118] S Floerchinger, N Tetradis and U A Wiedemann. Accelerating cosmological expansion from shear and bulk viscosity. *Phys rev lett,* **114**(9): 091301, 2015.

[119] M F Ghiti, N Mebarki and H Aissaoui. Quantum entanglement of fermions–antifermions pair creation modes in noncommutative Bianchi I space–time. *Inter J Mod Phys A,* **30**(24): 1550141, 2015.

[120] M F Ghiti, N Mebarki and H Aissaoui. The von Neumann boson-antiboson pair creation quantum entanglement entropy in noncommutative Bianchi I space-time. *J Phys Conf Ser*, **1766**(1): 012023, 2021.

Abstract

In this thesis, we studied two different phenomena. In the first part, the accelerated expansion of the universe without introducing the concept of dark energy was examined. We studied it in the FRW metric with perfect and non-perfect fluid, as well as the Kantowski-Sachs Space-Time and F (R) Gravity models. We confirmed that geometry has an impact on the accelerated expansion of the universe. In the second part, the effect of gravitational force on entanglement is explored, where the elaboration of a general formalism for quantum spin entanglement in curved space-time is presented. This formulation permits the study of different models in curved space-time. The Schwarzschild-de Sitter Space-Time was investigated as well by inspecting the role of the cosmological constant and its influence on entanglement. The Kerr and the Reissner-Nordström models are also considered, where the effect of the non-commutativity of space as well as black hole rotation on entanglement are discussed by using concurrence and the spin entanglement of a system of two spin-1/2 particles. With particularity, it contains multiple and various physical parameters, allowing for a detailed study of this purely quantum phenomenon in different frames of space and geometry or both at the same time.

Key words: FRW model, Kantowski model, extra dimension**,** Non commutative space, Quantum information, Curved space-time, Spin Entanglement.

Resumé

Dans cette thèse, nous avons étudié deux phénomènes différents. Dans la première partie, l'expansion accélérée de l'univers sans introduire le concept d'énergie noire a été examinée. Nous l'avons étudié dans la métrique FRW avec un fluide parfait et non parfait, l'espace-temps de Kantowski-Sachs et la gravité F (R). on a confirmé que la géométrie a un impact sur l'expansion accélérée de l'univers. Dans la deuxième partie, l'effet de la force gravitationnelle sur l'intrication est exploré, où l'élaboration d'un formalisme général pour l'intrication de spin quantique dans l'espace-temps courbe est présentée. Cette formulation permet l'étude de différents modèles dans un espace-temps courbe. De plus, l'espace-temps de Schwarzschild-de Sitter a été étudié en examinant le rôle de la constante cosmologique et son influence sur l'intrication. Les modèles de Kerr et de Reissner-Nordström sont également considérés, où l'effet de la non-commutativité de l'espace ainsi que la rotation du trou noir sur l'intrication sont discutés en utilisant la concurrence et l'intrication de spin d'un système de deux particules de spin-1/2. Avec la particularité, il contient des paramètres physiques multiples et variés, permettant une étude détaillée de ce phénomène purement quantique dans différents cadres d'espace et de géométrie ou les deux à la fois.

Mot clé : Modèle FRW, Modèle de Kantowski, dimension supplémentaire, Espace non commutatif, Information quantique, Espace-temps courbé, Intrication de spin.

ملخص

في هذه الأطروحة درسنا ظاهرتين مختلفتين. في الجزء الأول، تم فحص تمدد الكون المتسارع دون إدخال مفهوم الطاقة المظلمة. درسنا التوسع المتسارع في مترية FRW مع سائل مثالي و غير مثالي, بالإضافة إلى نماذج Kantowski-Sachs للزمكان, و F (R) للجاذبية, أكدنا أن الهندسة لها تأثير على التوسع المتسارع للكون. في الجزء الثاني , تم فحص تأثير قوة الجاذبية على التشابك حيث يتم تقديم صيغة عامة لتشابك اللف الكمي في الزمكان المنحني. تسمح هذه الصيغة بدراسة نماذج مختلفة في الزمكان المنحني. تم التحقيق في نموذج Schwarzschild-de Sitter للزمكان من خلال فحص دور الثابت الكوني وتأثيره على التشابك. يتم أيضًا النظر في نماذج Kerr و Reissner-Nordström, حيث تم بحث تأثير عدم تبادلية الفضاء على التشابك. تمت مناقشة سلوك Concurrence وكذلك تشابك السبين لنظام مكون من جسيمين لهما سبين-1/2. بخصوصية أكثر ، النموذج يحتوي على معلمات فيزيائية متعددة ومتنوعة ، مما يسمح بدراسة مفصلة لهذه الظاهرة الكمومية البحتة في إطارات مختلفة من الفضاء والهندسة أو كليهما في نفس الوقت.

كلمات مفتاحية: نموذج FRW, نموذج Kantowski, الأبعاد الإضافية, الفضاء غير التبادلي, المعلومات الكوانتية, الزمكان المنحني, تشابك السبين.